ENGR 1100

Week 03 – Vector, Matrix & Linear Equations

Learning Objectives

Upon completion this module, students will be able to:

1. Explain the difference between Scalars, Vectors and Matrices
2. Demonstrate how to multiply Vectors by Scalars
3. Demonstrate how to multiply Vectors by Vectors
    - The inner product (produces a scalar)
    - The outer product (produces a matrix)
4. Explain how to multiply Vectors by Matrices
5. Demonstrate how to multiply Matrices by Matrices
6. Apply different methods to solve linear equations
7. Use MATLAB to solve linear algebra related problems

Engineering Applications

- Vectors are the heart and soul of Cartesian geometry and therefore are used in various of fields such as Mechanics, Fluid mechanics, thermodynamics, etc.

- Matrix can be used in graphic software to process linear transformations to render images.

- Matrix is essential to solve linear equations.

SCALARS , VECTORS & MATRIX DEFINITION

Scalars, Vectors

[Vectors on Khan Academy]

1) Scalar: A single number (integer or real)

2) Vector: An *ordered* list of scalars

   [12 10] ≠ [10 12]

Row vectors

   [1 2 3 4 5] [0.4 1.2 0.07 8.4 12.3] [12 10]   [2]

Column Vectors $\longrightarrow$ $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 12.0 \\ 17.1 \end{bmatrix}$ $\begin{bmatrix} 1.5 \\ 0.3 \\ 6.2 \end{bmatrix}$

Transposing Vectors

If u is a row vector…

$u = [\ 1\ 2\ 3\ 4\ 5\ ]$

…then u^T ("u-transpose") is a column vector

$$\mathbf{u}' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

… and vice-versa.

Matrices

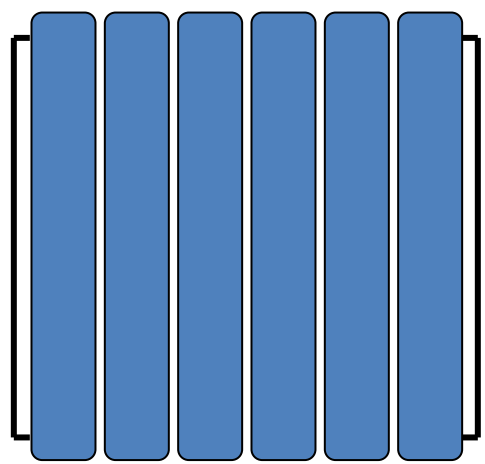
[Matrices on Khan Academy](#)

$$\begin{bmatrix} 1 & 2 & 6 & 1 & 7 & 8 \\ 2 & 5 & 9 & 0 & 0 & 3 \\ 3 & 1 & 5 & 7 & 6 & 3 \\ 2 & 7 & 9 & 3 & 3 & 1 \end{bmatrix}$$

Matrix: An ordered list of *vectors*:

Row vectors

Column vectors



Elements of Matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 6 & 1 & 7 & 8 \\ 2 & 5 & 9 & 0 & 0 & 3 \\ 3 & 1 & 5 & 7 & 6 & 3 \\ 2 & 7 & 9 & 3 & 3 & 1 \end{bmatrix}$$

Matrices are indexed (*row*, *column*)

M(1,3) = 6 (row 1, column 3)

M(3,1) = 3 (row 3, column 1)

Variable Naming Conventions

1) Scalars: Lowercase, *italics*

   x, y, z…

2) Vectors: Lowercase, **bold**

   $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$…

3) Matrices: Uppercase, **bold**

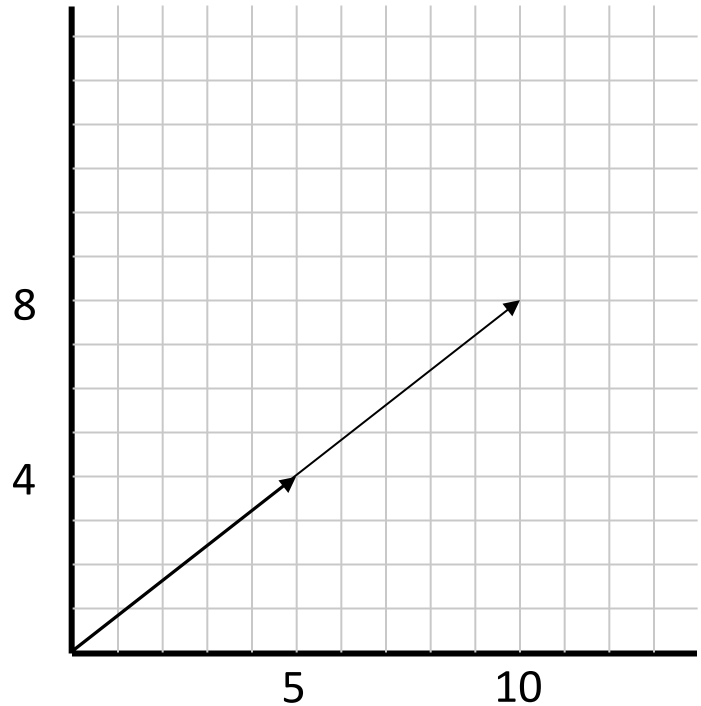
   $\mathbf{M}$, $\mathbf{N}$, $\mathbf{O}$ …

4) Constants: Greek

   α, β, δ, γ, λ …

VECTOR AND MATRIX OPERATIONS

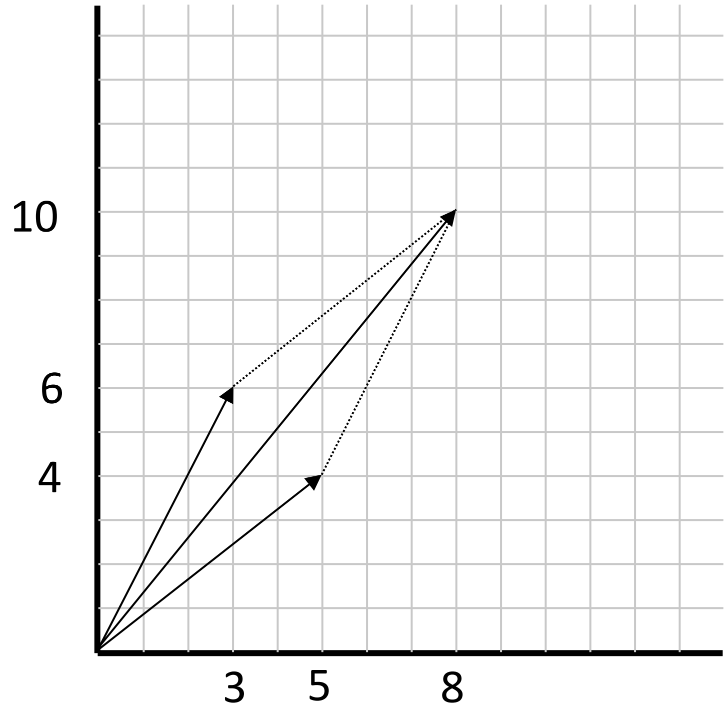
Multiplying a Vector by a Scalar



[5 4] * 2 = [10 8]

Lengthens the vector but does not change its orientation

Adding a Vector to a Vector



[5 4] + [3 6] = [8 10]

Forms a parallelogram.

Basic concepts

- Transpose:

$$\begin{pmatrix} a \\ b \end{pmatrix}^T = \begin{pmatrix} a & b \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- Vector products: $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
  - Dot product:

$$u \bullet v = u^T v = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

  - Outer product:

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

Basic concepts

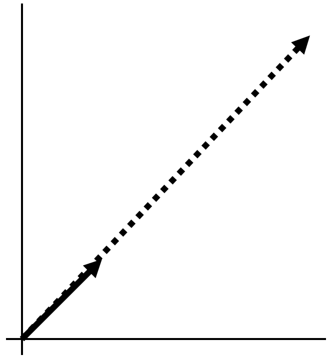
- Matrix product:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

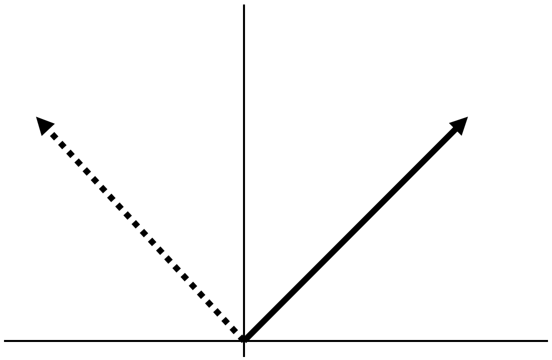
Matrices as linear transformations

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



(stretching)

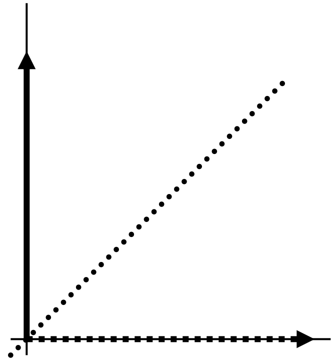
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(rotation)

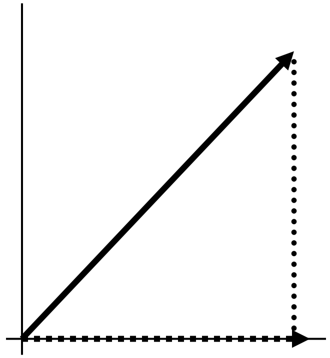
Matrices as linear transformations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



(reflection)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(projection)

Special matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$ diagonal

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$ upper-triangular

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix}$$ tri-diagonal

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$ lower-triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ I (identity matrix)

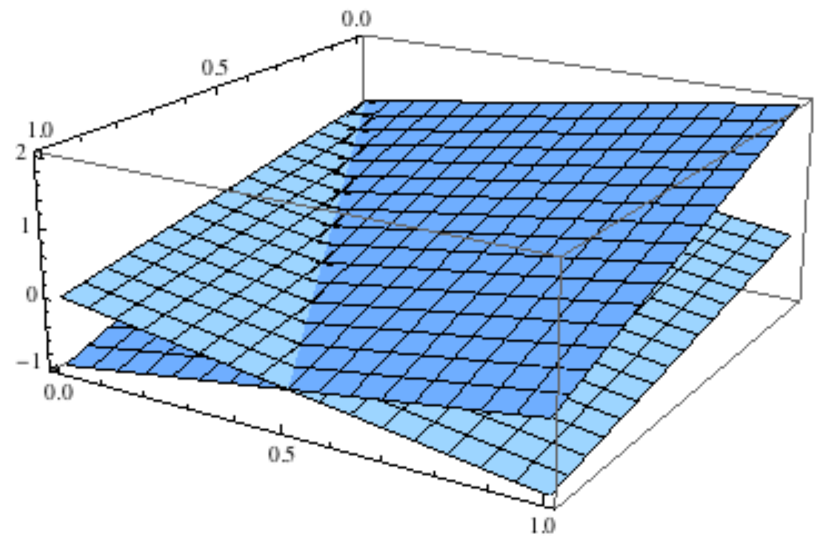
SOLVE LINEAR EQUATIONS

Matrices as sets of constraints

[Systems of Linear Equations on Khan Academy]

$$x + y + z = 1$$
$$2x - y + z = 2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Solving linear equations Ax=b

x + 2y + z = 0
y - z = 2
x +2z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

Write system of equations in matrix form.

----

x + 2y + z = 0
y - z = 2
-2y + z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

Subtract first row from last row.

----

x + 2y + z = 0
y - z = 2
- z = 5

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

Add 2 copies of second row to last row.

Now solve by back-substitution: z = -5, y = 2-z = 7, x = -2y-z = -9

Cramer's Rule

In Linear Algebra, **Cramer's rule** is an explicit formula for the solution of a system of linear equations ($Ax = b$) with as many equations as unknowns, valid whenever the system has a unique solution.

$$x_i = \frac{\det(A_i)}{\det(A)}$$

A_i is the matrix formed by replacing the i-th column of A by the column vector of *b.*

To compute Matrix determinants:

– Simple example: $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

– Matlab: det(A)

Matrix Inversion Method

- A matrix B is called the inverse of matrix A if, when the two matrices are multiplied, the product is the identity matrix.

$$AB = I, \qquad BA = I$$

- The inverse of A is typically written as A^{-1}
- The solution of linear equations can be written as

$$Ax = b$$
$$=> A^{-1}Ax = A^{-1}b$$
$$=> x = A^{-1}b$$

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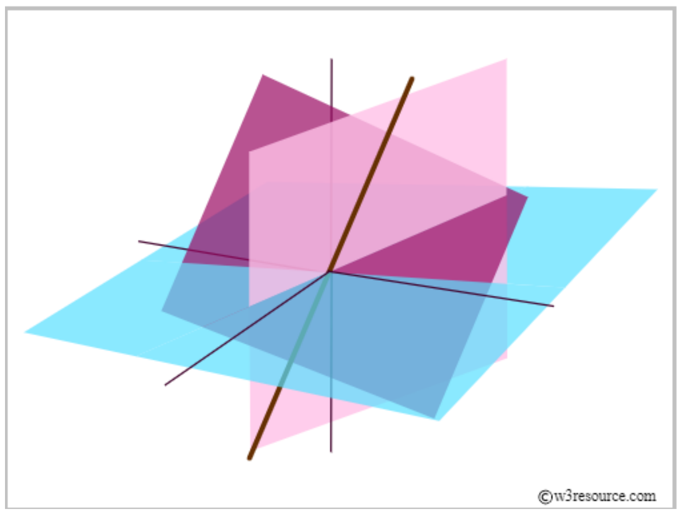
Solve Linear Equations

1. Solve a linear equation by hand
2. Use Cramer’s rule to compute the solution of linear equations
3. Use Matrix Inverse to find solution to linear equations
4. Use Matrix Division to solve linear equations
5. Use MATLAB built-in function linsolve(A,b) to solve linear equations

Why is Linear Algebra important?

[Linear Algebra for Engineers]

Promotional Video by Jeffrey Chasnov



Linear Algebra in MATLAB Resources

MATLAB: Array and Matrix Operations

Matrices in MATLAB Environment

MATLAB: Solving systems of linear equations

MATLAB Exercise 1

1. Create row vectors and column vectors
2. Appending vectors
3. Add & delete elements of a vector
4. Create a matrix
5. Add & delete elements of a matrix
6. Other built-in functions

MATLAB Exercise 2

1. Matrix transpose
2. Array Addition and Subtraction
3. Array Multiplication (inner product & outer product)
4. Array Division
5. Element-wise operations
6. Built-in functions

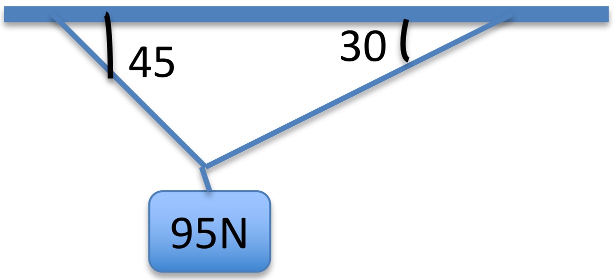
SOLVE LINEAR EQUATIONS

How to Solve Linear Equations?

1. Solve a linear equation by hand (using eliminations and substitutions)
2. Use Cramer's rule to compute the solution of linear equations
3. Use Matrix Inverse to find solution to linear equations
4. Use Matrix Division to solve linear equations
5. Use MATLAB built-in function linsolve(A,b) to solve linear equations

Example 1- Statics

Find the balanced forces on the two cords.



Example 2: Solve using MATLAB

Use matrix operations to solve the following system of linear equations.

$$
\begin{aligned}
4x - 2y + 6z &= 8 \\
2x + 8y + 2z &= 4 \\
6x + 10y + 3z &= 0
\end{aligned}
$$

Example 3

A football stadium has 100,000 seats. In a game with full capacity people with the following ticket and associated cost attended the game:

	Student	Alumni	Faculty	Public	Veterans	Guests
Cost	$25	$40	$60	$70	$32	$0

Determine the number of people that attended the game in each cost category if the total revenue was $4,897,000, there were 11,000 more alumni than faculty, the number of public plus alumni together was 10 times the number of veterans, the number of faculty plus alumni together was the same as the number of students, and the number of faculty plus students together was four times larger than the number of gusts and veterans together.

Solution

Step 1: Problem formulation

$\boldsymbol{S}$ = number of students; $\boldsymbol{A}$ = number of alumni; $\boldsymbol{F}$ = number of faculty;
$\boldsymbol{P}$ = number of public; $\boldsymbol{V}$ = number of veteran; $\boldsymbol{G}$ = number of guest

Total seats

$$S + A + F + P + V + G = 100000$$

Total revenue

$$25S + 40A + 60F + 70P + 32V + 0G = 4897000$$

Other constraints

$$F + 11000 = A$$

$$P + A = 10V$$

$$F + A = S$$

$$F + S = 4(V + G)$$

Solution-Cont'd

Step 2: Convert the constraints into standard linear equation form $Ax = b$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 25 & 40 & 60 & 70 & 32 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -10 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -4 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} 100000 \\ 4897000 \\ -11000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 3: Use MATLAB to find the solution