ENGR 1100

Week 02- Trigonometry

Agenda

1. Review of Trigonometry
2. Trigonometric functions in MATLAB
3. Examples of Trigonometry in Engineering

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°\left(\frac{\pi}{6}\right), 45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
- Use equal cofunctions of complements.
- Evaluate trigonometric functions with MATLAB.
- Use right triangle trigonometry to solve applied problems.
- Trigonometric functions for any angles.
- Inverse Trigonometric functions

Trigonometric Functions

[Trigonometric functions on Khan Academy]

The six trigonometric functions are:

Function	Abbreviation
sine	sin
cosine	cos
tangent	tan
cosecant	csc
secant	sec
cotangent	cot

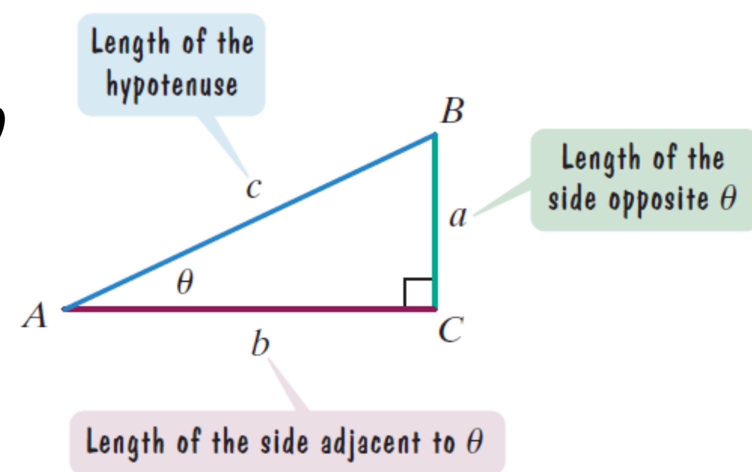
Right Triangle Definitions of Trigonometric Functions

$$\sin\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\cos\theta = \frac{\text{length of side adjacent to angle } \theta}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\tan\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta} = \frac{a}{b}$$

In general, the trigonometric functions of θ depend only on the size of angle θ and not on the size of the triangle.



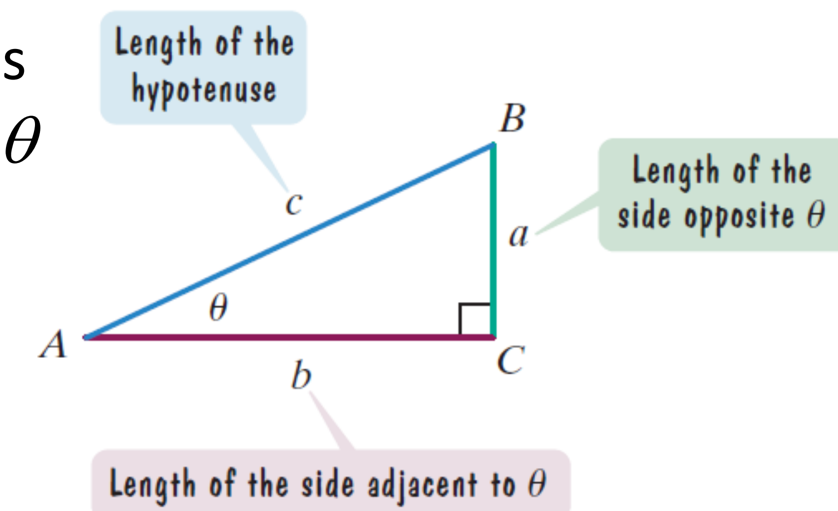
Right Triangle Definitions of Trigonometric Function *(continued)*

$$\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of side opposite angle }\theta} = \frac{c}{a}$$

$$\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to angle }\theta} = \frac{c}{b}$$

$$\cot\theta = \frac{\text{length of side adjacent to angle }\theta}{\text{length of side opposite angle }\theta} = \frac{b}{a}$$

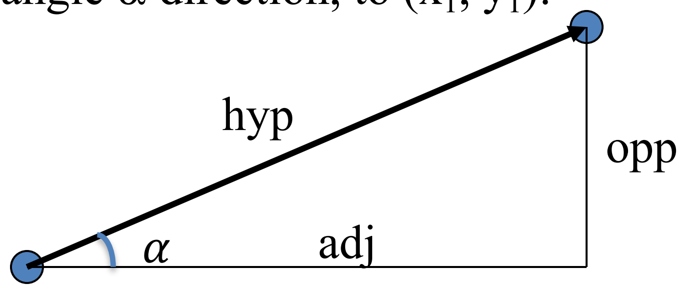
In general, the trigonometric functions of θ depend only on the size of angle θ and not on the size of the triangle.



Application: Finding locations

You want to move h units in the angle α direction, to (x_1, y_1):

You are at: (x, y)



So you make a right triangle...

And you label it...

And you compute:

$$x1 = x + adj = x + hyp * (adj/hyp) = x + hyp * \cos \alpha$$

$$y1 = y - opp = y - hyp * (opp/hyp) = y - hyp * \sin \alpha$$

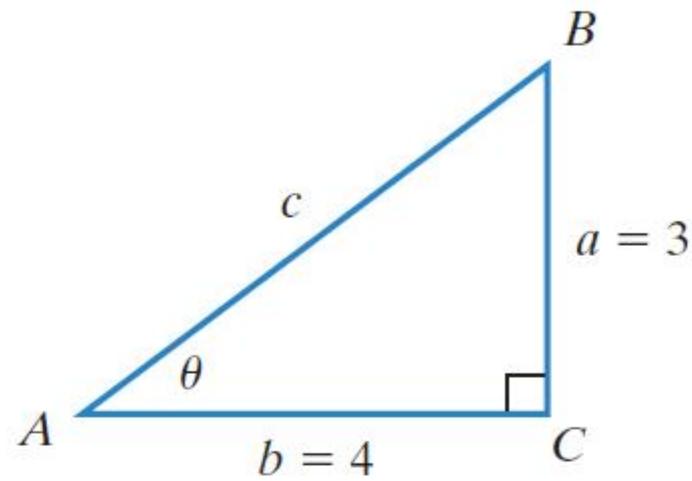
Trigonometry in Engineering

- Engineers routinely use trigonometric concepts to calculate angels. Civil and mechanical engineers use trigonometry to calculate torque and forces on objects, such as bridges or building girders. An example is the calculation of the static forces on an object that is not moving—such as a bridge.
- The sine and cosine functions are fundamental to the theory of periodic functions, those that describe the sound and light waves.

Example: Evaluating Trigonometric Functions

Find the value of the six trigonometric functions in the figure.

We begin by finding *c*.



$$a^2 + b^2 = c^2$$

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

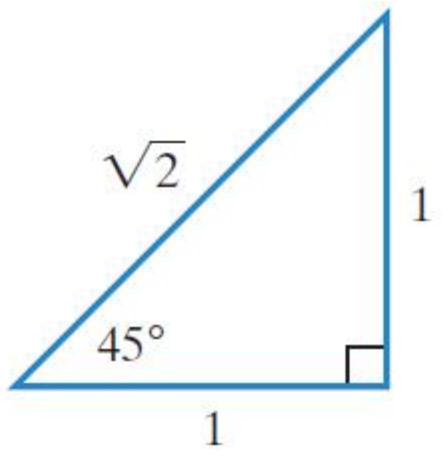
$$c = \sqrt{25} = 5$$

$$\sin\theta = \frac{3}{5} \qquad \tan\theta = \frac{3}{4} \qquad \sec\theta = \frac{5}{4}$$

$$\cos\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{3} \qquad \cot\theta = \frac{4}{3}$$

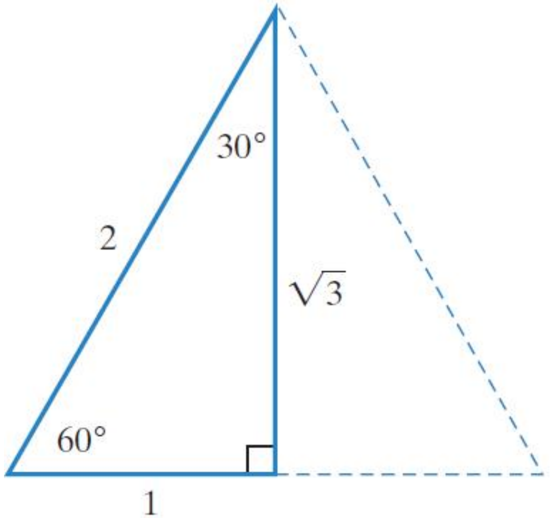
Function Values for Some Special Angles

- A right triangle with a 45°, or $\frac{\pi}{4}$ radian angle is isosceles – that is, it has two sides of equal length.



Function Values for Some Special Angles *(continued)*

A right triangle that has a 30°, or $\frac{\pi}{6}$ radian angle also has a 60°, or $\frac{\pi}{3}$ radian angle. In a 30-60-90 triangle, the measure of the side opposite the 30° angle is one-half the measure of the hypotenuse.

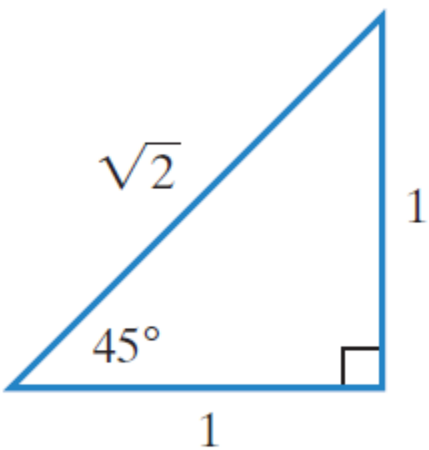


Example: Evaluating Trigonometric Functions

Use the figure to find csc 45°, sec 45°, and cot 45°.

$$\csc 45^\circ = \frac{\text{length of hypotenuse}}{\text{length of side opposite } 45^\circ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\sec 45^\circ = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to } 45^\circ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$



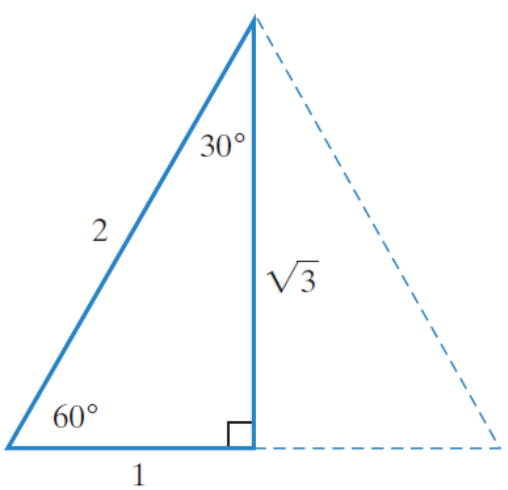
$$\cot 45^\circ = \frac{\text{length of side adjacent to } 45^\circ}{\text{length of side opposite } 45^\circ}$$

$$= \frac{1}{1} = 1$$

Evaluating Trigonometric Functions of 30°and 60°

Use the figure to find tan 60° and tan 30°. If a radical appears in a denominator, rationalize the denominator.

$$\tan 60° = \frac{\text{length of side opposite } 60°}{\text{length of side adjacent to } 60°} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



$$\tan 30° = \frac{\text{length of side opposite } 30°}{\text{length of side adjacent to } 30°}$$

$$= \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Trigonometric Functions of Special Angles

θ	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Fundamental Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example: Using Quotient and Reciprocal Identities

Given $\sin\theta = \frac{2}{3}$ and $\cos\theta = \frac{\sqrt{5}}{3}$, find the value of each of the four remaining trigonometric functions.

Ex:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2}{3} \times \frac{3}{\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Please calculate the value for other 3 trigonometric functions.

Example: Using Quotient and Reciprocal Identities *(continued)*

$$\sec\theta = \frac{1}{\cos\theta} = \frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

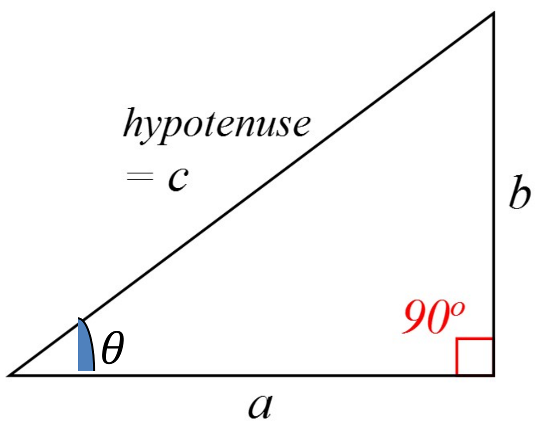
$$\cot\theta = \frac{1}{\tan\theta} = \frac{5}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{2\times 5} = \frac{\sqrt{5}}{2}$$

The Pythagorean Identities

[Pythagorean Theorem on Khan Academy]

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Pythagorean Theorem



$$c^2 = a^2 + b^2$$

Four different ways of proving Pythagorean Theorem!

Example: Using a Pythagorean Identity

Given that $\sin\theta = \frac{1}{2}$ and θ is an acute angle, find the value of $\cos\theta$ using a trigonometric identity.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{4}$$

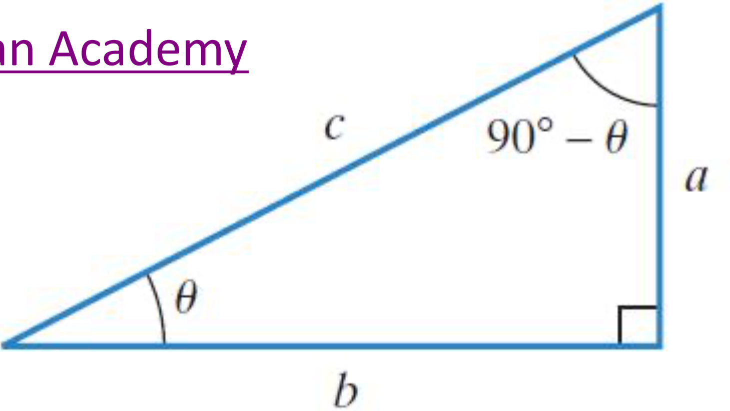
$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Trigonometric Functions and Complements

Two positive angles are **complements** if their sum is 90° or $\frac{\pi}{2}$. Any pair of trigonometric functions *f* and *g* for which $f(\theta) = g(90^\circ - \theta)$ and $g(\theta) = f(90^\circ - \theta)$ are called **cofunctions**.

[Complementary Angles on Khan Academy]



Cofunction Identities

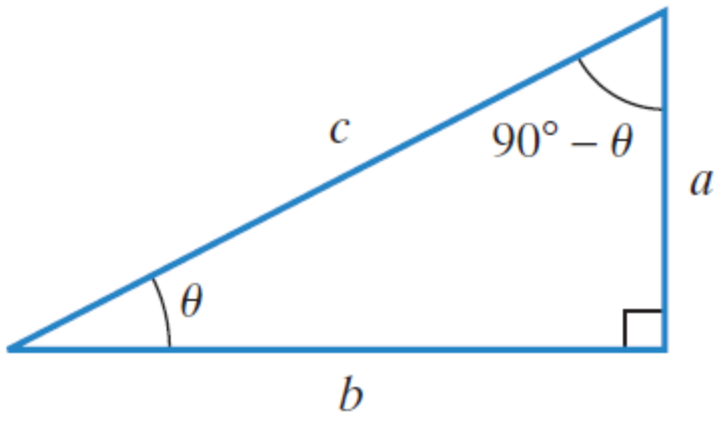
The value of a trigonometric function of θ is equal to the cofunction of the complement of θ. Cofunctions of complementary angles are equal.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$

$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

$$\sec\theta = \csc(90^\circ - \theta) \qquad \csc\theta = \sec(90^\circ - \theta)$$

If θ is in radians, replace 90° with $\frac{\pi}{2}$.



Using Cofunction Identities

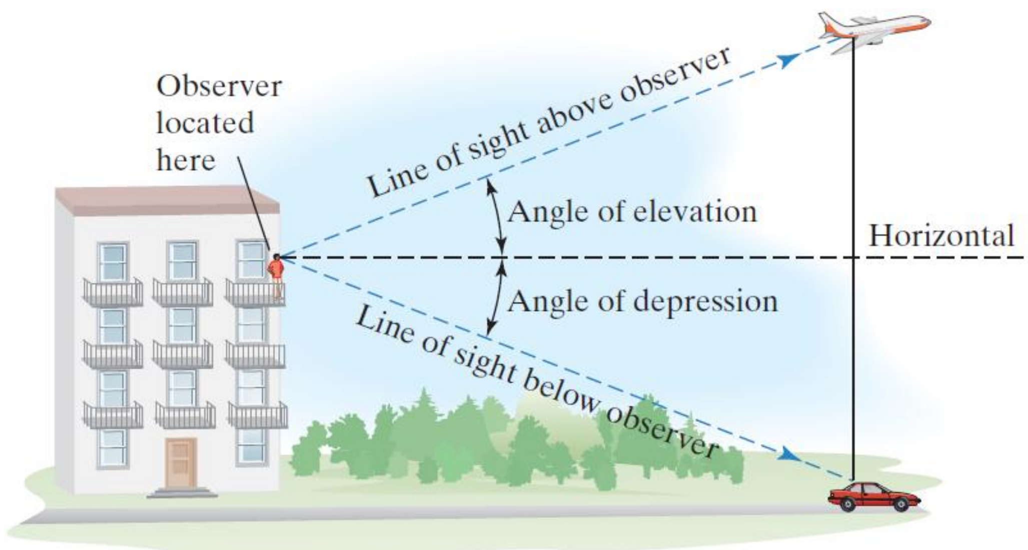
Find a cofunction with the same value as the given expression:

a. $\sin 46° = \cos(90° - 46°) = \cos 44°$

b. $\cot\dfrac{\pi}{12} = \tan\left(\dfrac{\pi}{2} - \dfrac{\pi}{12}\right) = \tan\left(\dfrac{6\pi}{12} - \dfrac{\pi}{12}\right) = \tan\dfrac{5\pi}{12}$

Applications: Angle of Elevation and Angle of Depression

An angle formed by a horizontal line and the line of sight to an object that is above the horizontal line is called the **angle of elevation**. The angle formed by the horizontal line and the line of sight to an object that is below the horizontal line is called the **angle of depression**.

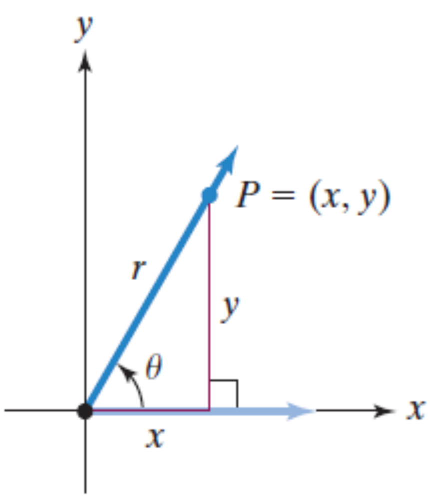


Trigonometric Functions of Any Angle

TRIGONOMETRY WITH GENERAL TRIANGLES

Definitions of Trigonometric Functions of Any Angle

- Let θ be any angle in standard position and let $P = (x, y)$ be a point on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from (0, 0) to (x, y), the **six trigonometric functions** of θ are defined by the following ratios:



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, x \neq 0$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$

The Graph of y = sin$x$

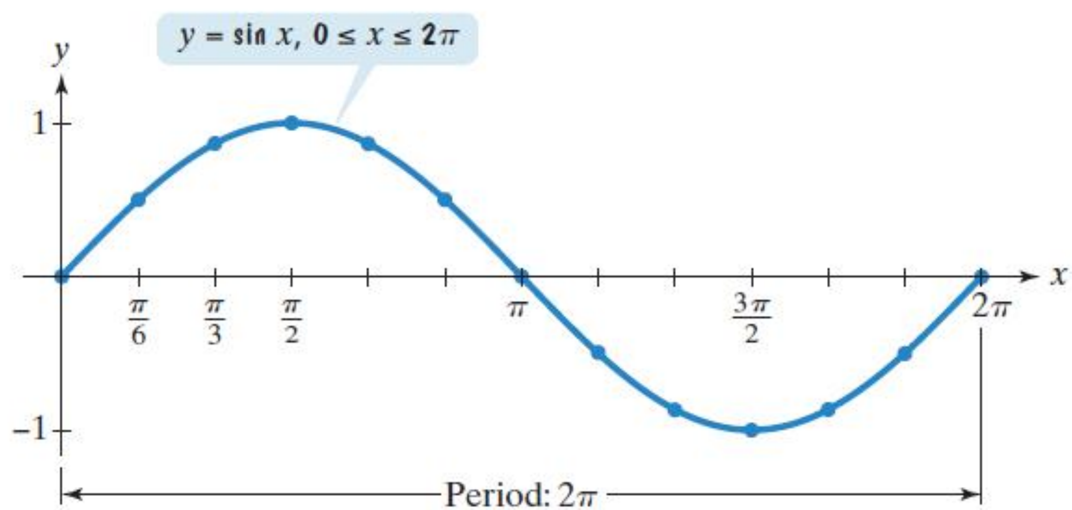
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

As x increases from 0 to $\frac{\pi}{2}$, y increases from 0 to 1.

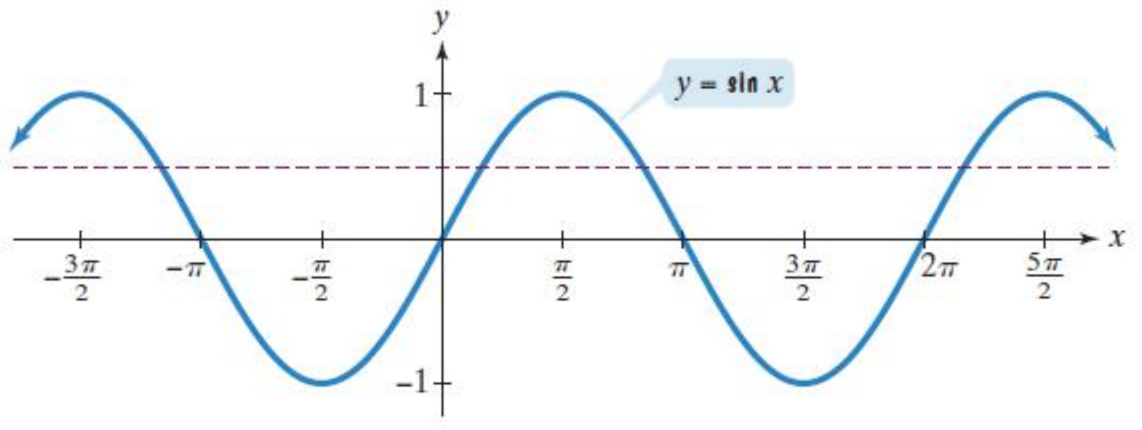
As x increases from $\frac{\pi}{2}$ to π, y decreases from 1 to 0.

As x increases from π to $\frac{3\pi}{2}$, y decreases from 0 to −1.

As x increases from $\frac{3\pi}{2}$ to 2π, y increases from −1 to 0.



The Inverse Sine Function



The horizontal line test shows that the sine function is not one-to-one; $y = \sin x$ has an inverse function on the restricted domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

The Inverse Sine Function *(continued)*

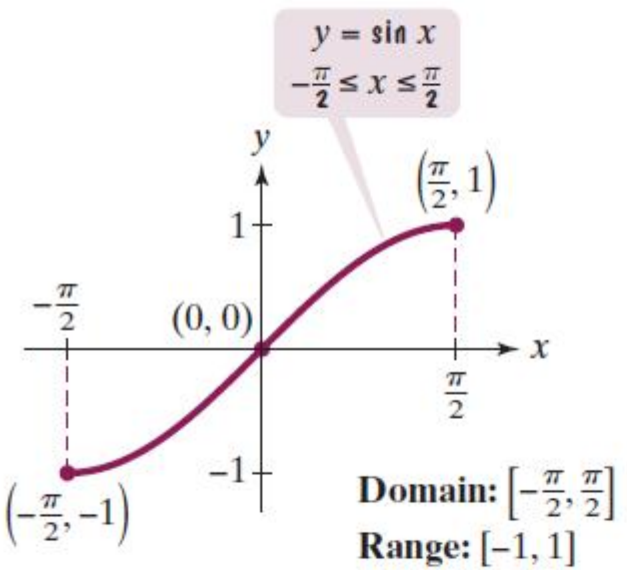
The **inverse sine function**, denoted by $\sin^{-1}$, is the inverse of the restricted sine function $y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Thus,

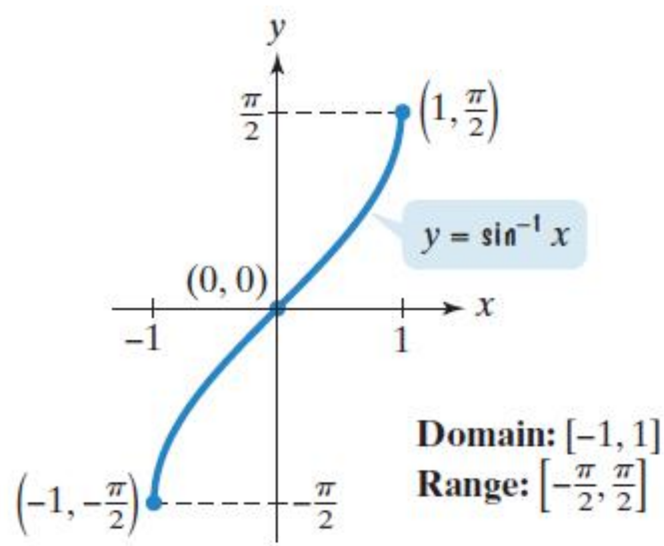
$$y = \sin^{-1} x \quad \text{means} \quad \sin y = x,$$

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-1 \leq x \leq 1$. We read $y = \sin^{-1} x$ as "y equals the inverse sine at x."

Graphing the Inverse Sine Function

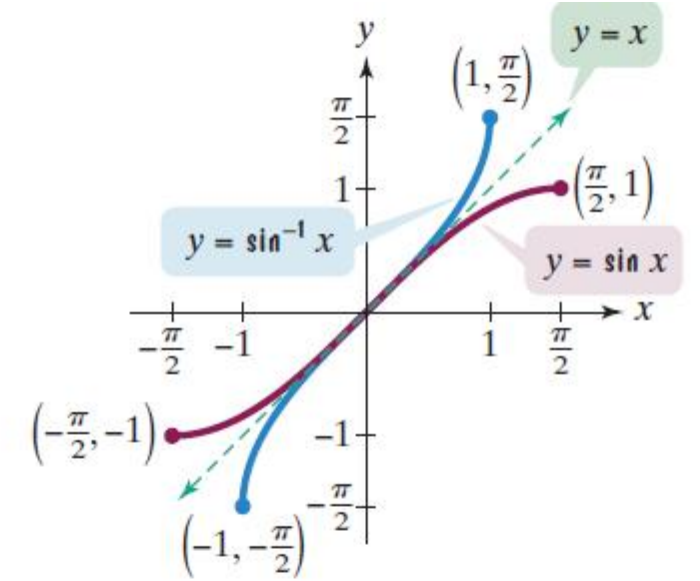
One way to graph $y = \sin^{-1} x$ is to take points on the graph of the restricted sine function and reverse the order of the coordinates.





Graphing the Inverse Sine Function *(continued)*

Another way to obtain the graph of $y = \sin^{-1} x$ is to reflect the graph of the restricted sine function about the line $y = x$.



Finding Exact Values of $\sin^{-1}x$

1. Let $\theta = \sin^{-1} x$.

2. Rewrite $\theta = \sin^{-1} x$ as $\sin\theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

3. Use the exact values in the table to find the value of θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that satisfies $\sin\theta = x$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Example: Finding the Exact Value of an Inverse Sine Function

Find the exact value of $\sin^{-1}\frac{\sqrt{3}}{2}$.

Solution:

Step 1. Let $\boldsymbol{\theta = \sin^{-1} x}$.

$$\theta = \sin^{-1}\frac{\sqrt{3}}{2}$$

Step 2. Rewrite $\boldsymbol{\theta = \sin^{-1} x}$ as $\boldsymbol{\sin\theta = x}$, where $-\frac{\boldsymbol{\pi}}{\mathbf{2}} \le \theta \le \frac{\boldsymbol{\pi}}{\mathbf{2}}$.

$$\sin\theta = \frac{\sqrt{3}}{2}$$

Example: Finding the Exact Value of an Inverse Sine Function *(continued)*

Step 3. Use the exact value in the table to find the value of θ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ that satisfies $\sin\theta = x$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

The angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

Useful Resources

https://www.khanacademy.org/math/trigonometry

Trig Functions in MATLAB

[MATLAB: Trigonometric functions]

sin	Sine of argument in radians	csc	Cosecant of input angle in radians
sind	Sine of argument in degrees	cscd	Cosecant of argument in degrees
asin	Inverse sine in radians	acsc	Inverse cosecant in radians
cos	Cosine of argument in radians	sec	Secant of angle in radians
cosd	Cosine of argument in degrees	secd	Secant of argument in degrees
acos	Inverse cosine in radians	asec	Inverse secant in radians
tan	Tangent of argument in radians	cot	Cotangent of angle in radians
tand	Tangent of argument in degrees	cotd	Cotangent of argument in degrees
atan	Inverse tangent in radians	acot	Inverse cotangent in radians

Evaluate Trigonometric Functions

Evaluate the following trig functions in MATLAB

a. csc 1.5 (radian mode)

b. sin 72.8° (degree mode)

[**Hint**: convert the degree into radian]

TRIGONOMETRY IN ENGINEERING EXAMPLES

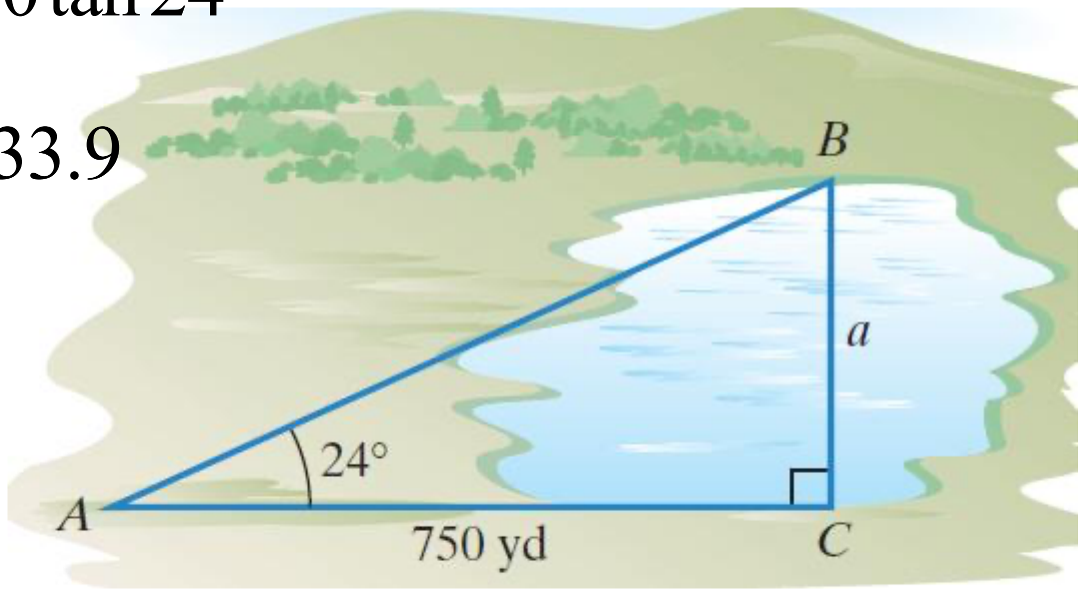
Example 1: Problem Solving Using an Angle of Elevation

The irregular blue shape in the figure represents a lake. The distance across the lake, *a*, is unknown. To find this distance, a surveyor took the measurements shown in the figure. What is the distance across the lake?

Solution:

$$\tan 24° = \frac{a}{750} \Rightarrow a = 750 \tan 24°$$

$$a \approx 333.9$$



The distance across the lake is approximately 333.9 yards.

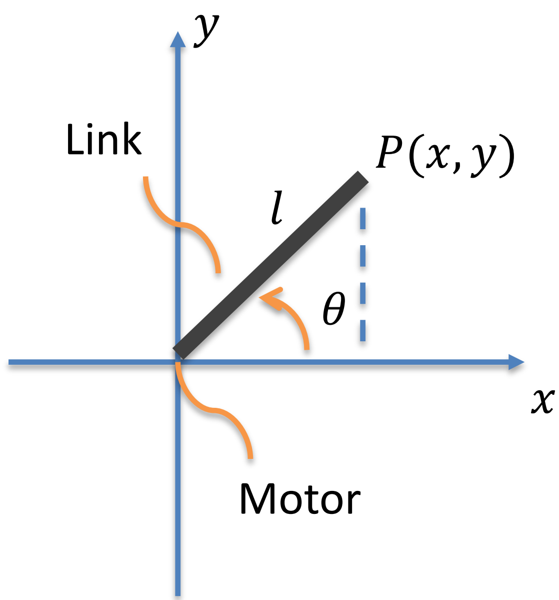
Example 2: Trigonometric Identity

A trigonometric identity is given by,

$$cos^2\frac{x}{2} = \frac{\tan x + \sin x}{2\tan x}$$

Verify that the identity is correct by calculating each side of the equation, substituting $x = \frac{\pi}{5}$.

Example 3: 1-Link Planar Robot

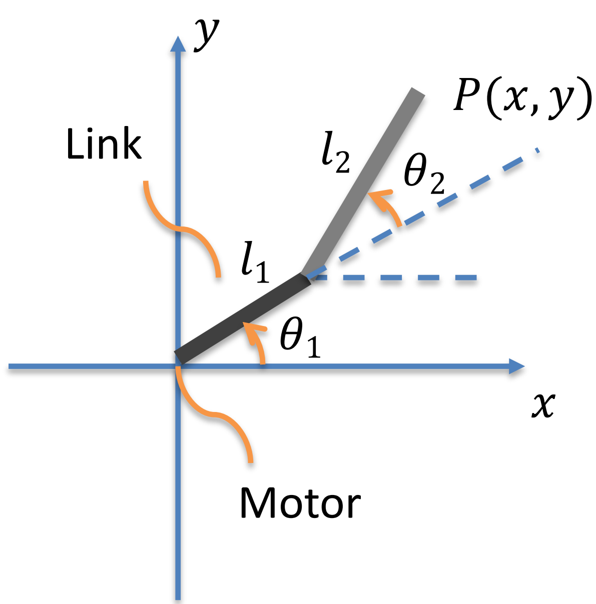


- Given l and θ, what are the coordinates of the end point $P(x, y)$?

$$l = \sqrt{x^2 + y^2}$$
$$x = l \cos\theta$$
$$y = l \sin\theta$$

$$x^2 + y^2 = (l\cos\theta)^2 + (l\sin\theta)^2 = l^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{\text{Trig identity}} = l^2$$

Example 4a: 2-Link Planar Robot



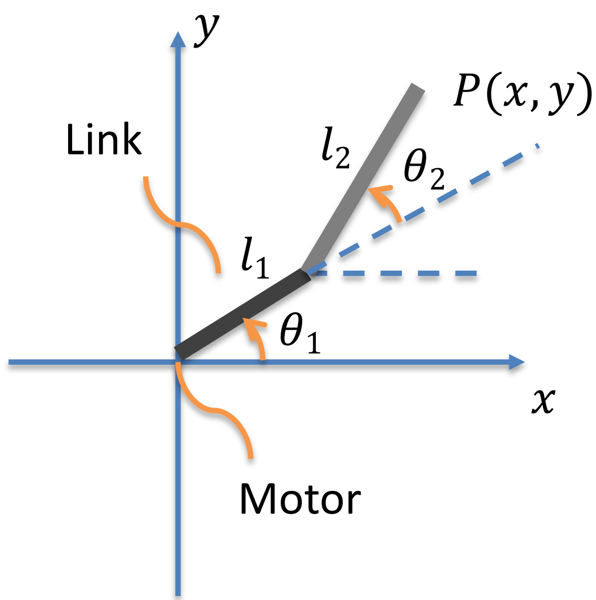
- Given l_1, l_2, $\theta_1$and θ_2, what are the coordinates of the end point $P(x, y)$?

$$x = x_1 + x_2, \quad y = y_1 + y_2$$

$$x_1 = l_1 \cos\theta_1, x_2 = l_2 \cos(\theta_1 + \theta_2)$$
$$y_1 = l_1 \sin\theta_1, y_2 = l_2 \sin(\theta_1 + \theta_2)$$

$$x^2 + y^2 = (l\cos\theta)^2 + (l\sin\theta)^2 = l^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{\text{Trig identity}} = l^2$$

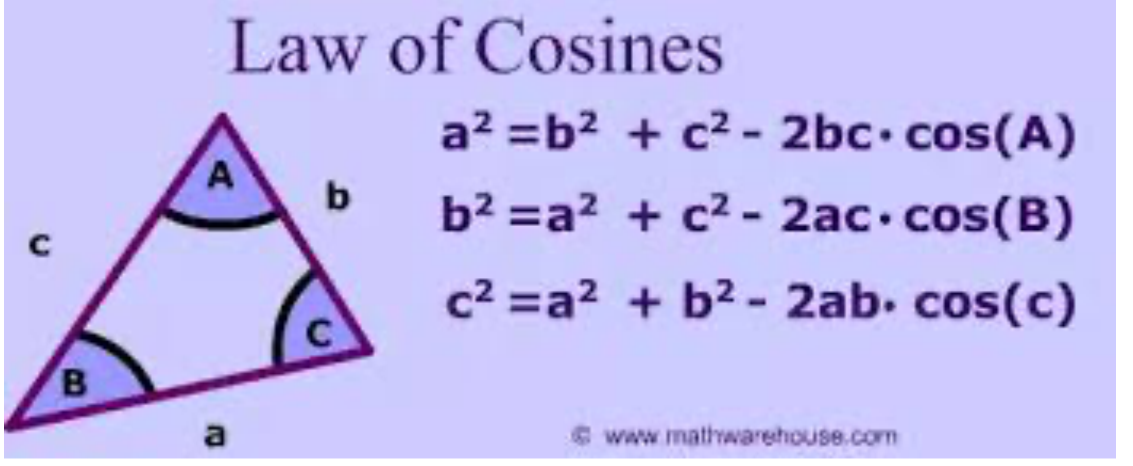
Example 4b: 2-Link Robot Inverse Problem



Given $P(x, y)$, l_1, l_2, what are the angles of the two links, $\theta_1$and θ_2, assuming that both $\theta_1$and θ_2 are acute angle ?

$P(x, y) = (12{,}6)$
$l_1 = l_2 = 5\sqrt{2}$

Law of Cosine



Proof can be found at, https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-law-of-cosines/v/law-of-cosines

Example 5: Geometry and Trigonometry

Four circles are place as shown in the figure. At each point where two circles are in contact, they are tangent to each other. Determine the distance between the centers C2 and C4. The radii of the circles are,

$R_1 = 16mm, R_2 = 6.5mm, R_3 = 12mm$ and $R_4 = 9.5mm$