ENGR 1100

Week 11- Numerical Analysis II

(Class lecture)

Learning Objectives

Upon completion this module, students will be able to:

1. Explain the definition and applications of Ordinary Differential Equations (ODE)
2. Solve first-order ODEs in MATLAB
3. Apply numerical analysis techniques to solve engineering problems

Ordinary Differential Equations

Go over the following resources to learn about Ordinary Differential Equations (ODE) and numerical numerical methods to solve ODE.

- MATLAB: Solve Differential Equation
- What are Differential Equations?

Ordinary Differential Equations(ODE)

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dy}{dx}$$

y is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2 y}{dx^2} + 6xy = 1$$

involve one or more <u>Ordinary derivatives</u> of unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more <u>partial derivatives</u> of unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODE) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

$$Examples:$$

$$\frac{dy}{dx} - y = e^x$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \cos(x)$$

y(x): unknown function

x: independent variable

Applications of ODE

- **Population Growth**

  The population of a given species is decreased at a constant rate of n per year. The population due to birth and death is increased at a constant rate of $\lambda\%$ of the existing population. If the initial population is N, then the population x after t year is given by, $\frac{dx}{dt} = \frac{\lambda}{100}x - n$

- **Law of cooling**

  The rate of change of temperature of a body is proportional to the difference between the temperature of the body and the temperature θ of the surrounding. Suppose T is the temperature of the body at time t, then, $\frac{dT}{dt} = k(T - \theta)$ where $k<0$

Order of a differential equation

The **order** of an ordinary differential equations is the order of the highest order derivative

Examples:

$$\frac{dy}{dx} - y = e^x$$ First order ODE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \cos(x)$$ Second order ODE

$$\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 2y^4 = 1$$ Second order ODE

Solution of a differential equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

$$Solution \quad x(t) = e^{-t}$$

$$\text{Proof}:$$

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one. No product of the unknown function and/or its derivatives

$$Examples:$$

$$\frac{dy}{dx} - y = e^{x}$$ Linear ODE

$$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 2x^2 y = \cos(x)$$ Linear ODE

$$\left(\frac{d^2 y}{dx^2}\right)^3 - \frac{dy}{dx} + \sqrt{y} = 1$$ Non-linear ODE

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\quad y'(0)=2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problem

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\quad y(2)=1.5$$

different

Classification of ODE

ODE can be classified in different ways

- **Order**
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- **Linearity**
  - Linear ODE
  - Nonlinear ODE
- **Auxiliary conditions**
  - Initial value problems
  - Boundary value problems

Solutions

- Analytical Solutions to ODE are available for linear ODE and special classes of nonlinear differential equations.
- Numerical method are used to obtain a graph or a table of the unknown function
- We focus on solving first order linear ODE
  - A first order differential equation is said to be ***linear*** if it can be written

$$\boxed{y' + p(x)y = g(x)}$$

Solving First-order ODE in MATLAB

Solve first-order ODE analytically in MATLAB

- Some ODE can be solve analytically with or without initial conditions.
- MATLAB uses `dsolve()` function to achieve this.
- Details can be found @MathWorks,
  https://www.mathworks.com/help/symbolic/solve-a-single-differential-equation.html
- This module focuses on solving first-order ODE **numerically**.

Steps for solving a single first-order ODE numerically

<u>Step 1:</u> **Write the problem in a standard form**

Write equation as

$$\frac{dy}{dt} = f(t,y) \text{ for } \mathrm{t}_0 \le t \le t_f, \text{ with } y = y_0 \text{ at } t_0$$

Three necessary pieces of information

1. Equation involving first derivative of the function
2. Interval of independent variable
3. Initial value of function

Solution is value of *y* as a function of *t*, for $t_0 \le t \le t_f$

Example

Example of an initial-value problem is

$$\frac{dy}{dt} = \frac{t^3 - 2y}{t}$$

for $1 \le t \le 3$

with $y = 4.2$ at $t = 1$

Step 2: Create a user-defined function

Either in a function file or an anonymous function

Write a function for the right side of

$$\frac{dy}{dt} = f(t, y)$$

e.g., `dydt = f( t, y )`

Function must have inputs in order shown

- `t` – scalar with value of independent variable
- `y` – scalar with value of *y* at *t*, i.e., *y(t)*
- `dydt` – first derivative of *y* at time *t*

Example

Example:

$$\frac{dy}{dt} = \frac{t^3 - 2y}{t}$$

Example script file

```
function dydt = ODEexp1(t,y)
dydt = ( t^3 - 2*y ) / t;
```

Example anonymous function

```
>> ode1 = @(t,y)(t^3 - 2*y)/t
```

General idea of numerical methods

- Assume that *y(t)* changes slowly enough so that within a small distance Δt from *t*, *y(t)* is approximately the same as tangent line at *y(t)*
- Slope *m* of tangent going through *y(t)* is just first derivative of *y(t)* and can compute that from $\frac{dy}{dt}(t)=f(t,y)$
- Equation of tangent line is $y=m\cdot t+b$ and since tangent passes through point *y(t)* at *t*, can determine *b*
- Approximate value of *y* a small time Δt in the future by $y(t+\Delta t) = m\cdot(t+\Delta t)+b$
- Repeat process at $t+2\Delta t$ to get $y(t+3\Delta t)$
- Keep repeating process until $t > t_f$

End up with

$$y(t_0),\ y(t_0+\Delta t)\ ,\ y(t_0+2\Delta t),\ldots,\ y(t_0+m\Delta t)$$

MATLAB ODE solvers

MATLAB's ODE solvers can be used,

- *stiff problem* – an ODE whose solution has parts that change slowly in time and parts that change rapidly
- *one-step solver* – a solver that uses only information from current step to get to next step
- *multistep solver* – a solver that uses information from current and previous steps to get to next step

General advice – try `ode45()` first. If it takes too long, try `ode15s()`

https://mathworks.com/help/matlab/math/choose-an-ode-solver.html

Step 3: Solve the ODE

Call solvers by

```
[t y] = solver_name(ODEfun,tspan,y0)
```

- `solver_name` - MATLAB built-in solvers, e.g., `ode45`, `ode23t`
- `ODEfun` – a function as described in Step 2
- `tspan` – a two-element vector of the first and last time values, e.g., `[t0 tf]`
  - `tspan` can have more than two elements. See Help for more information
- `y0` – the initial value of *y*
- `[t y]` - two column vectors of the same size
  - `t` is vector of time points, with `t(1)=t0` and `t(end)=tf`
  - `y` is value of function at corresponding times in `t`

MATLAB Example

$$\frac{dy}{dt} = \frac{t^3 - 2y}{t}$$ for $1 \le t \le 3$ with $y = 4.2$ at $t = 1$

```
>> [t y]=ode45(@ODEexp1,[1:0.5:3],4.2)
t =
    1.0000
    1.5000
    2.0000
    2.5000
    3.0000
y =
    4.2000
    2.4528
    2.6000
    3.7650
    5.8444
```

The initial value.

The vector `tspan`.

The handle of the user-defined function `ODEexp1`.

Higher-order ODE

A typical approach to solving higher-order ordinary differential equations is to convert them to systems of first-order differential equations, and then solve those systems.

$$y_1'' - \mu(1 - y_1^2)y_1' + y_1 = 0$$

Let $y_2 = y_1'$, then the above equation becomes a system of first-order ODE,

$$y_1' = y_2$$
$$y_2' = \mu(1 - y_1^2)y_2 - y_1$$

- More details on how to solve @ MathWorks

https://www.mathworks.com/help/matlab/ref/ode45.html

Summary - Solve the ODE in MATLAB

Call solvers by

```
[t y] = solver_name(ODEfun,tspan,y0)
```

- `solver_name` - MATLAB built-in solvers, e.g., `ode45, ode23t`
- `ODEfun` – a function as described in Step 2
- `tspan` – a two-element vector of the first and last time values, e.g., `[t0 tf]`
  - `tspan` can have more than two elements. See Help for more information
- `y0` – the initial value of *y*
- `[t y]` - two column vectors of the same size
  - `t` is vector of time points, with `t(1)=t0` and `t(end)=tf`
  - `y` is value of function at corresponding times in `t`

MATLAB Example

Solve the ODE $\frac{dy}{dt} = 2t$ use a time interval of [0,5] and the initial condition $y(0)=0$. Use MATLAB function "ode45()"

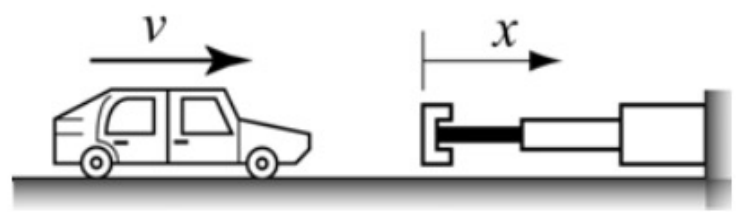
```
tspan = [0 5];
y0=0;
% Method 1: directly define the ODE in ode45()
[t,y]=ode45(@(t,y)2*t,tspan, y0)

% Method 2: define the ODE using anonymous function
myODE = @(t,y)2*t;
[t,y]=ode45(myODE,tspan, y0)

% Method 3: define the ODE function script
[t,y]=ode45(@myODE,tspan, y0)
```

```
Function dydt=myODE(t)
    dydt=2*t;
end
```

Example 1- Safety Bumper



A safety bumper is placed at the end of a racetrack to stop out-of-control cars. The bumper is designed such that the force that the bumper applies to the car is a function of the velocity v and the displacement x of the front edge of the bumper according to the equation:

$$F = Kv^3(x+1)^3$$

where $K = 30$ (s kg)/m^5 is a constant.

A car with a mass m of 1,500 kg hits the bumper at a speed of 90 km/h. Determine and plot the velocity of the car as a function of its position for $0 \le x \le 3$ m.

Solution

The deceleration of the car once it hits the bumper can be calculated from Newton's second law of motion,

$$ma = -Kv^3(x+1)^3$$

which can be solved for the acceleration a as a function of v and x:

$$a = \frac{-Kv^3(x+1)^3}{m}$$

The velocity as a function of x can be calculated by substituting the acceleration in the equation

$$vdv = adx$$

which gives

$$\frac{dv}{dx} = \frac{-Kv^2(x+1)^3}{m}$$

The last equation is a first-order ODE that needs to be solved for the interval $0 \le x \le 3$ with the initial condition $v = 90$ km/h at $x = 0$.

Example 2- Population Growth

The population growth of species with limited capacity can be modeled by the equation:

$$\frac{dN}{dt} = kN(N_M - N)$$

where N is the population size, N_M is the limiting number for the population, and k is a constant. Consider the case where $N_M = 5000$, $k = 0.000095$ 1/yr, and $N(0) = 100$. Determine N for $0 \le t \le 20$. Make a plot of N as a function of t.

SOLVING FIRST-ORDER ODE MANUALLY

First Order Linear Differential Equations

- How to solve first-order linear ODE ?

$$\boxed{y' + p(x)y = g(x) \quad (1)}$$

Solution:

Multiplying both sides by $\mu(x)$, called an integrating factor, gives

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \quad (2)$$

assuming $\mu(x)p(x) = \mu'(x), \quad (3)$ we get

$$\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)g(x) \quad (4)$$

First Order Linear Differential Equations

By product rule, (4) becomes

$$(\mu(x)y(x))' = \mu(x)g(x) \quad (5)$$

$$\Rightarrow \mu(x)y(x) = \int \mu(x)g(x)dx + c_1$$

$$\Rightarrow y(x) = \frac{\int \mu(x)g(x)dx + c_1}{\mu(x)} \quad (6)$$

Now, we need to solve $\mu(x)$ from (3)

$$\mu(x)p(x) = \mu'(x) \Rightarrow \frac{\mu'(x)}{\mu(x)} = p(x)$$

First Order Linear Differential Equations

$$\frac{\mu'(x)}{\mu(x)} = p(x) \Rightarrow (\ln \mu(x))' = p(x)$$

Derivative of logarithm function

$$\Rightarrow \ln \mu(x) = \int p(x)dx + c_2$$

$$\Rightarrow \mu(x) = e^{\int p(x)dx + c_2} = c_3 e^{\int p(x)dx} \qquad (7)$$

to get rid of one constant, we can use

$$\mu(x) = e^{\int p(x)dx} \qquad (8)$$

The solution to a linear first order differential equation is then

$$y(x) = \frac{\int \mu(x)g(x)dx + c}{e^{\int p(x)dx}} \qquad (9)$$

Summary of the Solution Process

1. Put the differential equation in the form (1)
2. Find the integrating factor, $\mu(x)$ using (8)
3. Multiply both sides of (1) by $\mu(x)$ and write the left side of (1) as $(\mu(x)y(x))'$
4. Integrate both sides
5. Solve for the solution $y(x)$

Example 1 $\boxed{y' - y = e^{2x}}$

Solution:

Example 1 $\boxed{y' - y = e^{2x}}$

Solution:

$$\boxed{\begin{aligned} y(x) &= e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} g(x)dx + c\right] \\ &= e^{-\int(-1)dx}\left[\int e^{\int(-1)dx} e^{2x}dx + c\right] \\ &= e^{x}\left[\int e^{-x}e^{2x}dx + c\right] \\ &= e^{x}\left[e^{x} + c\right] \\ &= ce^{x} + e^{2x} \end{aligned}}$$

Example 2 $\boxed{xy' + 2y = x^2 - x,\ y(1) = \frac{1}{2}}$

Solution:

$$\Rightarrow y' + \frac{2}{x} y = x - 1$$

$$\Rightarrow y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} g(x)dx + c\right]$$

$$= e^{-\int \frac{2}{x}dx}\left[\int e^{\int \frac{2}{x}dx}(x-1)dx + c\right] = x^{-2}\left[\int x^2(x-1)dx + c\right]$$

$$= x^{-2}\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + c\right] = \frac{1}{4}x^2 - \frac{1}{3}x + cx^{-2}$$

Apply the initial condition to get c,

$$\Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + c \Rightarrow c = \frac{7}{12}.$$