Calculus I Workbook

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For use with OpenStax Calculus, Volume 1

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Disclaimer about sources

This document presents lecture notes developed by the author while teaching Calculus at Texas State University, UWM, Berry College, and GSU in the years 2006 through 2020. The handwritten originals drew material from a variety of editions of a number of textbooks, including the Five College Consortium's *Calculus in Context*, George Thomas' *Calculus*, James Stewart's *Calculus*, Claudia Neuhauser's *Calculus for Biology and Medicine*, and Ross Middlemiss's *Differential and Integral Calculus*.

A serious effort has been made to remove direct quotations of expository material and exercises the author suspects to have been taken from these copyrighted textbooks, and to provide citations where the sources are known. As time allows, and to the best of her ability, she will continue to purge and replace material whose original attribution is unknown to her.

However, the author no longer has access to every edition of every textbook from which she taught Calculus over the years, and her memory regarding the origin of every exercise included in this document is far from infallible. Doubtless some of the material appearing herein can also be found in the sources from which she drew while compiling her handwritten notes.

The author also notes the exigency imposed by the COVID-19 pandemic, which was ongoing when this document was created and hastily deployed.

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Lesson 0: Sets and functions

Sets

A *set* is a well-defined collection of objects called **members** or **elements**.

If S is a set, the symbols

$$a \in S$$

mean that a is a member of S, and the symbols

$$a \notin S$$

mean that a is not a member of S.

A set can be specified in one of several ways, which we will now discuss.

Defining a set by a rule:

☞ $\mathbb{R}$ is the set of all numbers on the number line. The numbers on the number line are called **real numbers**.

☞ The symbols $(-\infty, \infty)$ mean the same thing as $\mathbb{R}$: that is, the set of all numbers on the number line.

Ex. 1. True/False/Makes no sense:
(a) $\pi \in \mathbb{R}$.
(b) $3 \notin \mathbb{R}$.
(c) $\mathbb{R} \in 2$.
(d) ∞ is a number on the number line.

Defining a set by listing its members ("roster notation"):

☞ Curly braces $\{\_\_\}$ mean "the set consisting of."

Example:

The statement

$$S = \{2, 4, 6, 8, \dots\}$$

means

"S is the set consisting of 2, 4, 6, 8, and so on."

This way of writing a set (that is, by listing its members) is called **roster notation**.

Ex. 2. True/False/Makes no sense:
(a) $10 \in \{2, 4, 6, 8\}$.
(b) $10 \in \{2, 4, 6, 8, \dots\}$.
(c) $10 \in 2, 4, 6, 8$.

Set-builder notation:

☞ The vertical bar | means "such that."

Example:

The statement

$$S = \{x \in \mathbb{R} \,|\, x < 0\}$$

means

"S is the set consisting of x in the set of real numbers such that $x < 0$."

This way of writing a set (that is, by specifying a test for membership after the vertical bar) is called **set-builder notation**.

Ex. 3. Write in set-builder notation.
(a) The set of numbers x such that $0 \leqslant x < 1$.
(b) The set of numbers x such that $x \geqslant 0$.

Some special sets:

The set of **integers** is

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set of **natural numbers** is

$$\mathbb{N} = \{n \in \mathbb{Z} \,|\, n \geqslant 1\}.$$

The **Cartesian plane** $\mathbb{R}^2$ is the set of **points** (or **ordered pairs**) (x, y), where x, $y \in \mathbb{R}$. We can write $\mathbb{R}^2$ in set-builder notation as follows:

$$\mathbb{R}^2 = \big\{(x, y) \,|\, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\big\}.$$

Ex. 4. True/False/Makes no sense:

(a) Every member of $\mathbb{Z}$ is a member of $\mathbb{R}$.

(b) Every member of $\mathbb{N}$ is a member of $\mathbb{R}$.

(c) Every member of $\mathbb{R}^2$ is a number.

Ex. 5. Write in set-builder notation:

The set of fractions n/d *such that* n *and* d *are real numbers, and* d *is not equal to* 0.

The set in Ex. 5 is called the set of **rational numbers**. It is denoted by $\mathbb{Q}$.

☞ You're not required to memorize the meaning of the symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}^2$, and $\mathbb{Q}$. (But you *should* memorize the meaning of the symbol $\mathbb{R}$.)

Intersections and unions:

☞ Let A and B be two sets. Their **intersection** $A \cap B$ is the set of members both of A and of B. Their **union** $A \cup B$ is the set of members either of A or of B. In set-builder notation,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Ex. 6. Write the set $[2, 4] \cap (3, \infty)$ in the simplest possible notation. (*Hint*: Draw the number line and, using two different colors, shade the regions corresponding to the two sets $[2, 4]$ and $(3, \infty)$.)

Ex. 7.

(a) How many numbers are members of the set $\{1, 2\}$?

(b) How many numbers are members of the set $(1, 2)$?

(c) How many numbers are members of the set $[1, 2)$?

Functions

Definition: A variable y is said to be a **function** of a second variable x if a relation exists between them such that to each of a certain set of values of x, there corresponds exactly one value of y.

- The amount y of postage you pay to ship a package is a function of the weight x of the package.
- The volume V of a cubical box (that is, the amount of space inside the box) is a function of the length s of its side.
- The position D of a car's gasoline gauge *(see image below)* is a function of the amount v of gas in the tank.

Definition: If y is a function of x, we call y the **dependent variable** (or **output**), we call x the **independent variable** (or **input**), and we say that y **depends on** x.

Ex. 8. The equation

$$4x + 2y = 12$$

is a relation between x and y such that to every value of x, there corresponds exactly one value of y.

- If $x = 0$, then $y =$ __.
- If $x = 1$, then $y =$ __.

It appears that y is a function of x. Is x a function of y?

Ex. 9. Consider the following relation between the two variables x and y.

$$y = x^2.$$

Is y a function of x? Is x a function of y?

Some functions are not described by algebraic formulas.

Instead, a procedure may be given for determining the output for a given input, as in the following example.

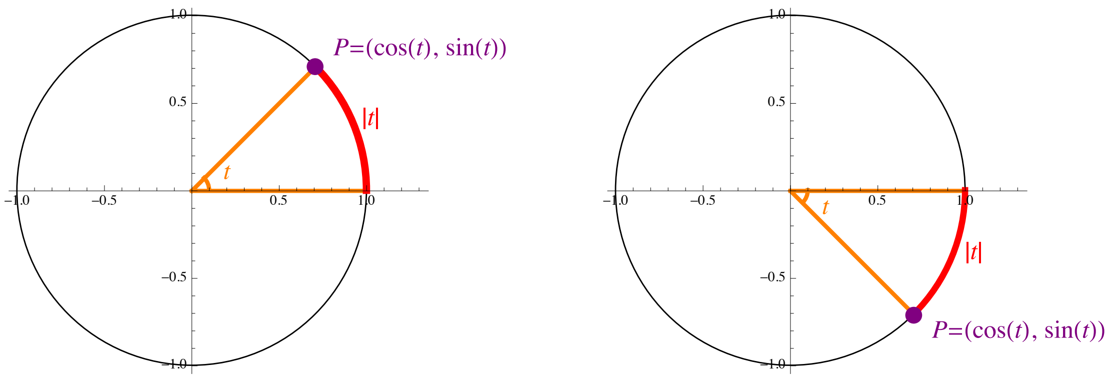
Ex. 10. Define two functions $\cos$ and $\sin$ as follows.

Recall that the **unit circle** is the circle of radius 1 in the Cartesian plane centered at $(0, 0)$.

Given a real number t, locate a point P on the unit circle by tracing an arc of length $|t|$ along the circle starting from the point $(1, 0)$.

- If t is positive, trace the arc counter-clockwise.
- Trace the arc clockwise if t is negative.

Then the outputs $\cos(t)$ and $\sin(t)$ are defined to be the coordinates of the point P, as shown:



Ex. 11. Explain why we can't solve the equation $\cos(t) = 1/t$ by writing $t = 1/\cos$.

Function notation

A function is a relationship between two variables—an input variable and an output variable whose value depends on the value of the input.

In Ex. 10, we saw that a function can be given a *name* (like $\cos$).

When we give a function a name, we often use **function notation**:

> If "f" is the name of a function, we write
>
> $$f(t)$$
>
> (read as "f of t") for the value of the output of f when the input value is t.

This does <u>*not*</u> *mean "f times* t*."* (See Ex. 11.)

Ex. 12. $V(s)$ = volume of a cube with side length s. Evaluate the expression $V(2)$.

Ex. 13. Define a function $y = f(x)$ by the relation $y = x^2$. Evaluate the expression

$$f(3 + h) - f(3).$$

> *Hint*: Start by writing
>
> $$f(3+h) - f(3) = [\qquad] - [\qquad]$$
>
> and then fill in the first blank with the value of $y = f(x) = x^2$ when $x = 3 + h$, and the second blank with the value of $f(x) = x^2$ when $x = h$. Then simplify.

Ex. 14. Define a function $S(n)$ by the relation

$$S(n) = \begin{cases} -n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(a) What's the value of $S(0)$?
(b) What's the value of $S(-3)$?
(c) *(Sneaky question:)* What's the value of $S(\pi)$?

The domain of a function

The output of a function f need not be defined for every input value. (See Ex. 14(c).)

Definition. The set of all input values for which f is defined is called the **domain** of f.

Ex. 15.

(a) What is the domain of the function $f(x) = x^2 - 1$?

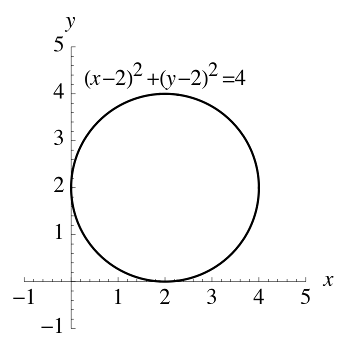
(b) What is the domain of the function $A(r)$ = area of a circle of radius r? (Is 0 a member of the domain of A?)

(c) What is the domain of the function $g(x) = -\sqrt{x}$? (Recall: the symbol $\sqrt{}$ means the *nonnegative square root*.)

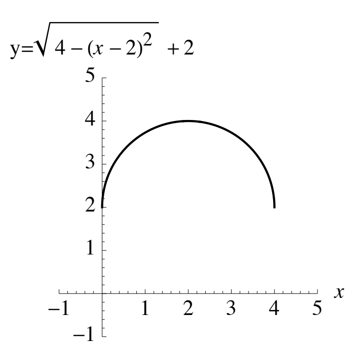
Convention: In this class, our variables will always represent *real numbers*, unless explicitly stated otherwise.

Graphs and the Vertical Line Test

Definition. The **graph of an equation** involving x and y is the set of ordered pairs $(x, y) \in \mathbb{R}^2$ satisfying the equation.



Definition. The **graph of a function** f is the graph of the equation $y = f(x)$.



☞ In symbols, the graph of a function f is the set $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$.

Recall: To determine whether a graph is the graph of a function, use the **Vertical Line Test**:

- If *no* vertical line meets the graph in more than one point, then the graph is the graph of a function.

Ex. 16.

(a) Graph $y = -x + 2$. Is y a function of x?

(b) Graph $y = x^4$. Is y a function of x?

(c) Graph $x = y^4$. Is y a function of x?

Symmetry

Definition.

- A function f is **even** if $f(x) = f(-x)$ for all $x \in \text{DOMAIN}(f)$.
- A function f is **odd** if $-f(x) = f(-x)$ for all $x \in \text{DOMAIN}(f)$.

To determine if a function f is even, odd, both, or neither using algebra only, simplify the expressions $-f(x)$ and $f(-x)$, and then check the definitions above.

Ex. 17. Using algebra only, determine whether the function is even, odd, both, or neither.

- $f(x) = \dfrac{1}{1 - x^2 - x^4}$
- $g(x) = x^7 + x^5 - x^4 + x$
- $h(x) = 2x^2 + x$
- $j(x) = 3$
- $k(x) = \sqrt{x}$

Ex. 18. Suppose f is a function with domain $\mathbb{R} = (-\infty, \infty)$, and $f(3) = 20$.

(i) If f is even, then $f(-3) =$ ____.

(ii) If f is odd, then $f(-3) =$ ____.

Solution:

(i) $f(-3) \overset{(\text{f even})}{=} f(3) = 20$.

(ii) $f(-3) \overset{(\text{f odd})}{=} -f(3) = -20$.

How can we tell whether a function is even or odd from its graph?

- A function is even if its graph has mirror symmetry in the y-axis.
- A function is odd if it is symmetric in the origin (has 180° rotational symmetry).

Ex. 19 (Challenge). *How many* functions with domain $\mathbb{R}$ are both even and odd?

Increasing and decreasing

Definition. Let I be an interval.

- A function f is **increasing on** I if
$$f(x_1) < f(x_2)$$
for every $x_1, x_2 \in I$ such that $x_1 < x_2$.
- A function f is **decreasing on** I if
$$f(x_1) > f(x_2)$$
for every $x_1, x_2 \in I$ such that $x_1 < x_2$.

Ex. 20. Consider the **absolute value function** $A(x) = |x|$,

$$A(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

On what interval(s) is this function increasing? On what interval(s) is it decreasing?

Repertoire of basic functions

Definition. A **power function** is a function defined by a relation of the form

$$f(x) = x^a$$

where a is a constant.

- When $a = 1$, we call f the **identity function**.
- When $a = -1$, we call f the **reciprocal function**.
- When $a = 1/n$ and $n \geqslant 2$ is a positive integer, we call f a **root function**. (Recall that $x^{1/n} = \sqrt[n]{x}$ for any natural number $n \geqslant 2$.)

Ex. 21. What is the domain of $f(x) = x^a$ when $a = -1$?

Solution.

$f(x) = x^{-1} = \dfrac{1}{x}$.

This is defined when $x \neq 0$, undefined when $x = 0$.

Answer: $\boxed{\{x \mid x \neq 0\}}$.

Definition. A **rational function** is a ratio of two polynomial functions,

$$R(x) = \frac{N(x)}{D(x)}.$$

Fact: The domain of a rational function is

$$\{x \mid D(x) \neq 0\}.$$

Ex. 22. Find the domains of $g(x) = \dfrac{x-1}{x^2-1}$ and $h(x) = 3x$.

What's wrong with the following reasoning?

Since $g(x) = \dfrac{x-1}{(x+1)(x-1)} = \dfrac{1}{x+1}$, *the domain of* g *is* $\{x \bigm| x \neq 1\}$.

Solution.

Set the denominator $D(x) = 0$:

$$\begin{aligned} x^2 - 1 &= 0 \qquad\qquad \text{makes } g(x) \text{ undefined} \\ x &= \pm 1 \end{aligned}$$

Answer: $\boxed{\{x \mid x \neq \pm 1\}}$

Now we'll look at $h(x) = 3x$. First we rewrite $h(x)$ to spell out what the denominator is...

$$h(x) = 3x = \frac{3x}{1}$$

...and then we set the denominator $= 0$:

$$1 = 0 \qquad\qquad \text{makes } h(x) \text{ undefined}$$

No choice of x makes $1 = 0$, obviously! So:

Answer: $\boxed{\mathbb{R}}$

Ex. 23. Find the domain of $F(x) = \dfrac{4x}{\sqrt{x-4}(x-2)}$.

Solution.

F is not a rational function.

But we can still ask the question, *for what values of* x *is* $F(x)$ *defined?*

For $F(x)$ to be defined, we need:

(1) the denominator $\sqrt{x-4}(x-2) \neq 0$, and

(2) $\sqrt{x-4}$ must be a real number.

First we'll check (1): when is the denominator equal to 0?

$$\begin{aligned} \sqrt{x-4}(x-2) &= 0 \\ \sqrt{x-4} &= 0 \text{ or } x = 2 \\ x &= 4 \text{ or } x = 2 \end{aligned}$$

So 4 and 2 are not in the domain of F. Now we check (2): when is $\sqrt{x-4}$ a real number?

$$\begin{aligned} x - 4 &\geqslant 0 \\ x &\geqslant 4 \end{aligned}$$

So only numbers x such that $x \geqslant 4$ can be in the domain of F. But note that we already knew that 4 is not in the domain of F.

Answer: $\boxed{\{x \mid x > 4\}}$

Definition. A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, and taking roots) starting with polynomials.

Definition. A function f is called a **transcendental function** if it is not an algebraic function.

Ex. 24. Give three examples of algebraic functions. Then do the same for transcendental functions. Is the absolute value function algebraic? Is exp algebraic? How about ln?

Composition of functions

A function may be viewed as a machine, or a "black box," that "eats" an input value x and "spits out" the corresponding output value $y = f(x)$:

$$x \quad \overset{f}{\mapsto} \quad y = f(x)$$

(the symbol $\mapsto$ is read as "maps to").

We can combine two functions f and g by doing them sequentially, one after another:

$$x \quad \overset{f}{\mapsto} \quad y = f(x) \quad \overset{g}{\mapsto} \quad z = g\big(f(x)\big)$$

We call this the composite of f and g. It is a new, third function, which we give the symbol $g \circ f$ (notice the order of f and g in this notation).

$$x \overset{g \circ f}{\mapsto} z$$

You may find it helpful to read the symbol $\circ$ as "after."

Definition. Given two functions f and g, the **composite** function $g \circ f$ (also called the **composition of** g **and** f) is defined by the relation

$$(g \circ f)(x) = g\big(f(x)\big).$$

☞ The domain of $g \circ f$ is the intersection of two sets:

$$\text{Domain}(f) \bigcap \{x \mid f(x) \in \text{Domain}(g)\}.$$

Ex. 25. Find formulas for $f \circ g$ and $g \circ f$ if f and g are defined by

$$f(x) = \sqrt{x - 2}, \qquad g(x) = x + 7.$$

Ex. 26. Find the domain of $f \circ g$ and the domain of $g \circ f$, where f and g are as in the previous example.

Ex. 27. Find a formula for $G \circ F$, where $F(x) = x + 3$ and $G(y) = \sqrt{y}$.

Domain of $F = \mathbb{R}$.

Domain of $G = [0, \infty)$.

When is $F(x)$ in the domain of G?

$$F(x) \geqslant 0$$
$$x \geqslant -3$$

Domain of $G \circ F$:

$$\text{Domain}(f) \bigcap \{x \mid f(x) \in \text{Domain}(g)\} = \mathbb{R} \cap [-3, \infty) = \boxed{[-3, \infty)}.$$

Function arithmetic

Functions can also be combined using the arithmetic operations $+$, $-$, $\cdot$, and $/$.

For example, if f and g are two functions, we can define a new function $h(x) = f(x) + g(x)$. The *name* of this new function is $f + g$:

$$\big(f + g\big)(x) = f(x) + g(x).$$

(We write parentheses around the name of the function $f + g$ because, if we didn't, we'd end up writing $f + g(x)$, which, if you think about it, doesn't make any sense—take the name of the function f and add it to the output of g?)

The domain of the new function $f + g$ is $\text{Domain}(f) \cap \text{Domain}(g)$.

Ex. 28.

- What's the domain of $\big(f - g\big)(x) = f(x) \cdot g(x)$? In other words, for what x-values is the expression $f(x) - g(x)$ defined?
- What's the domain of $\big(fg\big)(x) = f(x) \cdot g(x)$?
- What's the domain of $\big(f \,/\, g\big)(x) = \dfrac{f(x)}{g(x)}$? *(Careful!)*

More preliminaries

This document is a refresher of some—but certainly not *all*—of the concepts you'll need in Calculus 1.

If it's been a while since you've taken Precalculus, *it is strongly recommended that you brush up on all the material in Chapter 1 of our textbook* with the possible exception of the hyperbolic functions (see ⋆ below).

In addition to the material covered in this document, be sure you are comfortable with:

- Linear functions (Section 1.2 in the OpenStax textbook)
- Point-slope and slope-intercept form of the equation of a line (Section 1.2)
- Polynomial and power functions (Section 1.2)
- "Behavior at infinity," also known as "end behavior" (Section 1.2)
- Piecewise-defined functions (Section 1.2)
- Transformations of functions (Section 1.2)
- Radian measure (Section 1.3)
- The six basic trigonometric functions $\sin$, $\cos$, $\tan$, $\csc$, $\sec$, and $\cot$ (Section 1.3)
- Trigonometric identities (Section 1.3)
- Inverse functions (Section 1.4)
- One-to-one functions, the Horizontal Line Test, and restrictions of a function's domain (Section 1.4)
- The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\csc^{-1}$, $\sec^{-1}$, $\cot^{-1}$ (Section 1.4)
- The laws of exponents (Section 1.5)
- Exponential functions $f(x) = b^x$ (Section 1.5)
- The number e (Section 1.5)
- Logarithmic functions $f(x) = \log_b(x)$ (Section 1.5)
- Changing between logarithmic bases (Section 1.5)

⋆: Studying the hyperbolic trigonometric functions $\sinh$, $\cosh$, $\tanh$, csch, sech, and $\coth$ (Section 1.5) may be useful to students of electrical engineering, architecture, or physics, but we won't be discussing them in our course.

Workbook Lesson 1

§2.1, Introduction to Calculus

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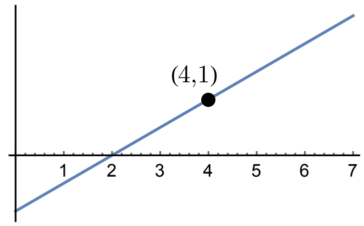
If the world and everything in it was flat—that is to say, *linear*—we wouldn't need calculus.

FIRST QUESTION: How fast is this line rising?



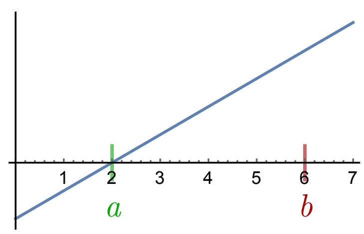
Algebra is all we need to answer this question.

The slope of the line, $\dfrac{\Delta y}{\Delta x}$, tells us the line rises Δy units for every Δx units we move to the right.

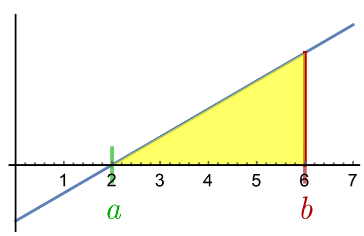
$$m = \frac{\Delta y}{\Delta x} = \frac{1}{2}$$

It doesn't matter what point on the line we start from. The slope is constant—the line everywhere rises at the same rate.

SECOND QUESTION: what's the area between the line and the x-axis, let's say between the two values of x marked on the graph below?



If we know the slope of the line is $\frac{1}{2}$, this is an easy geometry problem.



The base of the triangle is

$$b - a = 4$$

units long. The slope ("rise over run") tells us the height of the triangle must be 2.

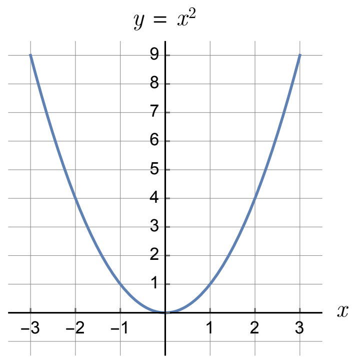
Once we know the base and the height of the triangle, we can easily find its area:

$$A = \tfrac{1}{2} \times \text{BASE} \times \text{HEIGHT} = 4$$

Average rate of change and the tangent problem

As soon as we start dealing with *curves*, it's less obvious how to proceed.

How fast is this curve rising?

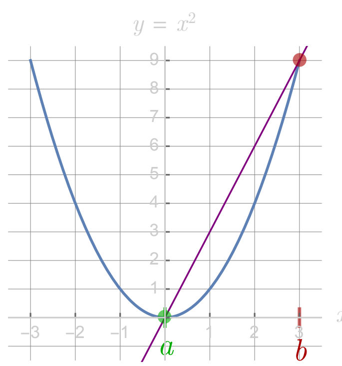


First of all, on one side of the y-axis, the curve isn't rising at all—it's *falling*.

Okay, so how fast is the curve rising for non-negative x?

Well, that depends on what point you're starting from. If you're starting at $x = 0$, the curve is rising slowly. If you're starting at, say, $x = 2$, the curve rises a lot faster.

We could use algebra to give a rough idea of how fast this curve rises. We could mark off two points on the curve and say something like, "the AVERAGE rate of change from $x = 0$ to $x = 3$ is the slope of the line through these two points."



Without calculus, that's the best we can do—an approximation of how fast the curve rises, depending on what part of the curve we're looking at.

This is exactly the idea of the *slope of the secant line*, which you may have heard about in a previous class.

Definition. Let $y = f(x)$ be a function.

- The **graph** of the function f is the set of points $(x, f(x))$ for all x in the domain of f.
- A **secant line** of the graph is a line that meets the graph in two points, $(a, f(a))$ and $(b, f(b))$.
- The **average rate of change in** $f(x)$ **from** $x = a$ **to** $x = b$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

☞ The symbol Δ in an expression of the form Δ__ means "the change in __." For a point on the graph of $y = f(x)$, the change in y, which we write as Δy, is the change in $f(x)$. So the slope of the secant line to the graph of $y = f(x)$ through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ is

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}.$$

<u>Ex. 1.</u> Find the average rate of change in $f(x) = x^2$ from $x = 0$ to $x = 3$.

<u>Solution to Ex. 1:</u>

To find the two points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ on the secant line, we take $a = 0$ and $b = 3$ and substitute the two values into the formula $f(x) = x^2$:

$$a = 0 \qquad f(0) = 0^2 = 0$$
$$b = 3 \qquad f(3) = 3^2 = 9$$

$$\big(a, f(a)\big) = (0, 0)$$
$$\big(b, f(b)\big) = (3, 9)$$

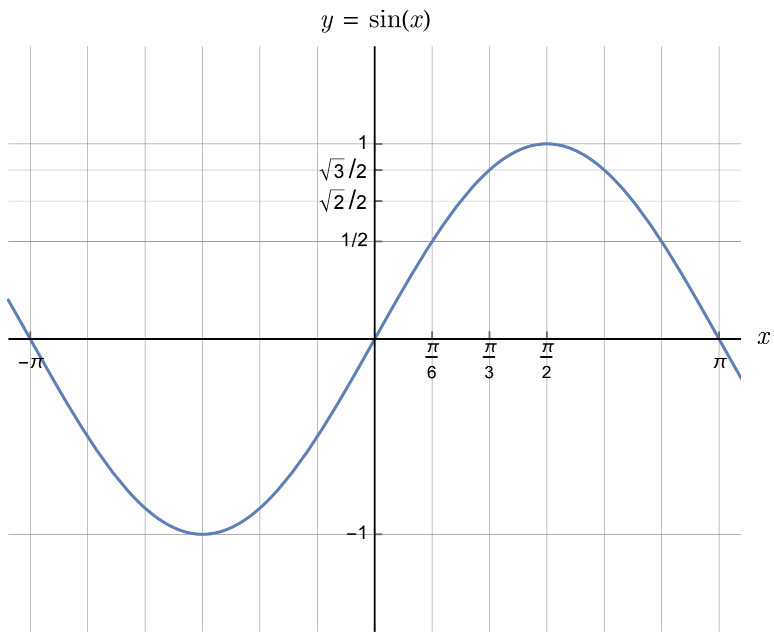
The average rate of change is the slope of the line through these two points:

$$\frac{\Delta y}{\Delta x} = \frac{9 - 0}{3 - 0} = \boxed{3}$$

The problem with the average rate of change is, although it always *approximates* how fast the curve rises or falls, sometimes it's a *bad* approximation of how fast the height changes starting from a given point (say, in the following example, starting from $x = 0$).

<u>Ex. 2.</u> Find the average rate of change in $y = \sin(x)$...

(a) ...from $x = 0$ to $x = \pi$.



We could get a better approximation if we take the two points of the secant line closer together.

Ex. 2 (continued). Find the average rate of change in $y = \sin(x)$.

(b) from $x = 0$ to $x = \pi/2$. (Round to four decimal places for parts **(b)**–**(d)**.)

(c) from $x = 0$ to $x = \pi/3$.

(d) from $x = 0$ to $x = \pi/6$.

Answers to Ex. 2:

The average rate of change in $\sin(x)\ldots$

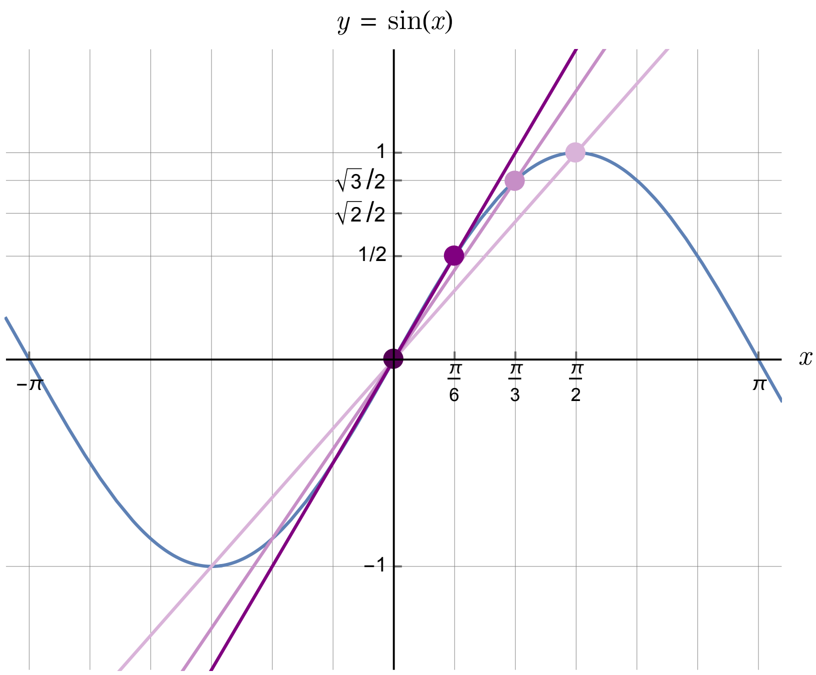
(a) …from $x = 0$ to $x = \pi$ is $\boxed{0}$.

(b) …from $x = 0$ to $x = \pi/2$ is $\boxed{0.6366}$.

(c) …from $x = 0$ to $x = \pi/3$ is $\boxed{0.8270}$.

(d) …from $x = 0$ to $x = \pi/6$ is $\boxed{0.9549}$.

What are these numbers telling us? It looks like as we take the rightmost point closer to $x = 0$, the slope gets larger—which means, the curve rises more and more sharply.



But although the slopes are increasing as the rightmost point gets closer to $x = 0$, we know from the graph that there's a *limit* to how large the slope can get. The secant line is never going to look anything like, say, a vertical line (which has infinite slope).

We now ask, will the slopes of the secant lines "home in" on some exact number as we take the rightmost point closer and closer to $x = 0$? And if so, what *is* that exact number? Our textbook calls this question the **tangent problem**.

To solve the tangent problem, we need more than algebra and geometry. We need calculus.

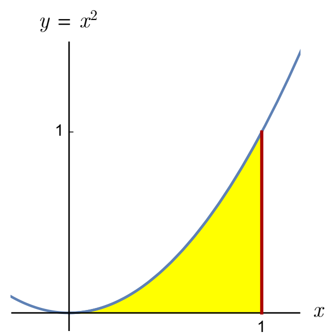
We're not going to solve the tangent problem in this lesson. For now, I just want you to get a sense of what algebra can't do that calculus *can*.

Definition. Let $y = f(x)$ be a function.

- The **instantaneous rate of change** of a function $f(x)$ at $x = a$ is the value TO WHICH the average rate of change on intervals $[a, b]$ APPROACHES as b approaches a, provided that there is such a value.
- The **tangent line at** $x = a$ is the line through the point $\big(a, f(a)\big)$ whose slope is the instantaneous rate of change at $x = a$.

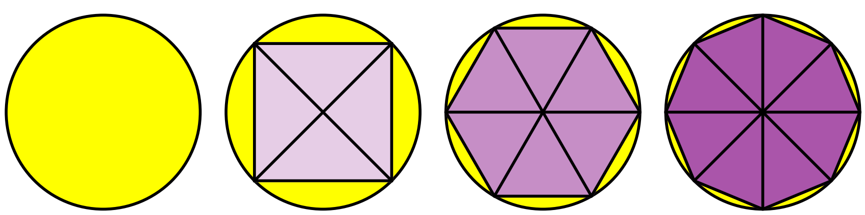
The method of exhaustion and the area problem

Let's ask another question for which algebra and geometry can't give us an exact answer. What's the area under the curve $y = x^2$ from $x = 0$ to $x = 1$?

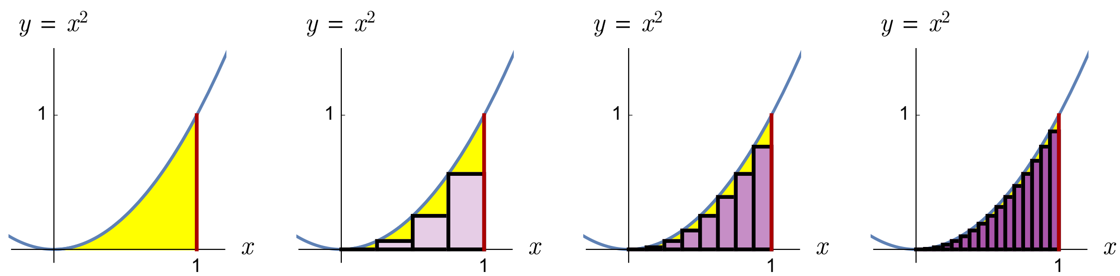


Again, we can come up with an approximation.

The idea here is an ancient one. The Greeks figured out how to approximate the area of a circle using what they called *the method of exhaustion*:



We can approximate the area under a curve by trying to "exhaust"—that is, cover all of—the area by rectangles.



We can't actually ever cover *all* the area with rectangles because one side of the region is curved.

But if you know what a pixel is, you understand that we can get pretty close to a curved shape using blocks, provided that we make the blocks small enough to fit in a large number of them.

In the case of finding the area under a curve, we take more and more rectangles, thinner and thinner. The total area of all the rectangles becomes a better approximation of the actual area.

As we take more and more, thinner and thinner rectangles, we know that the total area homes in on *some* number—namely, the actual area under the curve. But to find that number *exactly*, we need calculus. Geometry and algebra can't do the job.

Limiting processes

What we've seen today are two examples of what we might call *limiting processes*.

- In the case of the area under a curve, we ask whether the total area of the rectangles "homes in" on some number—which we'll learn is called a *limit*—as the number of rectangles becomes infinitely large.
- In the case of instantaneous velocity, we ask whether the slope of the secant line through two points homes in on some number—a *limit*—as the distance between the two points becomes infinitely small.

In calculus, we learn how to calculate with infinities. We find out that it's not as hard as it sounds. But you can understand why it took thousands of years for human beings to figure out how to do it. For the ancient Greeks, infinity was a mysterious, mystical concept. Honestly, infinity seems mysterious and mystical to *most* people—at least at first.

Infinity was tamed by Newton and Leibniz when they discovered the methods of calculus. You will learn how to tame the infinite, too. Does that sound impressive? Taming the infinite sounds pretty impressive to me.

But, you may wonder, what's the point of it? What's it useful for? How will calculus help me when I'm working at a job in marketing, or in engineering, or in psychology, or in the social sciences?

Calculus is the mathematics of change. Whatever career you're interested in, *change* plays some role.

- When your company raises the price of a product they're selling, how will that change the number of units you sell? That's the idea of *marginal demand*, which we calculate the same way as instantaneous velocity.

- When we ask someone to learn a more difficult task, how much longer will it take them to learn it? That question (of psychology) can be answered by finding a tangent line.
- How much water can a curved tank hold? In Calculus II and III, future engineers learn how to solve that problem, which is similar to finding the area under a curve.

Let's do some practice problems, now that we have an idea where this is all leading.

Average velocity and instantaneous velocity

Definition. Suppose the position of a particle in motion is given by a function $s(t)$. (For example, if a stone is dropped from a great height, its position function is the particle's height above ground level.)

- The particle's **average velocity** from time $t = a$ to time $t = b$ is the average rate of change in its position function:
$$\frac{s(b) - s(a)}{b - a}$$
- The **instantaneous velocity** of a particle in motion is the instantaneous rate of change in its position function—that is, the value TO WHICH the average velocity (from time a to time b) APPROACHES as b approaches a, provided that there is such a value.

Ex. 3 (§2.1—#1). Find the slope of the secant line to the graph of $f(t) = t^2 + 1$ passing through the point $P = (1, 2)$ and the point $Q = \big(t,\ f(t)\big)$ when...

(a) $t = 1.1$ **(b)** $t = 1.01$ **(c)** $t = 1.001$ **(d)** $t = 1.0001$

Then estimate the instantaneous velocity at $t = 1$ of a particle in motion whose position is given by $f(t) = t^2 + 1$.

Ex. 4. A stone is dropped from a window 300 feet above ground level. Its height in feet after t seconds is $h(t) = 300 - 16t^2$. Find the average velocity of the stone for the time period beginning when $t = 3$ and lasting 0.5 seconds.

Solution:

The average velocity is the slope of the secant line through two points.

- The first point is $\big(3, h(3)\big)$. It corresponds to the beginning of the time period.
- The second point, which corresponds to the end of the time period 0.5 seconds later, is $\big(3.5, h(3.5)\big)$.
- Substituting $t = 3$ and $t = 3.5$ into the position function $h(t)$ yields

$$h(3) = 156, \qquad h(3.5) = 104.$$

So the slope of the secant line through $\big(3, h(3)\big)$ and $\big(3.5, h(3.5)\big)$ is

$$\frac{h(3.5) - h(3)}{3.5 - 3} = \frac{104 - 156}{0.5} = -104$$

Answer: Falling at 104 feet per second

Ex. 5 (§2.1—#18). A stone is tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters after t seconds is $h(t) = 15t - 4.9t^2$. Find the average velocity of the stone for each of the given time intervals.

(a) $[1, 1.05]$ **(b)** $[1, 1.01]$ **(c)** $[1, 1.005]$ **(d)** $[1, 1.001]$

Ex. 6 (§2.1—#19). Use the preceding exercise to estimate the instantaneous velocity of the stone at $t = 1$ sec.

Ex. 7.

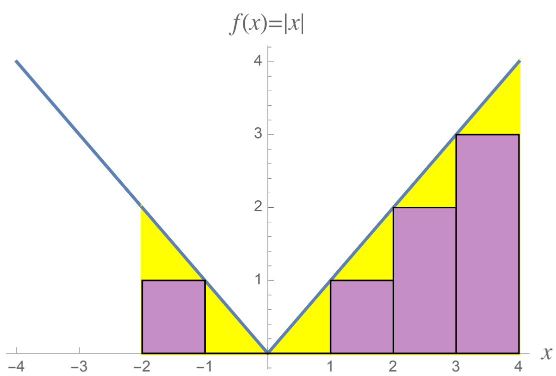
(a) Sketch the graph of $f(x) = |x|$ over the interval $[-2, 4]$ and shade the region above the x-axis.

(b) Approximate the area between the graph of f and the x-axis from $x = -2$ to $x = 4$ by drawing six rectangles and calculating their total area.

(c) Find the exact area between the graph of f and the x-axis from $x = -2$ to $x = 4$ using geometry.

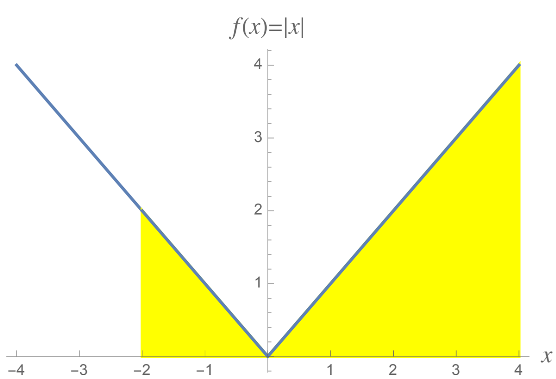
Solution:

First we shaded the region between the graph and the x-axis *(yellow)*. Then we drew rectangles contained in the shaded region.



Applying the formula for the area of a rectangle to each of the rectangles shown, we get

$$\text{TOTAL AREA} = 1 + 1 + 2 + 3 = \boxed{7}.$$



Applying the formula for the area of a triangle to the two triangles shown,

$$\text{TOTAL AREA} = \tfrac{1}{2}(2)(2) + \tfrac{1}{2}(4)(4) = \boxed{10}.$$

Ex. 8.

(a) **(§2.1—#26)** Sketch the graph of $y = \sqrt{1 - x^2}$ over the interval $[-1, 1]$. (*Hint*: Square both sides of the equation and recall the standard form of the equation of a circle.)

(b) Approximate the area between the graph of $y = \sqrt{1 - x^2}$ and the x-axis from $x = -1$ to $x = 1$ by drawing six rectangles and calculating their total area.

(c) Find the exact area between the graph of $y = \sqrt{1 - x^2}$ and the x-axis from $x = -1$ to $x = 1$ using geometry.

Workbook Lesson 2

§2.2, Limit of a Function (Informal Definitions)

Last revised: 2021-05-28 10:35

Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist. *(Moved to Lesson 5, §2.5)*
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

Ten simple rules (The Field Axioms)

The **real numbers** are the numbers on the number line. These include rational numbers and irrational numbers, but not imaginary numbers, and certainly not ∞, which isn't a number at all.

We denote the set of all real numbers either by the symbols $(-\infty, \infty)$ or the symbol $\mathbb{R}$ ("doublestruck" or "blackboard bold" R).

The real numbers, together with the operations of addition and multiplication, are an example of what algebraists call a **field**.

The rules for doing arithmetic with numbers in a field are called the **Field Axioms**:

For any real numbers a, b, and c:

name	addition	multiplication
associativity	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
commutativity	$a+b = b+a$	$ab = ba$
distributivity	$a(b+c) = ab+ac$	$(a+b)c = ac+bc$
identity	$a+0 = a = 0+a$	$a \cdot 1 = a = 1 \cdot a$
inverses	$a+(-a) = 0 = (-a)+a$	$aa^{-1} = 1 = a^{-1}a$ if $a \neq 0$

☞ Recall that a^{-1} is another way of writing the reciprocal of a, that is, $\dfrac{1}{a}$.

These rules are as natural to us as walking, or throwing a ball. We don't always think about the rules of a field when we do arithmetic, but we should—*all the alegbra we learn prior to calculus follows from these ten simple rules.*

Calculus begins with the definition of *additional* rules, defined in terms of what we call LIMITS. About 90% of undergraduate calculus involves a LIMIT of one kind or another (e.g., derivatives, integrals, limits of sequences, series), so it's worth taking the time to understand this concept.

The limit of a function: Informal definition

Let f be a function, and let a be a constant. The statement

$$\lim_{x \to a} f(x) = L$$

says that the **limit of** $f(x)$**, as** x **approaches** a**,** is the number L.

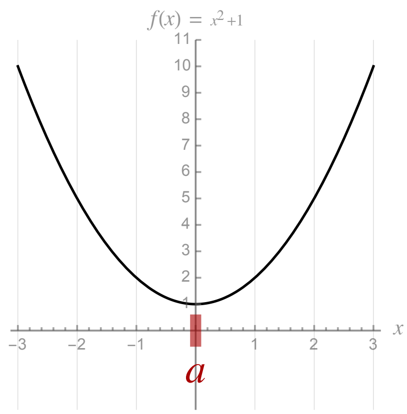
Informal Definition 1: This means that the output $f(x)$ can be made as near to L as we like, if we take x sufficiently near a, but not equal to a.

☞ It has nothing whatsoever to do with the value of $f(x)$ when x *equals* a.

For example, we can say

$$\lim_{x \to 0} x^2 + 1 = 1$$

because the value of $x^2 + 1$ is as close as we want to 1, whenever x is sufficiently near 0.



(*See applet on iCollege:* "`First example of a limit`")

It might be that the value of the function $x^2 + 1$ *is* 1 when x equals 0, but in general this has nothing to do with the idea of a limit.

Consider a less obvious example:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

What is the value of

$$\frac{x^2 - 4}{x - 2}$$

when $x = 2$?

Undefined: substituting gives $\frac{0}{0}$, a meaningless expression.

But for all $x \neq 2$ *near* 2, the value is *near* 4.

Indeed, since

$$\frac{x^2-4}{x-2} = \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = x+2$$

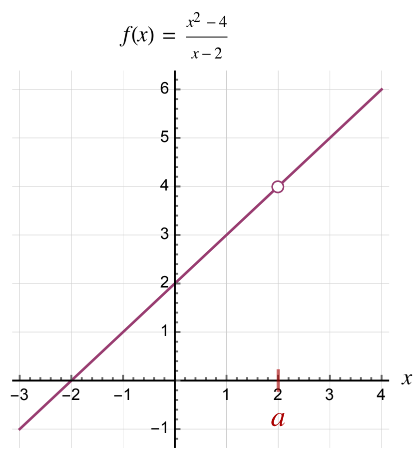
for any $x \neq 2$, the graph of

$$\frac{x^2-4}{x-2}$$

is identical to the graph of

$$x+2$$

everywhere except at $x = 2$.



From the graph, we see that the value of $\dfrac{x^2-4}{x-2}$ gets as close as we like to 4 if we take x close enough to 2.

The takeaway from Informal Definition 1:

- The expression "$\lim\limits_{x\to a} f(x)$" represents the number that the output $f(x)$ gets close to, whenever the input x is close to a.
- If we want to abbreviate $\lim\limits_{x\to a} f(x)$ by a letter—say, L—we can write:

$$\lim_{x\to a} f(x) = L$$

- When we say that $f(x)$ gets close to L, or x gets close to a, how "*close*" is close enough? We'll answer this question precisely in Section 2.5. For now, think of "*close*" as meaning "as close as you like."

Alternate notation for $\lim\limits_{x\to a} f(x) = L$:

$$\text{As } x \to a,\ \ f(x) \to L.$$

- The symbol $\to$ can be read aloud as "*approaches*" or "*gets close to*."

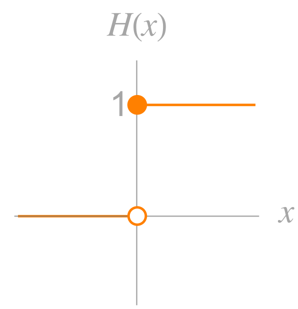
Determining a one-sided or two-sided limit of a function from its graph

It's important to be able to "eyeball" the limit of a function from its graph.

Ex. 1. The **Heaviside function** is

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Find $\lim_{x \to 2} H(x)$ by inspecting the graph.



Ex. 2. Does $\lim_{x \to 0} H(x)$ exist? That is, is there a number L such that $\lim_{x \to 0} H(x) = L$? If so, say what L is. If not, say why not.

Answer:

As x approaches 0 from the right, $H(x)$ approaches 1.

As x approaches 0 from the left, $H(x)$ approaches 0.

There is no single number that $H(x)$ approaches as x approaches 0.

Therefore, $\boxed{\lim_{x \to 0} H(x) \text{ does not exist.}}$

Informal Definition 2: The statement

$$\lim_{x \to a^-} f(x) = L$$

says that the **lefthand limit of** $f(x)$**, as** x **approaches** a**,** is the number L.

- It means that the output $f(x)$ can be made as near to L as we like, for $x < a$ sufficiently near a.

Informal Definition 3: The statement

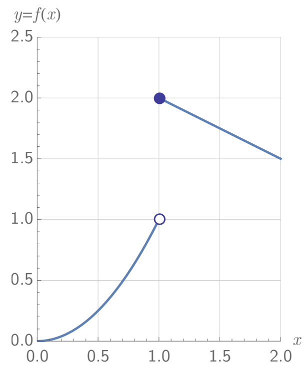
$$\lim_{x \to a^+} f(x) = L$$

says that the **righthand limit of** $f(x)$**, as** x **approaches** a**,** is the number L.

- It means that the output $f(x)$ can be made as near to L as we like, for $x > a$ sufficiently near a.

Theorem. $\lim_{x\to a} f(x) = L$ if, and only if, $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

If the lefthand limit $\lim_{x\to a^-} f(x)$ and the righthand limit $\lim_{x\to a^+} f(x)$ are not the same, then we say the "two-sided" limit $\lim_{x\to a} f(x)$ does not exist.

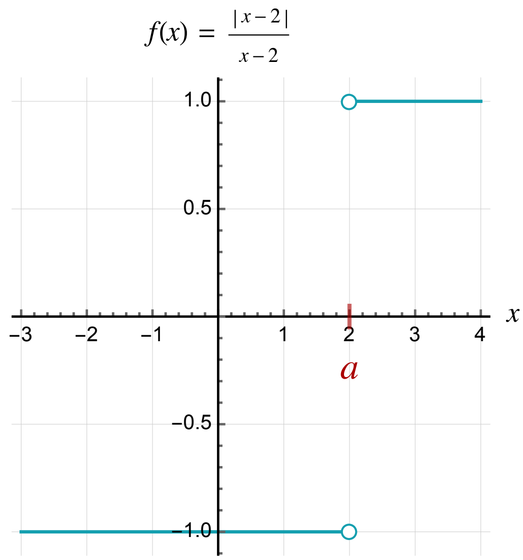


$$\lim_{x\to 1^-} f(x) = 1$$

$$\lim_{x\to 1^+} f(x) = 2$$

$\lim_{x\to 1} f(x)$ does not exist

Ex. 3. The graph of a function $y = f(x)$ is given below. Find the lefthand limit $\lim_{x\to 2^-} f(x)$ and the righthand limit $\lim_{x\to 2^+} f(x)$. If $\lim_{x\to 2} f(x)$ exists, state its value; if it does *not* exist, give your answer as "DNE."



(*See applet on iCollege:* "`Visualizing one-sided and two-sided limits`')

Ex. 4 (compare with §2.2, #46–49) The graph of a function $y = f(x)$ is given. State the value of each quantity, if it exists. If it does not exist, explain why not.

(a) $\lim_{x\to 6^-} f(x)$

(b) $\lim_{x\to 6^+} f(x)$

(c) $\lim_{x\to 6} f(x)$

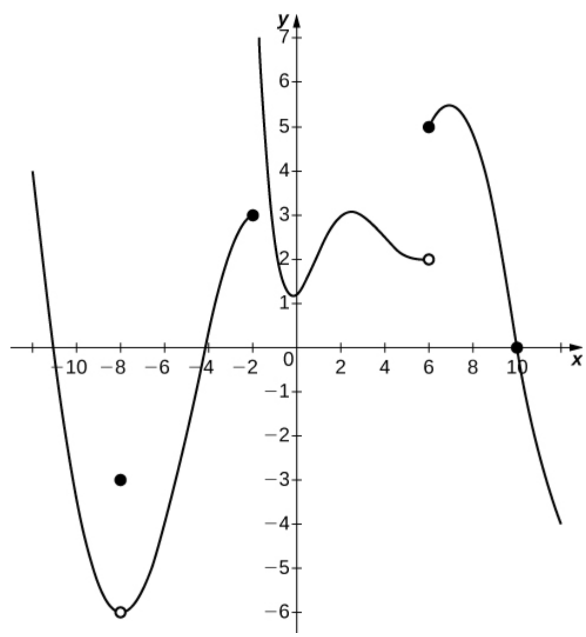
(d) $f(6)$

(e) $\lim_{x\to -8^-} f(x)$

(f) $\lim_{x\to -8^+} f(x)$

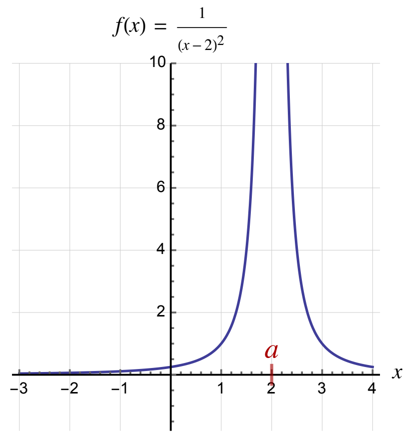
(g) $\lim_{x\to -8} f(x)$

(h) $f(8)$



Infinite limits

Consider the graph of $f(x) = \dfrac{1}{(x-2)^2}$. Does the limit of $f(x)$ as x approaches 2 exist?



The graph shows that the output $f(x)$ can be made as large as we like by taking x near 2. Although the outputs do not approach a *number*, we can say something about how f behaves close to 2.

Informal Definition 4:

- A (one-sided or two-sided) limit of $f(x)$ as x approaches a **is positive infinity** if the values of $f(x)$ increase without bound as x approaches a.
- A (one-sided or two-sided) limit of $f(x)$ as x approaches a **is negative infinity** if the values of $f(x)$ decrease without bound as x approaches a.
- ☞ The symbol ∞ is not a number. In particular, WE SHOULD NOT SUBSTITUTE $x = \pm\infty$ IN AN EQUATION OR AN EXPRESSION. We use the symbols $\pm\infty$ only to indicate the behavior of $f(x)$ as $x \to a$.

Given the previous graph, we see that $\lim_{x\to 2^-} f(x) = \infty$ and $\lim_{x\to 2^+} f(x) = \infty$.

Since the one-sided limits as $x \to 2$ are the same, we see that $\lim_{x\to 2} f(x) = \infty$.

Definition: The graph of a function $f(x)$ has a **vertical asymptote at** $x = a$ if one or both of the one-sided limits of $f(x)$ as $x \to a$ is positive or negative infinity.

Ex. 5. Graph $f(x) = \dfrac{1}{x-3}$. (*Hint*: $\dfrac{1}{x-3}$ is a transformation of $\dfrac{1}{x}$.)

Write the equation of its vertical asymptote.

Then state the value of each quantity, if it exists. If it does not exist, explain why not.

(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

Additional exercises

Ex. 6 (§2.2—#76). Sketch the graph of a function f with the given properties.

$\lim_{x\to 2} f(x) = 1$,

$\lim_{x\to 4^-} f(x) = 3$,

$\lim_{x\to 4^+} f(x) = 6$,

$f(4)$ is not defined.

Ex. 7 (§2.2—#77). Sketch the graph of a function f with the given properties.

As $x \to \infty$, $f(x) \to 0$.

$\lim_{x\to 0} f(x) = f(0)$.

$\lim_{x\to 1^-} f(x) = -\infty$.

$f(0) = 1$.

$\lim_{x\to 1^+} f(x) = \infty$.

As $x \to -\infty$, $f(x) \to -\infty$.

Ex. 8. Plot the graph of the function

$$f(x) = \frac{x+2}{x-3}$$

using a graphing calculator, an online graphing tool (like `Geogebra`), or software (like `Mathematica` or `Apple Grapher`). *(You may want to copy a sketch of the graph in the space provided.)* Use the graph to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 3^-} f(x)$ **(b)** $\lim_{x \to 3^+} f(x)$ **(c)** $\lim_{x \to 3} f(x)$

Ex. 9. Plot the graph of the function

$$f(x) = x\sqrt{\frac{x^2+1}{x^2}}$$

using technology. Use the graph to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ **(b)** $\lim_{x \to 0^+} f(x)$ **(c)** $\lim_{x \to 0} f(x)$

Ex. 10. Sketch the graph of the piecewise-defined function below. Then use the graph to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 0 \\ 4x & \text{if } 0 < x < 1 \\ (x+1)^2 & \text{if } x \geq 1 \end{cases}$$

Workbook Lesson 3

§2.3, The Limit Laws

Last revised: 2021-06-17 13:29

Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the Squeeze Theorem.
- Evaluate $\lim_{x\to 0} \frac{\sin(x)}{x}$.

To find the exact value of a limit, we often use the following properties, called the **Limit Laws**.

In the equations below, a and c are constants, n is a positive integer, and $f(x)$ and $g(x)$ are functions defined for all $x \neq a$ in some open interval containing a.

Identity Law $\lim_{x\to a} x = a$

Constant Law $\lim_{x\to a} c = c$

Sum Law $\lim_{x\to a} \left[f(x) + g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Difference Law $\lim_{x\to a} \left[f(x) - g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

Constant Multiple Law $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$

Product Law $\lim_{x\to a} \left[f(x)\, g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

Quotient Law $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Power Law $\lim_{x\to a} \left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$

Root Law $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

(If n is even, we assume $a > 0$.)

Ex. 1. Derive the rule $\lim_{x\to a} x^n = a^n$ (from the Identity Law and Power Law).

Ex. 2. Use the Limit Laws to find a formula for $\lim_{x\to a} \frac{1}{f(x)}$. When does this limit exist?

Ex. 3. Calculate $\lim_{x\to -1}(x^4 - 3x)(x^2 + 5x + 3)$. Justify each step by indicating the appropriate Limit Law.

$$\begin{aligned}
\lim_{x\to -1}(x^4 - 3x)(x^2 + 5x + 3) &= \lim_{x\to -1}(x^4 - 3x) \cdot \lim_{x\to -1}(x^2 + 5x + 3) && \text{(Product)}\\
&= \left[\lim_{x\to -1} x^4 - \lim_{x\to -1} 3x\right] \cdot \left[\lim_{x\to -1} x^2 + \lim_{x\to -1} 5x + \lim_{x\to -1} 3\right] && \text{(Sum, Diff.)}\\
&= \left[\big(\lim_{x\to -1} x\big)^4 - 3\big(\lim_{x\to -1} x\big)\right] \cdot \left[\big(\lim_{x\to -1} x\big)^2 + 5\big(\lim_{x\to -1} x\big) + \lim_{x\to -1} 3\right] && \text{(C. Mult., Power)}\\
&= \big[(-1)^4 - 3(-1)\big]\,\big[(-1)^2 + 5(-1) + 3\big] && \text{(Const., Id.)}\\
&= (1+3)(1-5+3)\\
&= 4(-1)\\
&= -4.
\end{aligned}$$

Ex. 4. Evaluate $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2 + x}$, if it exists.

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2+x} &= \lim_{x\to 0} \frac{1}{x} - \frac{1}{x(x+1)} && \text{Can't use Quotient Law... Try simplifying.}\\
&= \lim_{x\to 0} \frac{(x+1)-1}{x(x+1)}\\
&= \lim_{x\to 0} \frac{x}{x(x+1)}\\
&= \lim_{x\to 0} \frac{1}{x+1} && \text{Why is this equality justified?}\\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} x + \lim_{x\to 0} 1}\\
&= \frac{1}{0+1}\\
&= 1.
\end{aligned}$$

Ex. 5. Evaluate $\lim_{h \to 0} \frac{(a+h)^3 - a^3}{h}$, if it exists.

$$\begin{aligned}\lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} &= \lim_{h \to 0} \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} \\ &= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &= \lim_{h \to 0} 3a^2 + 3ah + h^2 \\ &= 3a^2.\end{aligned}$$

Are we done?

Ex. 6. Find $\lim_{t \to -3} \frac{t^2 + 6t + 9}{t^2 + 3t}$.

$$\begin{aligned}\lim_{t \to -3} \frac{t^2 + 6t + 9}{t^2 + 3t} &= \lim_{t \to -3} \frac{(t+3)^2}{t(t+3)} \\ &= \lim_{t \to -3} \frac{1}{t} \frac{(t+3)^2}{t+3} \\ &= \lim_{t \to -3} \frac{1}{t} \cdot \lim_{t \to -3} (t+3) \\ &= -\frac{1}{3} \cdot 0 \\ &= 0.\end{aligned}$$

Direct Substitution: Special case (Polynomial and Rational Functions)

It is often the case that $\lim_{x \to a} f(x) = f(a)$.

- This is *not* always true.
- But it *is* true for polynomial and rational functions.
- We will see in Section 2.4 it is true for a larger class of functions (called *continuous* functions).

Theorem (Direct Substitution for Polynomial and Rational Functions).

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then

(Polynomials) $\lim_{x \to a} p(x) = p(a)$ and

(Rational Functions) $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ if $q(a) \neq 0$.

A proof of this Theorem can be found in the textbook.

Definition: A limit as x approaches a is called an **indeterminate form** if substituting $x = a$ results in a undefined expression.

- The examples of indeterminate forms we encountered in Exercises 4, 5, and 6 demonstrate that such limits can sometimes be evaluated by tinkering around with algebra until a Limit Law can be applied.
- When substituting $x = a$ yields the meaningless expression $\dfrac{0}{0}$, we say the indeterminate form is of **type** $0/0$.

  More examples of indeterminate forms of type $0/0$:

  $$\lim_{x\to 0} \tfrac{x^2}{x} = 0 \qquad \lim_{x\to 0} \tfrac{x-5}{x-5} = 1 \qquad \lim_{x\to 0} \tfrac{x}{x^3} = \infty$$

- Other types of indeterminate forms include the meaningless expressions $\dfrac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0.

Additional techniques for finding limits

We have already seen (in Exercises 4, 5, and 6) that, before the Limit Laws can be applied, a certain amount of algebraic "tinkering" must be done first.

We now introduce additional techniques. The first involves the *conjugate* of a sum or difference.

Recall:

- The **conjugate** of a sum $(+)$ or difference $(-)$ $w \pm A$ is $w \mp A$.

  That is, the conjugate of $w + A$ is $w - A$, and the conjugate of $w - A$ is $w + A$.

- To **rationalize the denominator** of a fraction

  $$\frac{N}{w \pm A}$$

  means to multiply it by the following "special form of 1":

  $$\frac{w \mp A}{w \mp A}$$

- To **rationalize the numerator** of a fraction

  $$\frac{w \pm A}{D}$$

  means to multiply it by

  $$\frac{w \mp A}{w \mp A}.$$

Ex. 7. Evaluate $\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$, if it exists.

Solution:

$$\begin{aligned}
\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} &= \lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1} \\
&= \lim_{x \to -1} \frac{x+2-1}{(x+1)\left(\sqrt{x+2}+1\right)} \\
&= \lim_{x \to -1} \frac{\cancel{x+1}}{\cancel{(x+1)}\left(\sqrt{x+2}+1\right)} \\
&= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} \\
&= \frac{1}{2}.
\end{aligned}$$

Can we apply the Limit Laws now?

The trick of "multiplying by a special form of 1" can also be useful for evaluating limits of **complex fractions** (fractions that contain fractions).

Ex. 8. Evaluate $\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$, if it exists.

Hint: As written, the limit is an indeterminate form of type $0/0$. Rewrite it by multiplying by the complex fraction by the following special form of 1:

$$\frac{2(x+1)}{2(x+1)}$$

Do you see where the expression $2(x+1)$ came from, and why the hint will result in simplifying the numerator of the complex fraction?

The Limit Laws for one-sided limits

In the fine print of the Limit Laws, we required that:

$f(x)$ and $g(x)$ are functions defined for all $x \neq a$ over some open interval containing a.

The Limit Laws can be applied to one-sided limits by changing the condition to:

defined for all x in some open interval of the form (a, b)

(for righthand limits) or

defined for all x in some open interval of the form (b, a)

(for lefthand limits).

Ex. 9. Two limits are given below.

$$\lim_{x \to 3^-} \sqrt{x-3} \qquad\qquad \lim_{x \to 3^+} \sqrt{x-3}$$

(a) Sketch the graph of the function $f(x) = \sqrt{x-3}$.

(b) Which of the one-sided limits exists?

(c) Evaluate the limit(s) for which the Limit Laws can be applied.

Before we move on to the next topic, we'll do one more exercise that requires the use of one-sided limits.

<u>Ex. 10.</u> Evaluate $\lim_{x\to 0} |x|$, if it exists.

<u>Solution:</u>

Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Sidebar: Is it always true that $-x < 0$? No! Not if x is negative...

Resuming the problem,

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \qquad \text{since } |x| = x \text{ whenever } x > 0.$$

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0 \qquad \text{since } |x| = -x \text{ whenever } x < 0.$$

Therefore, by the Theorem given in Section 2.2,

$$\boxed{\lim_{x\to 0} |x| = 0}$$

since

$$\lim_{x\to 0^-} |x| = 0 = \lim_{x\to 0^+} |x|.$$

The meaning of "if, and only if"

In mathematics, what is the meaning of the phrase "*if, and only if*"?

P: You did poorly in school.

Q: You end up serving in the military.

"$P \Longrightarrow Q$" *means* "**If** you did poorly in school, **then** you end up serving in the military."

—Al Gore during the 2000 presidential campaign

"$P \Longleftarrow Q$" *means* "**If** you end up serving in the military, **then** you did poorly in school."

—A misinterpretation of Gore's statement by his critics

"$P \Longleftrightarrow Q$" *means* $P \Longrightarrow Q$ AND $P \Longleftarrow Q$.

We pronounce the symbol "$\Longleftrightarrow$" as "**if, and only if**."

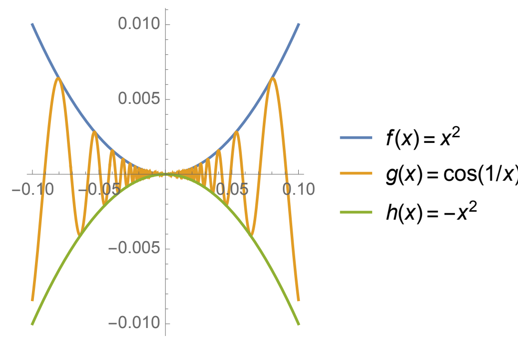
Two of these statements are indefensible insults to members of the Armed Services. One is a warning to students, and not necessarily an insult.

All three of these statements have different meanings.

Squeeze Theorem

Another useful property of limits is as follows.

> **Squeeze Theorem.** Suppose
> $$f(x) \le g(x) \le h(x)$$
> for all $x \neq a$ in an open interval containing a.
> If
> $$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x),$$
> then
> $$\lim_{x\to a} g(x) = L.$$



We can use the Squeeze Theorem to evaluate limits when other methods fail.

In the next exercise, we'll apply the Squeeze Theorem to find the important limit

$$\lim_{x\to 0} \frac{\sin(x)}{x},$$

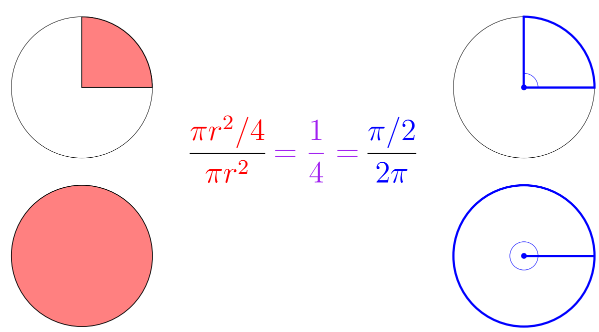
which cannot be found by applying the Limit Laws.

First, *recall*:

The region bounded by a circle and one of its central angles is called a **sector** of the circle.

A circle and its sector have areas proportional to their angle measures:

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Angle measure of sector}}{\text{Angle measure of circle}}. \qquad (\star)$$



$$\frac{\pi r^2/4}{\pi r^2} = \frac{1}{4} = \frac{\pi/2}{2\pi}$$

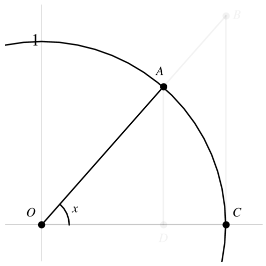
Ex. 11. Find $\lim_{x\to 0} \frac{\sin x}{x}$.

First let's take a guess. When $x \approx 0$, we see by graphing that $\sin x \approx x$, so $\frac{\sin x}{x} \approx 1$—at least for x near 0.

Our guess is, the limit is 1... but this is just a guess. We need a proof to be convinced.

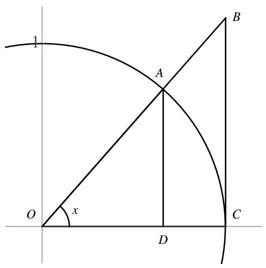
We can't use the Quotient Rule to evaluate the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$. *(Why not?)*

We'll make a geometric argument.



Let x be the radian measure of a sector of the unit circle in the first quadrant. (So $0 < x < \frac{\pi}{2}$.)

Label $O = (0,0)$, $A = (\cos x, \sin x)$, and $C = (1,0)$.



For no apparent reason, we construct a right triangle ΔOBC that contains the sector.

We'll also drop a perpendicular $\overline{AD}$ to the x-axis from A.

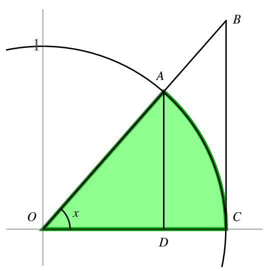
CLAIMS TO BE PROVEN:

(1) $\text{Area}(\text{sector } OAC) = \frac{x}{2}$

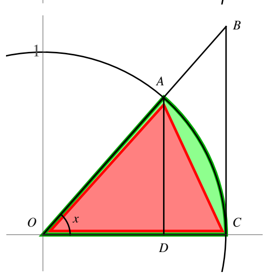
(2) $\text{Area}(\Delta OAC) = \frac{\sin x}{2}$

(3) $\text{Area}(\Delta OBC) = \frac{\tan x}{2}$

PROOFS OF CLAIMS:

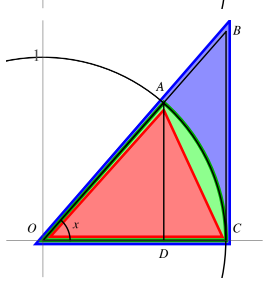
(1) 

$$\text{Area}(\text{sector } OAC) = \frac{x}{2\pi} \cdot \pi(1)^2 = \frac{x}{2}$$ by the above formula ($\star$).

(2) 

$$\text{Area}(\Delta OAC) = \frac{1}{2}(1)(AD) = \frac{1}{2}\sin x$$

because $A = (\cos x, \sin x)$.

(3) 

$$\tan x = \frac{\text{opp.}}{\text{adj.}} = \frac{BC}{1} = BC.$$

$$\text{Area}(\Delta OBC) = \frac{1}{2}(1)(BC) = \frac{\tan x}{2}.$$

The figures on the previous page show that

$$\textsc{Area}(\Delta OAC) \quad < \quad \textsc{Area}(\text{sector } OAC) \quad < \quad \textsc{Area}(\Delta OBC).$$

Let's tinker with this until we get something we can use...

$$\frac{\sin x}{2} \quad < \quad \frac{x}{2} \quad < \quad \frac{\tan x}{2} = \frac{\sin x}{2\cos x}$$

$$\frac{2\cos x}{\sin x} \quad {\color{red}<} \quad \frac{2}{x} \quad {\color{red}<} \quad \frac{2}{\sin x}$$

...yields:

$$\cos x \quad < \quad \frac{\sin x}{x} \quad < \quad 1$$

By the Squeeze Theorem, $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$.

A similar argument shows that $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$, so

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}.$$

<u>Ex. 12.</u> Find $\lim_{x\to 0} \frac{\sin(23x)}{x}$.

<u>Solution:</u>

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(23x)}{x} &= \lim_{x\to 0} \left[\left(23 \cdot \frac{1}{23} \right) \cdot \frac{\sin(23x)}{x} \right] \\
&= 23 \cdot \lim_{x\to 0} \frac{\sin(23x)}{23x} \\
&= 23 \cdot 1 \\
&= 23.
\end{aligned}$$

<u>Ex. 13.</u> Find $\lim_{x\to 0} \frac{\sin 4x}{\sin 6x}$. *Hint*: $\frac{\sin 4x}{\sin 6x} = \frac{\sin 4x}{x} \frac{x}{\sin 6x}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0} \frac{\sin 4x}{x}\frac{x}{\sin 6x} \\
&= \lim_{x\to 0} \frac{4\sin 4x}{4x}\frac{6x}{6\sin 6x} \\
&= \lim_{x\to 0} \frac{4\sin 4x}{4x} \cdot \lim_{x\to 0} \frac{6x}{6\sin 6x} \\
&= \left(4\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \cdot \left(\frac{1}{6}\lim_{x\to 0} \frac{6x}{\sin 6x}\right) \\
&= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \cdot \frac{1}{6}\left(\lim_{x\to 0} \frac{1}{\frac{\sin 6x}{6x}}\right) \\
&= 4(1) \cdot \frac{1}{6}(1) \\
&= \frac{2}{3}.
\end{aligned}$$

Any ideas on what might help now?

Additional exercises

Ex. 14. Use Direct Substitution to find the value of the limit.

(a) (§2.3—#87) $\lim_{x\to 7} x^2$

(b) (§2.3—#89) $\lim_{x\to 0} \frac{1}{1+\sin(x)}$

(c) (§2.3—#91) $\lim_{x\to 1} \frac{2-7x}{x-6}$

(d) (§2.3—#92) $\lim_{x\to 3} \ln(e^{3x})$

Ex. 15. Find the value of the limit. (*Hint*: See Example 2.19 in the textbook.)

$$\lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

Ex. 16. Find the value of the limit. (*Hint*: See Example 2.20 in the textbook.)

$$\lim_{x\to 0} \frac{1}{x} - \frac{3}{x(x+3)}$$

Ex. 17. Find the value of the limit. (*Hint*: See Example 2.23 in the textbook.)

$$\lim_{x\to -2^-} \frac{-x^2 + 5x + 14}{x^2 + x - 2}$$

Ex. 18 (§2.3—#107–114). Given that $\lim_{x\to 6} f(x) = 4$, $\lim_{x\to 6} g(x) = 9$, and $\lim_{x\to 6} h(x) = 6$, find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 3} 2f(x)g(x)$

(b) $\lim_{x\to 3} \left[f(x) + \frac{1}{3}g(x)\right]$

(c) $\lim_{x\to 3} \frac{\left(h(x)\right)^3}{2}$

(d) $\lim_{x\to 3} \sqrt{g(x) - f(x)}$

(e) $\lim_{x\to 3} \left[(x+1)\cdot f(x)\right]$

(f) $\lim_{x\to 3} \left(f(x)\cdot g(x) - h(x)\right)$

Ex. 19. Use Direct Substitution to show that the limit leads to the indeterminate form $0/0$. Then evaluate the limit.

(a) (§2.3—#93) $\lim_{t\to 4} \frac{x^2-16}{x-4}$

(b) (§2.3—#95) $\lim_{t\to 4} \frac{3x-18}{2x-12}$

(c) (§2.3—#96) $\lim_{h\to 0} \frac{(h+1)^2-1}{h}$

(d) (§2.3—#97) $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3}$

(e) (§2.3—#99) $\lim_{\theta\to \pi} \frac{\sin(\theta)}{\tan(\theta)}$

(f) (§2.3—#100) $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

(g) (§2.3—#101) $\lim_{t\to 4} \frac{2x^2+3x-2}{2x-1}$

(h) (§2.3—#102) $\lim_{x\to -3} \frac{\sqrt{x+4}-1}{x+3}$

Ex. 20. Evaluate the limit.

(a) $\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$ **(b)** $\lim_{x\to -1} \frac{x^2 + 2x + 1}{x^4 - 1}$ **(c)** $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

Ex. 21 (Example 2.24). Use the Squeeze Theorem to show that $\lim_{x\to 0} x\cos(x) = 0$.

Ex. 22. Show that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

satisfies $\lim_{x\to 0} f(x) = 0$.

Workbook Lesson 4

§2.4, Continuity

Last revised: 2021-08-20 19:53

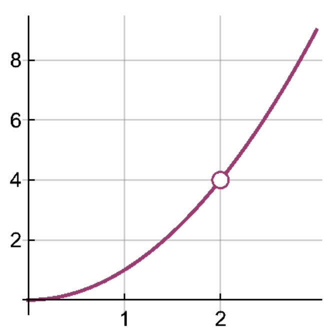
Objectives

- Explain the three conditions for continuity at a number.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the Composite Function Theorem.
- Apply the Intermediate Value Theorem.

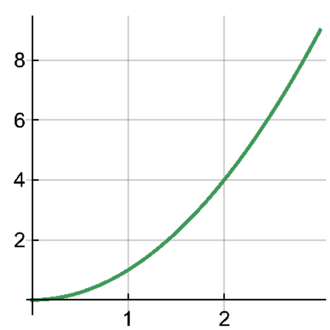
Introduction to continuity

Recall that, in general, the limit of $f(x)$ as x approaches a has nothing to do with the value of $f(x)$ when x REACHES a. For example, in the graph below on the left, the limit as $x \to 2$ of $f(x)$ is 4. The output value AT $x = 2$—that is, $f(2)$—isn't even defined.

$\lim_{x \to 2} f(x) = 4$



$\lim_{x \to 2} g(x) = 4$



The graph on the right shows a modified version of f. Let's call the modified version g. The only difference between f and g is, there's no hole at $x = 2$ in the graph of g. The limit of $g(x)$ as $x \to 2$ is the same as for $f(x)$. But this time, the output at 2,

$$g(2) = 4,$$

is defined. Moreover, $g(2)$ is the number to which the values of $g(x)$ are approaching as x approaches 2.

Another word for "unbroken" is *continuous*. We can express the difference between the behavior of f and g at $x = 2$ by saying that g is *continuous* at $x = 2$, and f is not.

When Calculus was first being developed, a distinction was drawn between graphs you can draw without lifting your pencil—which were called continuous—and ones you can't. Later, in the 1800s, that rough idea of *continuity* (that is, the property of being continuous) was refined and defined more precisely. We'll use the modern, more precise definition of "continuous."

At first glance, our definition doesn't seem to say anything about the graph of $f(x)$ being unbroken:

Definition. A function f is **continuous at a number** a if $\lim_{x \to a} f(x) = f(a)$.

In fact, this definition describes only what happens near a single point on the graph. It says that, as x approaches a, the value of $f(x)$ doesn't just approach a limit—it actually reaches the value of the limit when x is equal to a. That is, as the input x approaches a, the output $f(x)$ approaches $f(a)$—and *gets* there.

We'll see in the next subsection (see $\star$) how this definition relates to the idea that a graph has no breaks in it.

But first, we need to unpack the meaning of the equation in the definition of continuity:

$$\lim_{x\to a} f(x) = f(a).$$

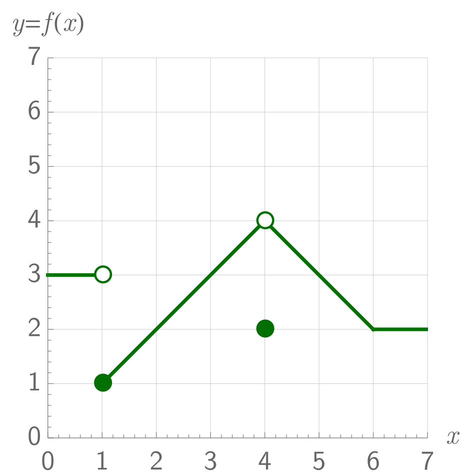
This equation comes up so often, we'll give it a special name: "the **Direct Substitution Property**."

What is required for this equation to be a meaningful and true statement?

- a is in the domain of f ← (that is, $f(a)$ is a real number)
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$ ← (and since $f(a)$ is a real number, $\lim_{x\to a} f(x) \neq \pm\infty$)

☞ To decide whether a function f is continuous at a number a, verify that *all three* of the bulleted statements are true.

Ex. 1. The graph of a function f is given.



(a) At what numbers a does $\lim_{x\to a} f(x)$ NOT exist?

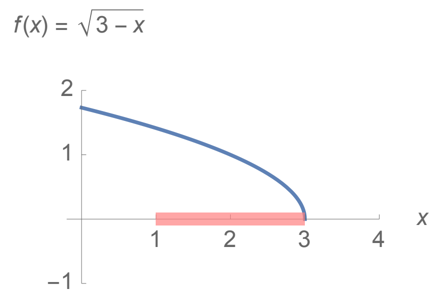
(b) At what numbers is $f(x)$ NOT continuous?

(c) At what numbers a does $\lim_{x\to a} f(x)$ exist but f is NOT continuous at a?

Continuity on an interval

We can extend the idea of continuity at a *single* number $x = a$ to continuity over an entire *interval* of x-values. ($\star$): If (but *not only if!*) you can draw a function's graph over an entire interval without lifting your pencil, then the function is continuous "on" (or "over") that interval:

<u>Definition.</u> A function f is **continuous on an <u>open</u> interval** (a, b) if f is continuous at every x in (a, b).



$f(x) = \sqrt{3-x}$ is continuous at each x-value such that $1 < x < 3$

Notice that we specified that the interval has to be *open* (that is, endpoints are not included). To define continuity on a *closed* interval (endpoints included), we need to define what it means to be *continuous from one side*, say, at an endpoint. The definition we just gave won't work, because if a function's output isn't *defined* on the far side of an endpoint, then one of the one-sided limits at the endpoint won't exist.

<u>Definition.</u> f is **continuous from the left at** a if $\lim_{x \to a^-} f(x) = f(a)$, and **continuous from the right at** a if $\lim_{x \to a^+} f(x) = f(a)$.

<u>Definition.</u> A function f is **continuous on an <u>closed</u> interval** $[a, b]$ if f is continuous at every x in (a, b), continuous from the right at a, and continuous from the left at b.

That is, "continuous on $[a, b]$" means "continuous on both sides everywhere in the interior (a, b) of the interval, and continuous from one side at each endpoint."

Finally, you are invited to consider the problem of defining "continuous everywhere." We need to clarify what we mean by "everywhere." Do we mean "at every number on the number line," or do we mean "at every number in the domain of f"? If the former, then $f(x) = 1/x$ is not continuous "everywhere." If the latter, then $f(x) = 1/x$ *is* continuous "everywhere."

Our textbook does not define "continuous everywhere." But sometimes, speaking casually, we may say that a function "is continuous," or "is continuous everywhere." This is imprecise—technically, in this class, we should always specify *on what interval* (or at what number) the function is continuous. Be mindful that, according to our definitions, "continuous" always means "continuous on an interval" or "continuous at a number $x = a$".

Discontinuities

Our word for "not continuous" is *discontinuous*.

Definition. If a is in the domain of f, but f is not continuous at a, we say f is **discontinuous at** a (or f has a **discontinuity** at a).

☞ Our book has a slightly different definition. For f to be discontinuous at a, they do not require $f(a)$ to be defined. During our exams, you can safely ignore this subtle distinction. We'll only give you problems in which this issue doesn't come up.

Ex. 2. The Heaviside function (see Lesson 2 for its graph) has a discontinuity at 0. This type of discontinuity is called a *jump discontinuity*.

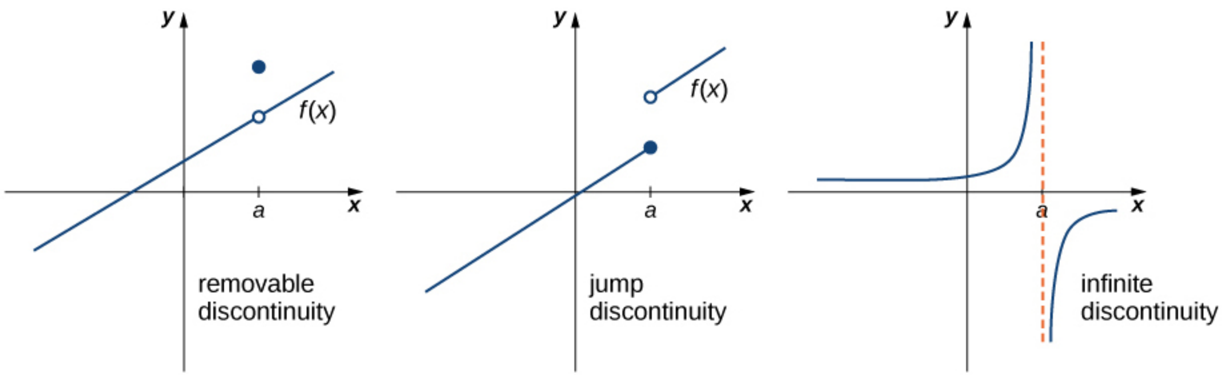
Ex. 3. The function $f(x) = \dfrac{x^2 - 3x - 4}{x - 4}$ has one discontinuity. What is it? This is an example of a *removable discontinuity*.

Sketch the graph:

Ex. 4. The function $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ has one discontinuity. What is it? This is an example of an *infinite discontinuity*.

Sketch the graph:

Ex. 5 (Challenge). The function $I(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is discontinuous at every real number x. Can you explain why?



Definition: If $f(x)$ is discontinuous at a, then

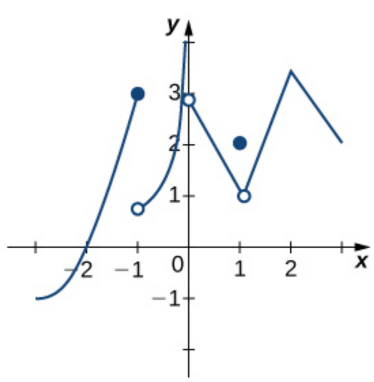
- f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists.
  (Note: When we state that $\lim_{x\to a} f(x)$ exists, we mean that $\lim_{x\to a} f(x) = L$ for some real number L.)
- f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
  (Note: When we state that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, we mean that both limits are equal to real numbers.)
- f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ (or both).

☞ A demonstration of the concept of *discontinuity* is provided on iCollege. Don't sweat the technical details referenced in the applet! Your instructor may cover them during lecture.

(*See applet on iCollege:* "`Formal meaning of discontinuity`")

☞ *Recall*: if you can draw the graph of a function over an interval without lifting your pencil, then the function is continuous on that interval.

Ex. 6. The graph of a function is given below



(a) From the graph, state the intervals on which the function is continuous.

(b) Classify each discontinuity as jump, removable, or infinite.

Ex. 7. Sketch the graph of the function $f(x) = \dfrac{1}{x+2}$ and explain why f is discontinuous at the number $a = -2$.

Theorems for evaluating limits

Theorem. If f is a rational function, polynomial, or trigonometric function, then f is continuous at every number in its domain.

Corollary: Direct Substitution for polynomials, rational functions, and trigonometric functions.

If

- f is a rational function, polynomial, or trigonometric function, and
- a is in the domain of f,

then

$$\lim_{x \to a} f(x) = f(a).$$

(*Recall*:) This equation is called the **Direct Substitution Property**.

Ex. 8. Find the limit, if it exists.

(a) $\lim_{x \to 0} \cos(x)$ **(b)** $\lim_{x \to \pi/2} \tan(x)$ **(c)** $\lim_{x \to 3} \dfrac{x^2 - 1}{2x - 6}$ **(d)** $\lim_{x \to 0} \dfrac{x^2 - 1}{2x - 6}$

Composite Function Theorem. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

Corollary to the Composite Function Theorem. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Ex. 9. Use the Composite Function Theorem to prove its Corollary.

Theorem.

If f and g are continuous at $x = a$, then the following functions are continuous at $x = a$.

$$f \circ g \qquad f \pm g \qquad f \cdot g \qquad f/g \quad (\text{if } g(a) \neq 0)$$

Ex. 10. Find $\lim_{x\to\pi} h(x)$, where $h(x) = x + \sin\left(x - \frac{\pi}{2}\right)$.

Solution:

The function h is the combination of continuous functions using the operations in the previous theorem, so h is continuous.

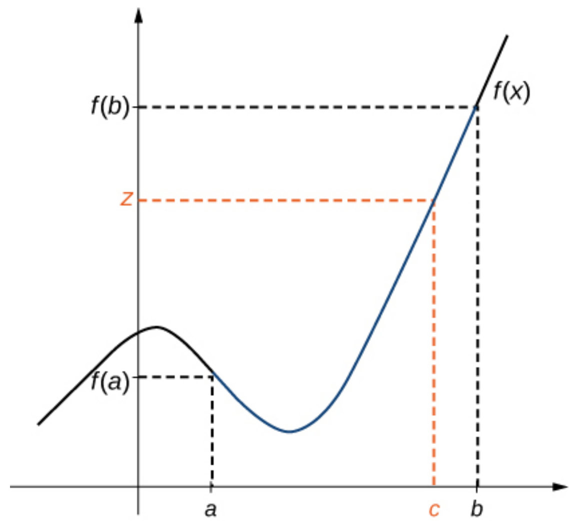
Since h is continuous, we can use the Direct Substution Property:

$$\lim_{x\to\pi} h(x) = h\left(\frac{\pi}{2}\right) = \pi - \sin(\frac{\pi}{2}) = \boxed{\pi - 1}$$

Intermediate Value Theorem

The following is an example of an *existence theorem*. It asserts that *there exists* a number (call it c) satisfying certain properties, but it does *not* tell us the value of c.

Intermediate Value Theorem (IVT). Suppose f is continuous on $[a, b]$. Suppose $f(a) \neq f(b)$. If z is any number between $f(a)$ and $f(b)$, then there exists a number c in the interval (a, b) such that $f(c) = z$.



<u>Ex. 11.</u> Let $f(x) = 1 - x^6$, and note that

$$f(0) = 1 \qquad \text{and} \qquad f(2) = 1 - 64 = -63.$$

Identify the numbers a, b, and z in the following statement.

By the Intermediate Value Theorem, $f(c) = 0$ for some number c between 0 and 2.

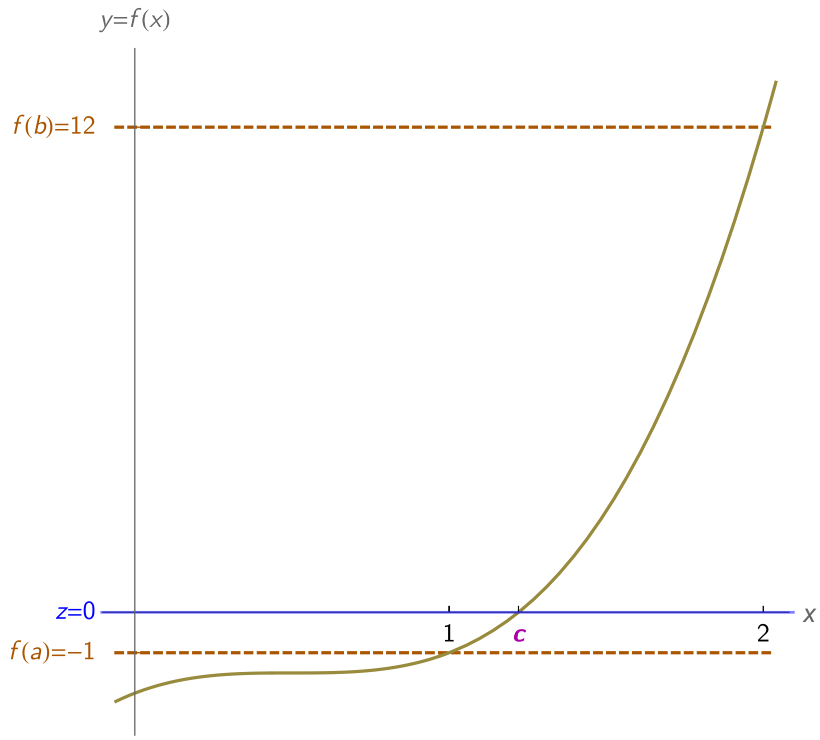
Recall: A **zero** (or **root**) of a function f is a value c such that $f(c) = 0$.

Ex. 12. Show there is a zero of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

- Since f is a polynomial, it is continuous, so the Intermediate Value Theorem can be used.
- Take $a = 1$, $b = 2$, and $z = 0$ for the Theorem.
- Check that $f(a) \neq f(b)$, and that z is between $f(a)$ and $f(b)$:

$$\left.\begin{array}{l} f(1) = -1, \\ f(2) = 12. \end{array}\right\} \quad -1 \neq -12, \text{ and } 0 \text{ is between } -1 \text{ and } -12 \ \checkmark$$

- Now the Theorem guarantees there exists a number c in $[1, 2]$ such that $f(c) = z = 0$.



Since f is continuous and $z = 0$ is between $f(1) = -1$ and $f(2) = 12$,
the IVT guarantees there is some number c between $a = 1$ and $b = 2$ such that $f(c) = z = 0$.

Additional exercises: Limits and continuity

Ex. 13. Explain why each function is continuous or discontinuous.

(a) The temperature in your home as a function of time.

(b) The cost of an Uber ride as a function of the distance traveled.

(c) The electrical current supplied to a household appliance prior to, during, and after a blackout.

Ex. 14. Determine the value(s), if any, at which each function is discontinuous. Classify each discontinuity as removable, jump, or infinite.

- $f(x) = \dfrac{1}{\sqrt{x}}$
- $g(x) = \dfrac{x}{x^2 - x}$
- $h(t) = \dfrac{1}{t} + 1$
- $j(t) = \dfrac{5}{e^t - 2}$
- $k(x) = \tan(2x)$
- $\ell(t) = \dfrac{t+3}{t^2 + 5t + 6}$

Ex. 15 (§2.4—#151). A particle moving along a line has a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$.

(a) Explain why there must be a value c such that $2 < c < 5$ and $s(c) = 4$.

(b) Now suppose a second particle has a position function $h(t) = s(t) - t$. Explain why there must be a value d such that $2 < d < 5$ and $h(d) = 0$.

Ex. 16. Let

$$h(x) = \begin{cases} 3x^2 - 4 & \text{if } x \leq 2, \\ x^2 & \text{if } x > 2. \end{cases}$$

Although $h(0) < 10$ and $h(4) > 10$, there is no value of x in the interval $[0, 4]$ such that $h(x) = 10$. Explain why this does not contradict the Intermediate Value Theorem.

Ex. 17. Define

$$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x & \text{if } x > 1. \end{cases}$$

Sketch the graph of f. Can you pick a value of k so that defining $f(1) = k$ makes $f(x)$ continuous on the interval $(-\infty, \infty)$?

Workbook Lesson 5

§2.5, The Epsilon-Delta Definition of a Limit

Last revised: 2021-06-15 07:33

Objectives

- Interpret an inequality of the form $0 < |x - a| < c$ as a statement about the distance between x and a.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist. *(Moved from Lesson 2, §2.2)*
- Describe the idea behind the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

Inequalities representing distance

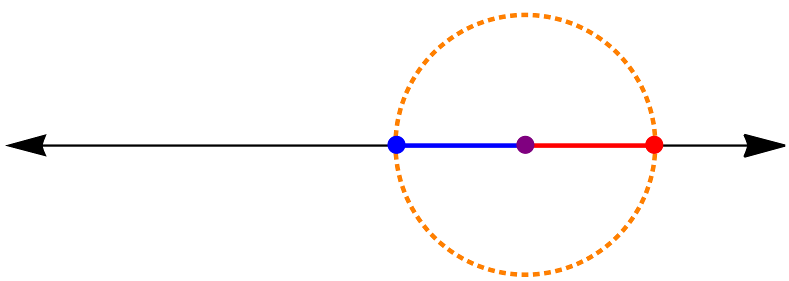
The **distance** between two numbers w and a is $|w - a| \geq 0$.

Let $c > 0$. The inequality

$$|w - a| < c$$

means that the distance $|w-a|$ between w and a is less than c. (We use the absolute value bars because the **difference** $w-a$ might be negative, while distance is by definition never negative.)

Anchoring one end of a piece of string at the purple point on the number line below, pinch off a length of string—call the length c—and swing it around the purple point like a compass to see why c is sometimes called the "radius" of the inequality.



Ex. 1. Where can x be on the number line if

$$0 < |x - 7| < 2?$$

Answer in words, or by graphing on the number line.

Ex. 2. If $|w - a| = 0$, what must be true about w and a?

Ex. 3. Graph the set of numbers w on the number line such $|w-3| \leq 1$.

Guessing the limit of a function using a table of values

Ex. 4. Guess the value of $\lim_{x \to 0} \dfrac{\sin x}{x}$ using only a calculator.

Taking x closer and closer to 0, we find that:

x	$\dfrac{\sin(x)}{x}$
$\pm 1.$	0.841471
± 0.5	0.958851
± 0.4	0.973546
± 0.3	0.985067
± 0.2	0.993347
± 0.1	0.998334
± 0.05	0.999583
± 0.01	0.999983
± 0.005	0.999996

Guess: $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

(Notice that the function $\dfrac{\sin x}{x}$ is undefined when $x = 0$.)

We showed in an earlier Lesson this guess is correct.

Ex. 5. Guess the value of $\lim_{x \to 0} \sin \dfrac{1}{x}$.

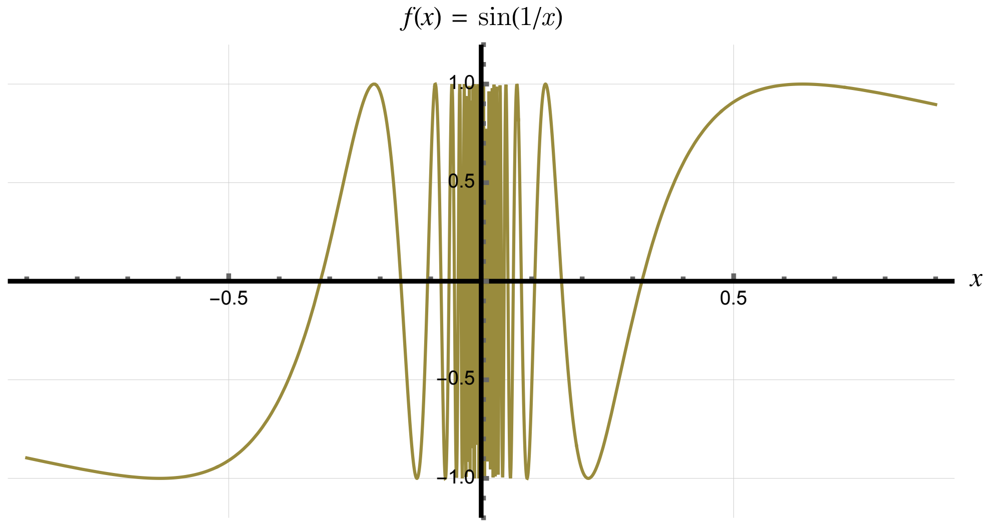
Taking x closer and closer to 0, we find that:

x	$\sin \dfrac{1}{x}$
$\pm \frac{1}{\pi}$	0
$\pm \frac{1}{2\pi}$	0
$\pm \frac{1}{3\pi}$	0
$\pm \frac{1}{4\pi}$	0
$\pm \frac{1}{5\pi}$	0
$\pm \frac{1}{10\pi}$	0
$\pm \frac{1}{100\pi}$	0

Guess: $\lim_{x \to 0} \sin \dfrac{1}{x} = 0$

This time our guess is wrong.

Can you explain why by looking at the graph of $\sin \dfrac{1}{x}$?



There's something seriously wrong with our "informal" definition of a limit—it misleads us into giving an incorrect answer. *The limit of* $\sin\dfrac{1}{x}$ *as* $x \to 0$ *does not exist.*

In the next section of this Lesson, we will revise our informal definition of

$$\lim_{x\to a} f(x)$$

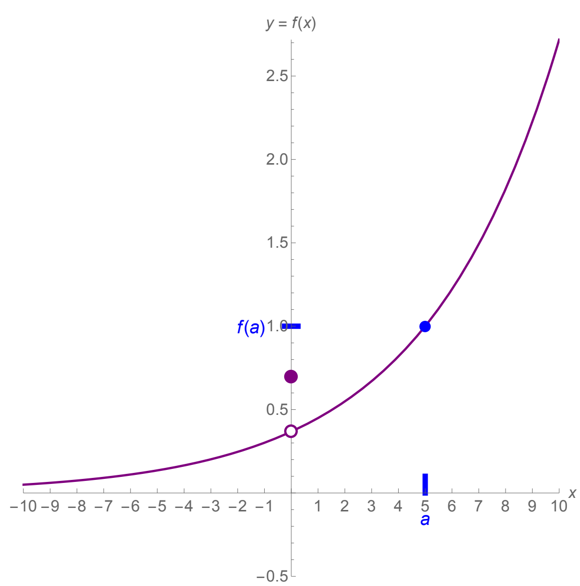
and give a precise definition that is reliable in all cases.

Formal definition of limit

Let's get a more intimate understanding of the concept of a limit before we look at the true definition.

(*See applet on iCollege:* "`Epsilon-delta definition of limit`")

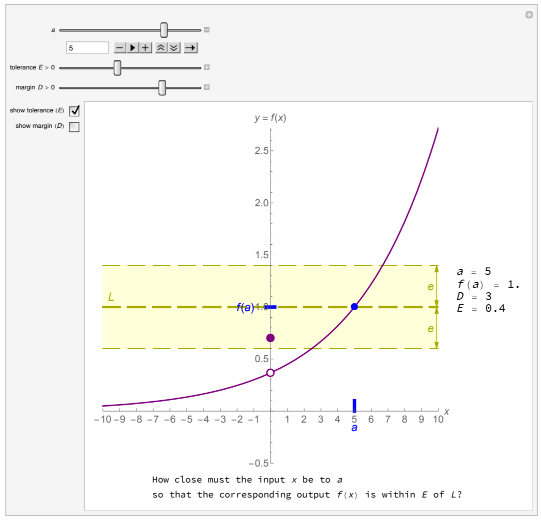
Here is the graph of a function. Let's not worry about what the formula for this function is.



A point $(a, f(a))$ on the graph is marked.

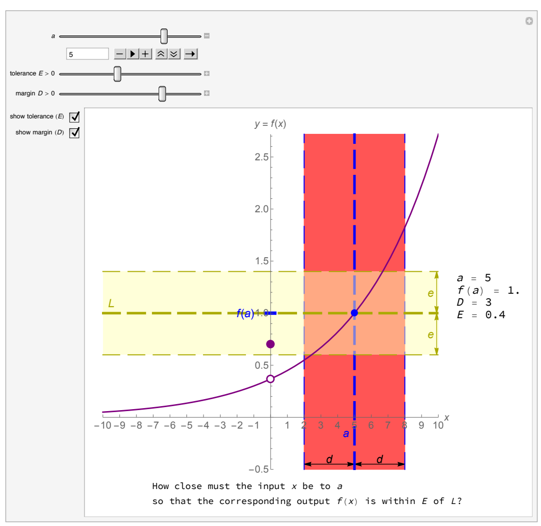
For what x-values is the output $f(x)$ $\underline{\text{NEAR}}$ $f(a)$?

You might say, well, how near do you want it? *(Set $a = 5$ in the applet.)* Let's say I want the output to be within *four tenths* of $f(a) = 1$. *(Set $E = .4$ in the applet.)*



Imagine an old-fashioned radio with a knob you turn to change the station. You don't have to tune the knob to exactly the right frequency. Within a certain tolerance will be close enough to make the radio station come in clearly.

So how close to a do our x-values have to be to give us output values that are all within the tolerance shown? Is it enough to be within 3 units? *(Set $D = 3$ in the applet.)*



What about within 1 unit? *(Set $D = 1$ in the applet.)*

But 2 is no good—there are x values that give us outputs that aren't within the tolerance. *(Set $D = 2$ in the applet to see, then go back to $D = 1$.)*

Let's give this margin of error along the x-axis a name—we'll call $D = 1$ the *margin*. Clearly, any smaller number will serve as a suitable margin, too.

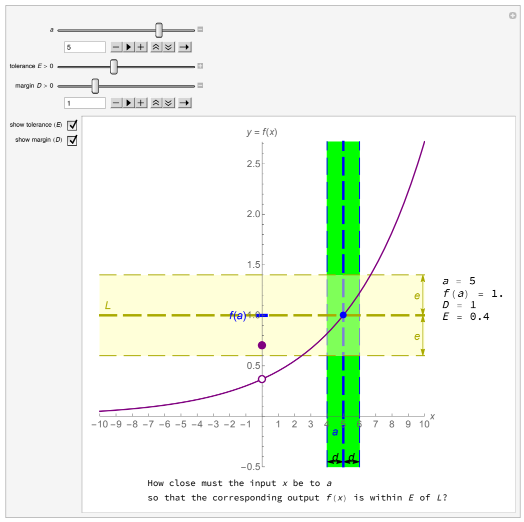
The point is, ALL *the inputs close enough to* 5*—that is, within the margin of* 5*—give us outputs that are within the tolerance of four tenths.*

Write:

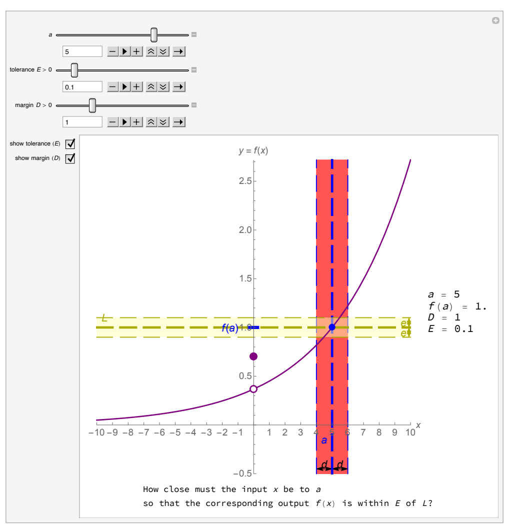
$$\begin{array}{r}\text{The distance between the output } f(x) \text{ and } L \text{ is less than the tolerance } E \\ \text{whenever the distance between the input } x \neq a \text{ and } a = 5 \text{ is less than the margin } D.\end{array} \quad (\star)$$

That's a lot of writing. Let's rewrite this fact using symbols.

$$\big|f(x) - L\big| < E \quad \text{whenever} \quad 0 < |x - a| < D. \qquad (\star)$$



But now suppose I want the output to be closer than 0.4 away. Suppose I change the tolerance to some smaller number—say 0.1. *(Set* $E = 0.1$ *in the applet.)*



Now this fact ($\star$) that I wrote (in both English and symbols) isn't true anymore.

This statement *guarantees* that every x within D of $a = 5$ gives us an output that's within the tolerance of 1. But there are x values within the margin of 5 that yield outputs that are more than 0.1 away from the desired value, 1.

Well, is there *some* D I could choose to make this fact true again? Is there some D so that the portion of the graph within the blue stripe, lies entirely within the red stripe?

Tinker with the slider and try to find a D that works...

Now let me ask you this. Is it the case that, *no matter what* E *I pick*, I can *always* pick a margin D so small that the fact on the board holds true?

That is, can I always make the margin stripe so small that the portion of the graph it contains is entirely contained in the yellow stripe—*no matter how narrow I make the yellow stripe*?

Yes. And this is the idea of a limit.

Formal definition. The statement

$$\lim_{x \to a} f(x) = L$$

(in words: "the **limit of** $f(x)$ **as** x **approaches** a is L") means that, given any tolerance $E > 0$, there exists some margin $D > 0$ such that

$$\left| f(x) - L \right| < E$$

whenever

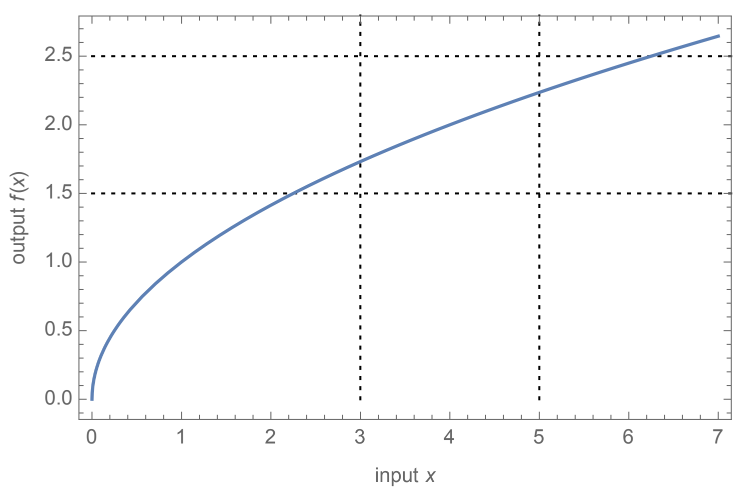
$$0 < |x - a| < D.$$

☞ We don't care what happens when $x = a$.

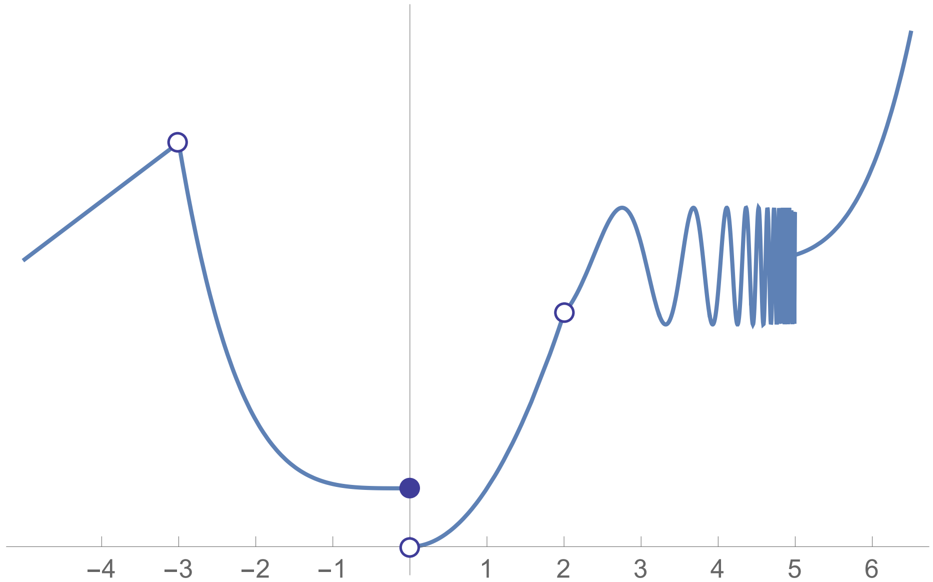
Note: Most authors use the Greek letters δ and ε in the above definition rather than the Roman letters D and E. (Of course, the names of variables don't matter in mathematics!)

Ex. 7. Use the graph provided below to complete the statement:

$$|f(x) - 2| < \_\_\_\_ \text{ whenever } \_\_\_\_ < x < \_\_\_\_.$$



Ex. 6. The graph of a function f is given. State whether or not each quantity exists. If it does not exist, explain why not.



(a) $\lim_{x\to -3^-} f(x)$

(b) $\lim_{x\to -3^+} f(x)$

(c) $\lim_{x\to -3} f(x)$

(d) $f(-3)$

(e) $\lim_{x\to 0^-} f(x)$

(f) $\lim_{x\to 0^+} f(x)$

(g) $\lim_{x\to 0} f(x)$

(h) $f(0)$

(i) $\lim_{x\to 2} f(x)$

(j) $f(2)$

(k) $\lim_{x\to 5^-} f(x)$

(l) $\lim_{x\to 5^+} f(x)$

Formal definitions of infinite limits

Let us revisit our definitions of *infinite limits* in order to make them more precise.

Formal definition (Infinite limits). The statement

$$\lim_{x\to a} f(x) = \infty$$

means that for any number M, there is a $D > 0$ such that

$$f(x) > M$$

whenever x satisfies

$$0 < |x - a| < D.$$

Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

means that for any number M, there is a $D > 0$ such that

$$f(x) < M$$

whenever x satisfies

$$0 < |x - a| < D.$$

The statements

$$\lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a^+} f(x) = -\infty, \qquad \lim_{x\to a^-} f(x) = \infty, \qquad \lim_{x\to a^-} f(x) = -\infty$$

have similar definitions.

Ex. 8. Find $\displaystyle\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\displaystyle\lim_{x\to 3^-} \frac{2x}{x-3}$. Then find $\displaystyle\lim_{x\to 3} \frac{2x}{x-3}$, if it exists.

Solution:

As $x \to 3$ from the right, $x - 3 \to 0$ through positive values, and $2x \to 6 > 0$. So

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \lim_{x\to 3^+} \Big(\underbrace{2x}_{\to 6} \cdot \underbrace{\frac{1}{x-3}}_{\to \infty} \Big) = \infty.$$

(This is an intuitive explanation. To evaluate this limit without appealing to intuition, we could write a formal proof—but you are not expected to do so.)

As $x \to 3$ from the left, $x - 3 \to 0$ through negative values, and $2x \to 6 > 0$. So

$$\lim_{x\to 3^-} \frac{2x}{x-3} = \lim_{x\to 3^-} \Big(\underbrace{2x}_{\to 6} \cdot \underbrace{\frac{1}{x-3}}_{\to -\infty} \Big) = -\infty.$$

Since $\displaystyle\lim_{x\to 3^+} \frac{2x}{x-3} \neq \lim_{x\to 3^-} \frac{2x}{x-3}$, the limit

$$\lim_{x\to 3} \frac{2x}{x-3}$$

does not exist.

Ex. 9. Find $\displaystyle\lim_{x\to 2} \frac{1}{(x-2)^4}$.

Solution:

Solution:

As $x \to 2$ from the right, $(x-2)^4 \to 0$ through positive values. That is, $\dfrac{1}{(x-2)^4} > 0$ for $x > 2$ near 2. Thus

$$\lim_{x\to 2^+} \frac{1}{(x-2)^4} = \infty.$$

As $x \to 2$ from the left, $(x-2)^4 \to 0$ through positive values. That is, $\dfrac{1}{(x-2)^4} > 0$ for $x < 2$ near 2. Thus

$$\lim_{x\to 2^-} \frac{1}{(x-2)^4} = \infty.$$

Since the one-sided limits as $x \to 2^{\pm}$ are both ∞, we have

$$\lim_{x\to 2} \frac{1}{(x-2)^4} = \infty.$$

Epsilon-delta proofs (optional)

The formal definition of a limit can be thought of as a challenge. If for example

$$\lim_{x\to 0} f(x) = 1,$$

this means:

Given any $E > 0$, you must be able to find a number $D > 0$
such that $|f(x) - 1| < E$ for any x such that $0 < |x - 0| < D$.

An **epsilon-delta proof** is an argument that *proves* it's always possible to meet the challenge no matter what $E > 0$ is.

Ex. 10. Let $f(x) = 4x - 5$. Let $E = 1$. Find $D > 0$ such that $|f(x) - 7| < E$ whenever $0 < |x - 3| < D$.

Solution:

To find d, begin with the inequality $|f(x) - 7| < E = 1$.

$$\begin{aligned}
|f(x) - 7| &< 1 \\
|(4x - 5) - 7| &< 1 \\
|4x - 12| &< 1 \\
|4 \cdot (x - 3)| &< 1 \\
4|x - 3| &< 1 && (|A \cdot B| = |A| \cdot |B|) \\
|x - 3| &< \tfrac{1}{4}
\end{aligned}$$

Therefore, $|f(x) - 7| < 1$ whenever $0 < |x - 3| < \boxed{\frac{1}{4}}$.

(Notice that, in this example, $x = 3$ makes $|f(x) - 7| < 1$, so the '$0 <$' isn't used.)

Can we conclude from the work above that $\lim_{x\to 3} f(x) = 7$?

No... this has to work for any $E = 1$, not only the single particular choice $E = 1$.

Ex. 11. Let $f(x) = 1 - 2x$. Show that $\lim_{x\to 4} f(x) = -7$.

Solution:

The definition of a limit requires that we can find a $D > 0$ such that, for any given $E > 0$, the following is true whenever $0 < |x - 4| < D$.

$$\begin{aligned} |f(x) - (-7)| &< E \\ |8 - 2x| &< E \\ 2|x - 4| &< E \\ |x - 4| &< \frac{E}{2} \end{aligned}$$

Our work proves the following statement: *for any $E > 0$, we have*

$$|f(x) - (-7)| < E$$

whenever

$$0 < |x - 4| < D = \frac{E}{2}.$$

That is, we have proven that

$$\lim_{x\to 4} f(x) = -7.$$

Workbook Lesson 6

§3.1, Definition of the Derivative

Last revised: 2021-06-17 13:27

Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Find an equation for the tangent line to the graph of a function f at the point $(a, f(a))$.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.

Definition of the derivative

Recall:

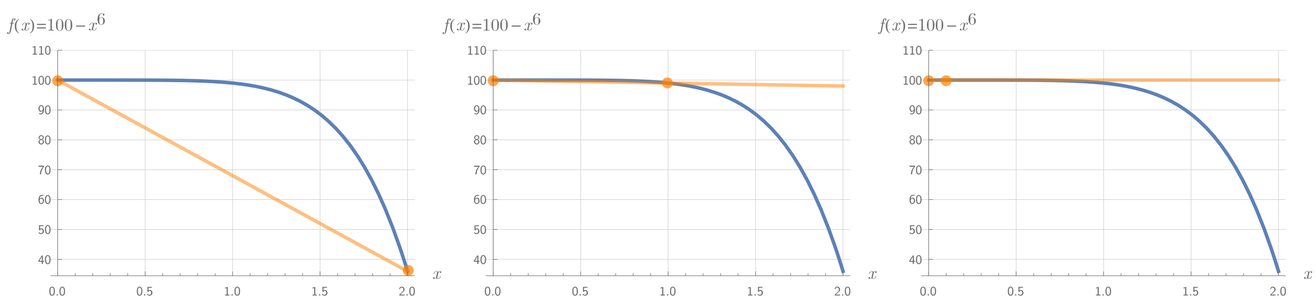
A **secant line** to the graph of a function f is a line through two points $(a, f(a))$ and $(b, f(b))$ on the graph.

Its slope

$$\frac{f(b) - f(a)}{b - a} \qquad (\dagger)$$

is the **average rate of change in f from $x = a$ to $x = b$**.

Definition: The expression $(\dagger)$ is also known as a **difference quotient**.



Ex. 1. Find the average rate of change in the function $f(x) = 100 - x^6$ from $a = 0$ to b.

(i) $b = 2$ **(ii)** $b = 1$ **(iii)** $b = 0.1$

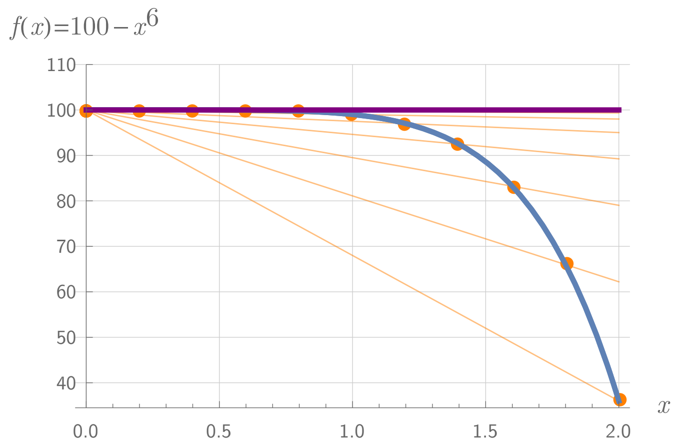
Solution:

(i) $\dfrac{[100 - 2^6] - [100 - 0^6]}{2 - 0} = -2^5 = -16.$

(ii) $\dfrac{[100 - 1^6] - [100]}{1 - 0} = -1.$

(iii) $\dfrac{[100 - (0.1)^6] - [100]}{1 - 0} = -0.000001.$

The derivative at $x = a$ is the limit of the slopes of secant lines to the graph through the points $P = \big(a, f(a)\big)$ and $Q = \big(b, f(b)\big)$, taking the limit as b approaches a.



Definition: Let f be a function. The **derivative of** f **at** $x = a$ is

$$f'(a) \overset{\text{def}}{=} \lim_{b \to a} \frac{f(b) - f(a)}{b - a}. \tag{*}$$

Since the names of variables don't make any difference, we can also write this as

$$f'(a) \overset{\text{def}}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{*}$$

If this limit exists, we say f is **differentiable** at a.

If we set $x = a + h$, we get an equivalent definition of the derivative of f at $x = a$:

Definition: Let f be a function. The **derivative of** f **at** $x = a$ is

$$f'(a) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{**}$$

An alternate notation for the derivative, called **Leibniz notation**, is $\dfrac{dy}{dx}$. Taking

$$\begin{aligned} \Delta y &= f(x) - f(a), \\ \Delta x &= x - a = h, \end{aligned}$$

we get a third way to write the definition of the derivative.

$$\frac{dy}{dx} \overset{\text{notation}}{=} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{***}$$

Visually, the secant lines through P and nearby points $Q = (x, f(x))$ "limit" as $Q \to P$ to a line that "kisses" the graph of f at P, meeting the graph at P and at no other nearby points. We call this line the *tangent line at* P to the graph of f:

Definition: If $f'(a)$ exists, the **tangent line** to the curve $y = f(x)$ at a point $P = (a, f(a))$ is defined to be the line with slope $f'(a)$ through the point P.

☞ A very old and very imprecise definition of a tangent line is:

a straight line which touches a curve, but does not cut it.

Here, "tangent" is opposed to a line which "cuts" the curve, as opposed to 'kissing" it. (Compare the definition of a continuous function as "a function whose graph can be drawn without lifting one's pencil"...)

Ex. 2. Find the derivative of $f(x) = x^2$ at $x = 5$, using the definition of the derivative.

Solution:

We substitute $a = 5$ into definition (**) of the derivative of f at a, which we've copied here:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad (**)$$

We get:

$$\begin{aligned} f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} &= \lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h} \\ &= \lim_{h\to 0} \frac{25 + 10h + h^2 - 25}{h} \\ &= \lim_{h\to 0} \frac{h(10+h)}{h} \\ &= \lim_{h\to 0} (10+h) \\ &= 10. \end{aligned}$$

Ex. 3. Find an equation for the tangent line to the graph of $f(x) = x^2$ at $x = 5$.

Solution:

The slope of the tangent line to $f(x)$ at $x = 5$ is $f'(a)$, which we found in the previous exercise.

We now use the point-slope form of a linear equation,

$$y - y_0 = m(x - x_0),$$

substituting the point $\big(5, f(5)\big) = (5, 25)$ for (x_0, y_0).

Both the equation

$$\boxed{y - 25 = 10(x - 5)}$$

and its simplified form

$$\boxed{y = 10x - 25}$$

are acceptable answers.

Ex. 4.

(a) Find the derivative of $f(x) = 3x - 1$ at $x = a$, using the definition of the derivative.

(b) Write an equation of a tangent line to the graph of $f(x)$ at any point without doing any additional scratchwork.

☞ Notice that, for a linear function $f(x) = mx+b$, the slope of the tangent line and the slope of all secant lines coincide. The derivative of a linear function is constant, $f'(x) = m$ for all x.

Ex. 5. Find the tangent line to the curve $y = 3/x$ at the point $(3, 1)$.

Solution:

Check: The graph of f—that is, the set containing all points of the form $\big(x, f(x)\big)$—contains the point $\big(a, f(a)\big) = (3, 1)$. ✓

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(3+h)} = -\lim_{h\to 0} \frac{1}{3+h} = -\frac{1}{3}.$$

We need the equation of a line with slope $f'(3) = -\dfrac{1}{3}$ through the point $(3, 1)$.

$$y - y_1 = f'(a) \cdot (x - a)$$
$$y - 1 = f'(3) \cdot (x - 3)$$
$$\boxed{y = -\tfrac{1}{3}(x - 3) + 1}$$
$$3y = -(x - 3) + 3$$
$$x + 3y - 6 = 0$$

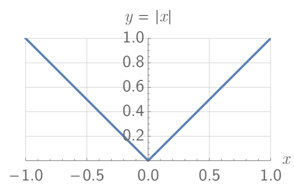
Determining differentiability from a graph, and the formula for the tangent line

A function f is differentiable at $x = a$ (that is, $f'(a)$ exists) if its graph looks like a non-vertical straight line when we zoom in sufficiently around the point $(a, f(a))$.

That is, the tangent line at $x = a$ is a close approximation of the graph of f at points $\big(x, f(x)\big)$ for x near a.

(*See applet on iCollege:* "`The differentiation microscope`")

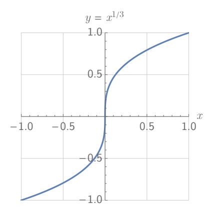
Counterexample. The function $f(x) = |x|$ is NOT differentiable at $x = 0$.



Intuitively: no matter how much we zoom in on the graph of f near $(0, 0)$, the graph will always have a "corner" (or "cusp") in it.

Therefore, no line is tangent to the graph of f at the point $(0, 0)$.

Counterexample. The function $f(x) = \sqrt[3]{x}$ is NOT differentiable at $x = 0$.



The graph of $y = x^{1/3}$ can be obtained by reflecting the graph of $y = x^3$ in the line $y = x$ because $y = \sqrt[3]{x}$ and $y = x^3$ are inverse functions.

As x approaches 0, the tangent lines to the curve at x become steeper and steeper.

When x *equals* 0, the tangent line is *vertical*.

That is, the slope of the tangent line, $f'(x)$, approaches infinity as $x \to 0$: in symbols,

$$\lim_{x \to 0} f'(x) = \infty.$$

Formula: The tangent line to the graph of $y = f(x)$ at the point $\big(a, f(a)\big)$ is

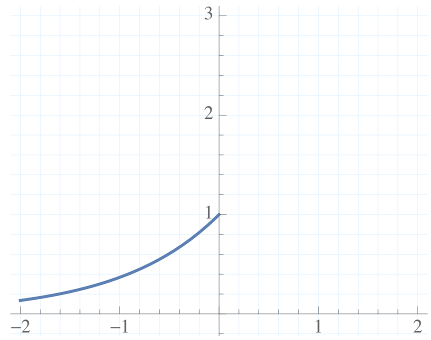
$$y = \underbrace{f'(a)}_{\text{slope}} \cdot (x - a) + f(a)$$

provided that

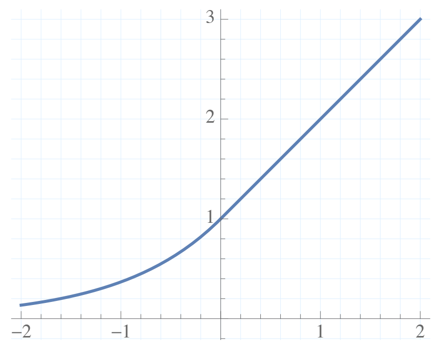
- a is in the domain of f *(so that $f(a)$ is defined),*
- f is differentiable at a *(so that $f'(a)$ is defined),*
- $\lim_{x \to a} f'(x) \neq \pm\infty$ *(that is, the tangent line cannot be* VERTICAL*).*

Finding a linear extension of a function that is differentiable and continuous

Suppose we are given a function that is defined only for $x \le 0 \ldots$ say, $f(x) = e^x$ for $x \le 0$.



We wish to extend the function so that its graph is a line for $x > 0$ in such a way that the resulting piecewise function is continuous *(e.g., no gaps)* and differentiable *(e.g., no sharp corners)*:



How should we pick the y-intercept and the slope of the line? That is, what should a and b be in order to make the piecewise function

$$f(x) = \begin{cases} e^x & \text{for } x \le 0, \\ ax + b & \text{for } x > 0 \end{cases}$$

continuous and differentiable?

- We can imagine the following real-world situation in which the above problem might need to be solved: Let $x = 0$ represent the present moment, so that $x < 0$ represents the past. Suppose a virus has been spreading faster and faster (exponentially) until now. We would like to predict the number of infected, $f(x)$, under the assumption that the rate of spread in the future (time $x > 0$) stays the same as it is at the present moment.
- A mathematically similar problem is encountered in graphic and industrial design, economics, and robotics: a curve is to be joined with a straight line at a point (called the **break point**) in such a way that the result is perfectly smooth, with no corners or gaps. In this situation, we might replace the exponential function e^x by some other type of function—for example, a polynomial. (Polynomials are often used in the design of fabricated objects and the modeling of economic phenomena.)

☞ *Use the applet "*`When is a piecewise function differentiable?`*" on iCollege to experiment with different values of a and b and different functions defined for $x \le 0$.*

The derivative as a rate of change

The definitions of *average velocity* and *instantaneous velocity* were given in Section 1.1.

Here we restate those definitions and introduce some new shorthand. (New shorthand in blue.)

Definition. Let $s(t)$ be a function that gives the position (or displacement) of a particle in motion at time t.

- The **average velocity** of a particle in motion from time $t = a$ to time $t = b$ is the average rate of change in its position function from $t = a$ to $t = b$.

  Let us denote the average velocity by v_{ave} for short, as follows:

  $$v_{\text{ave}} = \frac{s(b) - s(a)}{b - a}$$

- The **instantaneous velocity** of a particle in motion at time $t = a$ is the instantaneous rate of change in its position function at $t = a$.

  Thus the instantaneous velocity at $t = a$ is (by the definition of instantaneous rate of change in Lesson 1, and by the definition (*) of the derivative on p. 2 of this document):

  $$s'(a) = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}.$$

  We write:

  $$v(a) = s'(a)$$

  —that is, $v(a) = s'(a)$ is the *(instantaneous) velocity* at time a.

  Taking $b = a + h$, so that $h = b - a \to 0$ as $b \to a$, we get a more useful formula for $v(a)$ that should be used when working problems that ask for the instantaneous velocity:

  $$v(a) = s'(a) = \lim_{b \to a} \frac{s(b) - s(a)}{b - a} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}. \qquad (\star)$$

Ex. 6. A stone is tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters after t seconds is $s(t) = 15t - 4.9t^2$. Find the instantaneous velocity of the stone at $t = 1$ sec.

Ex. 7. A coffee shop determines that the daily profit on scones, $P(s)$, obtained by charging s dollars per scone is modeled by the following equation:

$$P(s) = -20s^2 + 140s - 240$$

The coffee shop currently charges \$3 per scone.

(a) Find $P'(3)$, the rate of change in profit when the price is \$3.

(b) Should the coffee shop consider raising or lowering its prices on scones?

Solution to part (a).

$$\begin{aligned}
P'(3.25) &= \lim_{s\to 3} \frac{P(s) - P(3)}{s-3} \\
&= \lim_{s\to 3} \frac{\left[-20s^2 + 140s - 240\right] - \left[-20(3)^2 + 140(3) - 240\right]}{s-3} \\
&= \lim_{s\to 3} \frac{-20s^2 + 140s - 240}{s-3} \\
&= \lim_{s\to 3} \frac{-20(s^2 - 7s + 12)}{s-3} \\
&= \lim_{s\to 3} \frac{-20(s-4)(s-3)}{s-3} \\
&= \lim_{s\to 3} -20s + 80 \\
&= 20
\end{aligned}$$

Additional exercises

Ex. 8 (§3.1—#1, 5, 7, 8). Find the slope of the secant line between the values x_1 and x_2 for each function.

- $f(x) = 4x + 7$, $x_1 = 2$, $x_2 = 5$
- $g(x) = \sqrt{x}$, $x_1 = 1$, $x_2 = 16$
- $h(x) = \sqrt{x-9}$, $x_1 = 10$, $x_2 = 13$
- $j(x) = \dfrac{4}{3x-1}$, $x_1 = 1$, $x_2 = 3$

Ex. 9 (§3.1—#11).

(a) Find the slope of the tangent line to the parabola $y = x^2 + x$ at the point $(1, 2)$.

(b) Find an equation for the tangent line to the parabola at the point $(1, 2)$.

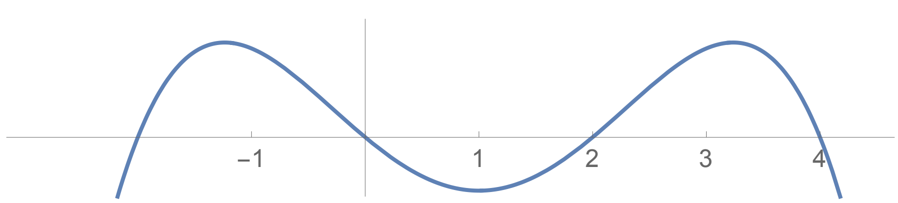
Ex. 10 (§3.1—#15).

(a) Find the slope $f'(a)$ of the tangent line to the curve $y = \dfrac{7}{x}$ at the point $\big(a, f(a)\big)$ for $a = 3$.

(b) Find an equation for the tangent line to the curve at $x = 3$.

Ex. 11. The graph of a function f is given. Arrange the following numbers from least to greatest.

$$0 \qquad f'(-2) \qquad f'(0) \qquad f'(2) \qquad f'(4)$$



Ex. 12 (§3.1—#21, 23, 25, 27). Find $f'(a)$ using the definition of the derivative.

(a) $f(x) = 5x + 4$, $a = -1$

(b) $f(x) = x^2 + 9x$, $a = 3$

(c) $f(x) = \sqrt{x}$, $a = 4$

(d) $f(x) = \dfrac{1}{x}$, $a = 2$

Workbook Lesson 7

§3.2, The Derivative as a Function

Last revised: 2021-05-03 08:53

Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

The derivative function

Let $f(x)$ be a function. A second function, called the **derivative of** f, is defined by

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This equation is just Equation (**) from Lesson 6 with the name of one variable changed.

☞ The domain of the function f'—that is, the set of input values x for which $f'(x)$ is defined—is the set of x-values for which the above limit exists and is a real number.

Notations for the derivative:

- Here we list all the most common notations used to denote the derivative of $y = f(x)$:

$$f'(x) \qquad \frac{dy}{dx} \qquad \frac{d}{dx}\big[f(x)\big] \qquad Df(x) \qquad y'$$

- In general, the notation

$$\Big|_{x=a}$$

  after an expression means "evaluate the expression by substituting $x = a$." So, for instance, the notation

$$\left.\frac{dy}{dx}\right|_{x=a}$$

  means the same thing as

$$f'(a).$$

Fail conditions for differentiability:

Recall (from the previous lesson) that a function f FAILS to be differentiable at $x = a$ (that is, $f'(a)$ does NOT exist) if...

- the graph of f contains a corner or
- the tangent line at $x = a$ is vertical.

The variable of differentiation ("differentiating with respect to")

Recall:

- When we discuss a function $y = f(x)$, the input variable x is called the **independent variable**.
- We call y the **dependent variable** because the value of $y = f(x)$ depends on x.
- For another example with different variable names, if r is a function of t, and we write $r = g(t)$, then t is the independent variable and r is the dependent variable.

We say that $\frac{dy}{dx} = f'(x)$ is the derivative of $y = f(x)$ **with respect to** x, the independent variable.

- $\frac{dy}{dx} = f'(x)$ is the instantaneous rate of change in $y = f(x)$ as x varies.
- $\frac{dr}{dt} = g'(t)$ is the instantaneous rate of change in $r = g(t)$ as t varies.
- The **variable of differentiation** is the independent variable, with respect to which the derivative is taken.

☞ When unfamiliar variable names are used, you may find it helpful to identify which variable is the dependent variable (what you take the derivative *of*) and the independent variable (what you take the derivative *with respect to*).

Graphing a derivative

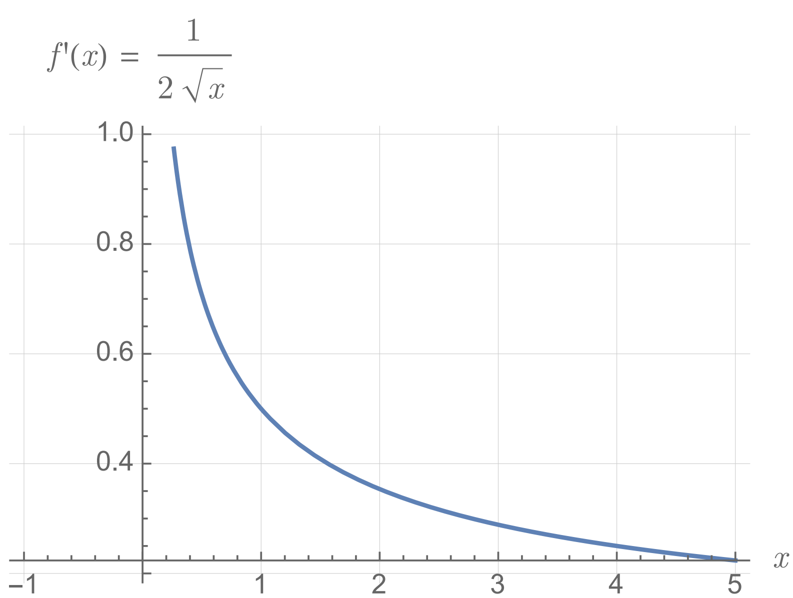
Since f' is a function, we can graph it.

Ex. 1. Find the derivative of $f(x) = x^2$ and sketch its graph.

Ex. 2. Find the derivative of $f(x) = \sqrt{x}$ and sketch its graph.

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$



Differentiability implies continuity

Theorem. If f is differentiable at a, then f is continuous at a.

Proof.

We need to show that $\lim\limits_{x\to a} f(x) = f(a)$. We'll do this by showing that $\lim\limits_{x\to a} f(x) - f(a) = 0$.

We can make $f(x) - f(a)$ look like the slope of the secant line if we divide it by $x - a$. We'll multiply by $(x - a)$ at the same time so that the value is not changed:

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Now we take the limit on both sides, and use the Limit Laws:

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}(x - a)\right]$$

$$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = \lim_{x\to a}[f(x) - f(a)] = \left[\lim_{x\to a}\frac{f(x) - f(a)}{x - a}\right] \cdot \left[\lim_{x\to a} x - a\right] \overset{(\dagger)}{=} f'(a) \cdot 0 = 0, \tag{*}$$

where the equality (†) is justified by the fact that, by definition of differentiability of f at a, the limit $f'(a) = \lim\limits_{x\to a} \dfrac{f(a) - f(x)}{x - a}$ exists.

Adding $\lim\limits_{x\to a} f(a)$ to both sides of (*) and then applying the Constant Law for Limits yields

$$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) = f(a).$$

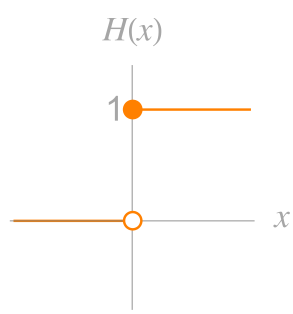
□

Ex. 3.

The graph of the Heaviside function $H(x)$ is shown below.

Is the Heaviside function differentiable at $x = 0$? (That is, does $H'(0)$ exist?)

Justify your answer.



Answer: No.

Justification: If $H'(0)$ did exist, then by the above Theorem, H would be continuous at $x = 0$. Since H is *not* continuous at $x = 0$, $H'(0)$ must not exist.

Higher derivatives

Recall:

If a function $y = f(x)$ is differentiable at every point of its domain, then $f'(x)$ is a new function with the same domain, called the (FIRST) DERIVATIVE of f.

This new function f' may (or may not) have a derivative of its own. If it does, the derivative of f' is denoted by

$$f'' \qquad \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} \qquad \frac{d^2}{dx^2}\big[f(x)\big] \qquad D^2 f \qquad \text{or} \qquad y''.$$

Definition: The derivative of the (first) derivative f' is called the **second derivative** of f.

Ex. 4. Find $f''(x)$ if $f(x) = x^2$.

Solution:

We know $f'(x) = 2x$ from Exercise 1.

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h} \\ &= 2. \end{aligned}$$

The **third derivative** f''' is the derivative of the second derivative f''.

Derivatives of higher order (for $n \geq 4$) are written $f^{(n)}$:

$$f', \quad f'', \quad f''', \quad f^{(4)}, \quad f^{(5)}, \quad \ldots$$

Ex. 5. Find all higher-order derivatives f'', f''', $f^{(4)}$, $f^{(5)}$, ... of $f(x) = x^2$.

Solution:

From Exercise 4, we know $f''(x) = 2$ for any value of the input x. Therefore,

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{2-2}{h} = 0.$$

Now

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h} = 0$$

and we easily see that

$$f^{(n)}(x) = 0 \qquad \text{for } n = 5, 6, 7, \ldots$$

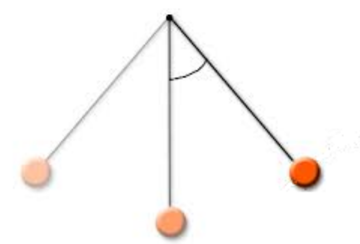
Velocity and acceleration

The concepts of derivative and second derivative have natural interpretations in terms of motion.

Our first example will be the motion of a swinging pendulum. In later sections, we will see that the same interpretation applies to a body undergoing one-dimensional "rectilinear" motion (that is, motion in a straight line), such as a ball thrown straight up in the air.

Let x be time. Let $f(x)$ be the displacement from an object's initial position at time $x = 0$.

For the pendulum, define $f(x)$ to be the positive or negative angle between the rest position of the pendulum and the angle of the pendulum at time x.



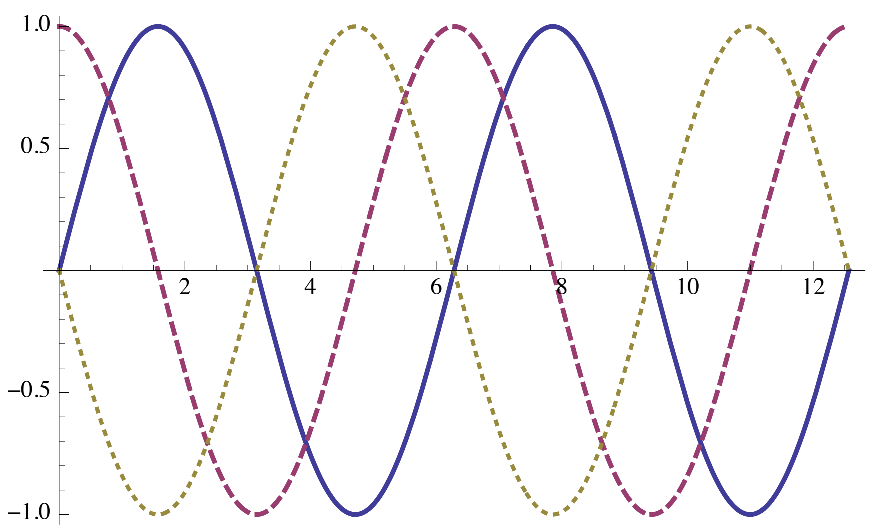
(*See applet on iCollege:* "`Pendulum`")

If $f(x) =$ displacement at time x, then:

$$\begin{aligned} & & & & v(x) &= f'(x) \quad \text{is velocity,} \\ & & a(x) &= v'(x) &&= f''(x) \quad \text{is acceleration, and} \\ j(x) &= a'(x) &&= v''(x) &&= f'''(x) \quad \text{is jerk (or lurch).} \end{aligned}$$

Let's look at a displacement function that is simpler than that of the pendulum in the applet.

Ex. 6. Let x be time and suppose $f(x) = \sin(x)$ is the displacement function for a pendulum. Identify the displacement, velocity and acceleration in the following graph.



Solution.

We recognize the solid blue graph as the graph of the function $\sin$ function.

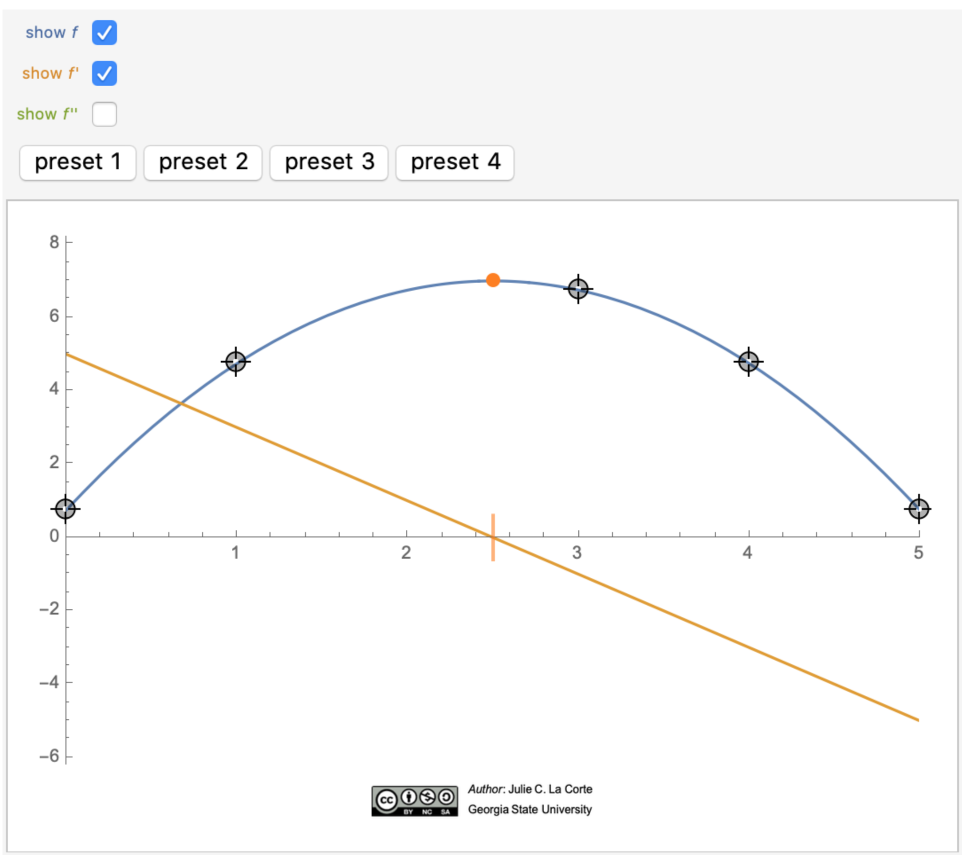
At time $t = \pi/2$, we have $f(x) =$ displacement $= 1$ radian $\approx 57°$. At that instant, the pendulum has stopped moving to the right, and is about to move to the left: its speed is 0. Thus the dashed purple line is velocity $f'(x)$.

By process of elimination, the dotted gold line must be acceleration. We verify that this makes sense: at time $t = \pi/2$, the pendulum is momentarily motionless, $f'(\pi/2) = 0$, and accelerating in the *leftward* direction from its rightmost position, $f(\pi/2) = 1$. At that instant, the acceleration is $f''(\pi/2) = -1$. The maximum acceleration in the *rightward* direction occurs when the pendulum is motionless and in its leftmost position.

How should we interpret the acceleration at the instant $t = \pi$, when the pendulum is in its rest position $f(\pi) = 0$?

Hands-on demo of the relationship between the graphs of f, f', and f''

The applet "`Derivative sandbox`," provided on iCollege, illustrates the relationship between the graph of a function f and the graph of its derivative f'. (The graph of f'' can also be displayed.) Interact with the applet by dragging the points indicated by crosshairs ⌖ to change the shape of the graph of f. Can you explain what the orange points on the graph of f represent, and how the orange points relate to the graph of f'?



Additional exercises

Ex. 7 (§3.2—#54, 55, 58, 63). Find the derivative of each function using the definition of the derivative. State the domain of the function and the domain of its derivative.

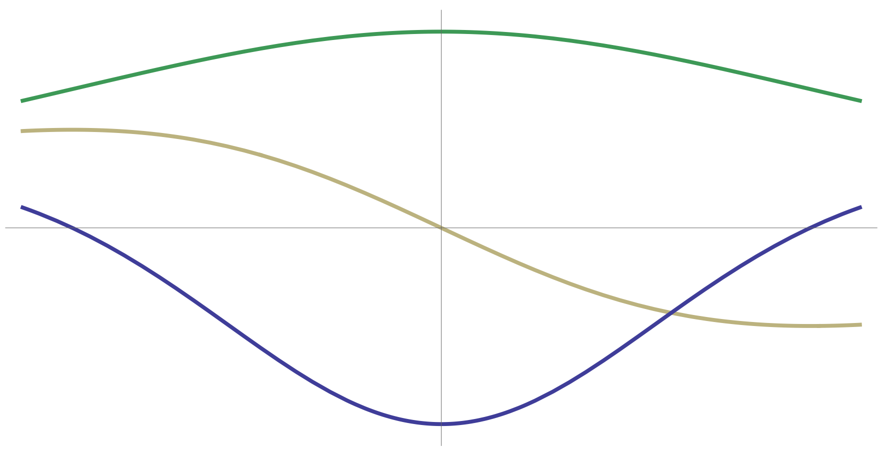
- $f(x) = 2 - 3x$
- $g(x) = 6$
- $h(x) = 5x - x^2$
- $j(x) = \dfrac{1}{\sqrt{x}}$

Ex. 8 (§3.2—#68, 73). The given limit is the derivative of a function $y = f(x)$ at $x = a$. What are $f(x)$ and a?

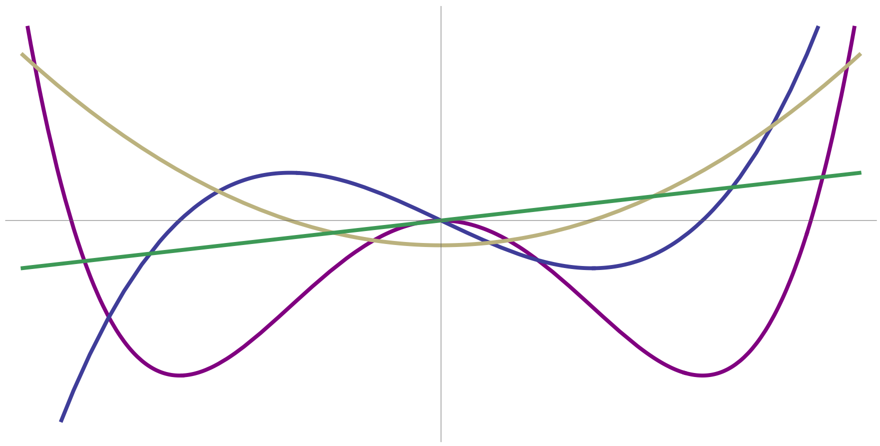
(a) $\lim\limits_{h \to 0} \dfrac{(1+h)^{2/3} - 1}{h}$

(b) $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h}$

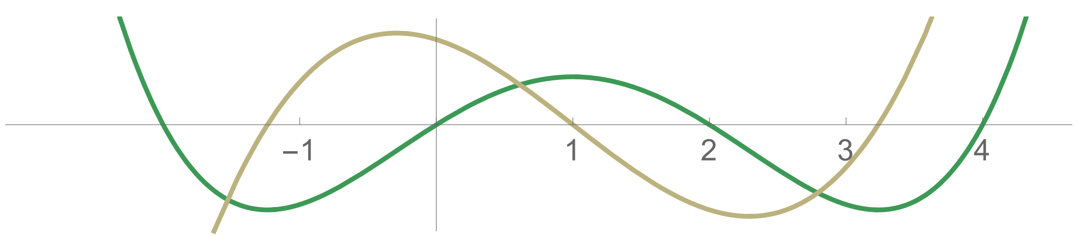
Ex. 10. The figure shows the graphs of f, f', and f''. Identify each curve.



Ex. 11. The figure shows the graphs of f, f', f'', and f'''. Identify each curve.



Ex. 12. The graphs of a function f and its derivative f' are shown. Which is larger, $f'(-1)$ or $f'(-1)$?



Workbook Lesson 8

§3.3, Differentiation Rules

Last revised: 2020-12-14 15:24

Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Basic differentiation formulas

The definition of the derivative can be difficult and tedious to use.

It's usually much faster and easier to use the Differentiation Rules (last page of this document).

- Once we've convinced ourselves that the derivative $\frac{d}{dx}[x^2]$ of the function x^2 is $2x$ (as we did in Lesson 7), we can treat this fact as a rule.
- *In general*, the derivative $\frac{d}{dx}[x^n]$ of any power function x^n is nx^{n-1}.
  ("The exponent moves out front and drops by 1.")
- This "General Power Rule" is listed among several other basic differentiation rules in the Differentiation Rules.
- All the Differentiation Rules can be proven using the definition of the derivative.
- Some of the rules on the handout won't be introduced until later in the course.

Let's concentrate on Rules 1, 2, 4, 5, 6, and 7 first.

Ex. 1. Find the derivative $\frac{d}{dx}[5]$ of the constant function $f(x) = 5$ using the definition of the derivative.

Since the average rate of change of a constant function is 0, we expect the instantaneous rate of change to be 0 also.

$$\begin{aligned}\frac{d}{dx}[5] &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{5-5}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= 0\end{aligned}$$

where the last equality is justified by the fact that the limit as $h \to 0$ depends only on NONZERO values of h.

Ex. 1 is an example of the Constant Rule (Rule 1 on the Handout).

Intuitively: the instantaneous rate of change in a constant function is 0.

Ex. 2. Calculate: $\dfrac{d}{dt}\left[\dfrac{e^{\pi}-e^{-\pi}}{2}\right]$.

Ex. 3. Find $\dfrac{d}{ds}[s]$.

Write $f(s) = s$.

$$\frac{d}{ds}[s] = \lim_{h\to 0}\frac{f(s+h)-f(s)}{h} = \lim_{h\to 0}\frac{s+h-s}{h} = 1.$$

Proofs of the Power, Constant Multiple, Sum, and Difference Rules (Rules 4–7 on the Handout) can be found in the textbook.

Let's look at some more problems similar to those you might see on an Exam.

Ex. 4. Find the derivative: $t = 5s + \dfrac{1}{s^3} + 2 + 3s^5$. Justify each step by stating which Rules were used.

The dependent variable is t, and the independent variable is s,
so we are differentiating with respect to t.

$$\begin{aligned}
\frac{d}{ds}\big[t(s)\big] &= \frac{d}{ds}\left[5s + s^{-3} + 2 + 3s^5\right] \\
&= \frac{d}{ds}[5s] + \frac{d}{ds}\left[s^{-3}\right] + \frac{d}{ds}[2] + \frac{d}{ds}\left[3s^5\right] && \text{(Rule 6: Sum Rule)} \\
&= 5\frac{d}{ds}[s] + \frac{d}{ds}\left[s^{-3}\right] + 0 + 3\frac{d}{ds}\left[s^5\right] && \text{(Rule 5 and Rule 1)} \\
&= 5 - 3s^{-4} + 15s^4 \\
&= 5 - \frac{3}{s^4} + 15s^4.
\end{aligned}$$

Ex. 5. Differentiate the function: $y = \dfrac{x^3 + 2x + 1}{\sqrt{x}}$.

The dependent variable is y. The independent variable is x.
So we are differentiating with respect to x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{x^3 + 2x + 1}{x^{1/2}}\right] \\ &= \frac{d}{dx}\left[x^{5/2} + 2x^{1/2} + x^{-1/2}\right] \\ &= \frac{5}{2}x^{3/2} + x^{-1/2} - \frac{1}{2}x^{-3/2} \\ &= \frac{3}{2}x\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}\end{aligned}$$

Ex. 6. Find the derivative: $f(t) = \dfrac{2}{\sqrt[5]{t}} - \sqrt[3]{27t} + \dfrac{t^2}{2}$.

Independent variable: t

$$\begin{aligned}\frac{d}{dt}f(t) &= \frac{d}{dt}\left[-2t^{-1/5} - 3t^{1/3} + \frac{1}{2}t^2\right] \\ &= -2\big(-\tfrac{1}{5}\big)t^{-6/5} - 3\big(\tfrac{1}{3}\big)t^{-2/3} + \frac{1}{2}(2)t \\ &= \frac{2}{5}t^{-6/5} - \frac{3}{3}t^{-2/3} + \frac{2}{2}t \\ &= \frac{2}{5t^{6/5}} - \frac{1}{t^{2/3}} + t \\ &= \frac{2}{5\sqrt[5]{t^6}} - \frac{1}{\sqrt[3]{t^2}} + t.\end{aligned}$$

Product and Quotient Rules

We have seen (Sum Rule) that

$$\frac{d}{dx}\big[f(x)+g(x)\big]=\frac{d}{dx}\big[f(x)\big]+\frac{d}{dx}\big[g(x)\big].$$

That is,

The derivative of a sum of two functions, is the sum of the derivative of the two functions.

A similar rule holds for subtraction (Difference Rule), but *not* for products:

$$\frac{d}{dx}\big[f(x)\cdot g(x)\big]=f(x)\cdot\frac{d}{dx}\big[g(x)\big]+\frac{d}{dx}\big[f(x)\big]\cdot g(x).$$

That is, the right-hand side of the Product Rule is

(first) · (derivative of the second) + (derivative of the first) · (second).

☞ Since addition and multiplication are commutative, we can write this rule in a few different ways, e.g.:

$$\begin{aligned}\frac{d}{dx}\big[f(x)\cdot g(x)\big]&=f(x)\cdot g'(x)+f'(x)g(x)\\&=f'(x)\cdot g(x)+f(x)g'(x)\\&=g(x)\cdot f'(x)+g'(x)f(x).\end{aligned}$$

Ex. 7. Find $\frac{dy}{dx}$ and $\frac{dy}{dt}$.

(a) $y=tx^2+t^3x$

(b) $y=\frac{t}{x^2}+\frac{x}{t}$

Hint: To avoid having to use the Quotient Rule in part **(b)**, rewrite the given formula without fractions by using negative exponents:

$$y=\frac{t}{x^2}+\frac{x}{t}=tx^{-2}+xt^{-1}$$

Ex. 8. Differentiate $(5x^{12}+2)\,(\pi-\pi^2x^4)$.

For the Product Rule:

$f(x)=5x^{12}+2$	$g(x)=\pi-\pi^2x^4$
$f'(x)=60x^{11}$	$g'(x)=-4\pi^2x^3$

$$\begin{aligned}\frac{d}{dx}\left[\left(5x^{12}+2\right)\left(\pi-\pi^2x^4\right)\right]&=\frac{d}{dx}\left[\left(5x^{12}+2\right)\left(\pi-\pi^2x^4\right)\right]\\&=\left(5x^{12}+2\right)'\left(\pi-\pi^2x^4\right)+\left(5x^{12}+2\right)\left(\pi-\pi^2x^4\right)'\\&=\boxed{60x^{11}(\pi-\pi^2x^4)-4\pi^2x^3(5x^{12}+2)}\\&=-80\pi^2x^{15}+60\pi x^{11}-8\pi^2x^3.\end{aligned}$$

Ex. 9. Prove that $\frac{1}{1-x^2}$ and $\frac{x^2}{1-x^2}$ have the same derivative.

$$\frac{d}{dx}\left[\frac{1}{1-x^2}\right] \overset{\text{(Recip. Rule)}}{=} \frac{-(1-x^2)'}{(1-x^2)^2} = \frac{-(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}.$$

For the Quotient Rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, take

$f(x) = x^2$	$g(x) = 1 - x^2$
$f'(x) = 2x$	$g'(x) = -2x$

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2}{1-x^2}\right] &\overset{\text{(Quot. Rule)}}{=} \frac{(1-x^2)(x^2)' - (x^2)(1-x^2)'}{(1-x^2)^2} \\ &= \frac{(1-x^2)(2x) - (x^2)(-2x)}{(1-x^2)^2} \\ &= \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} \\ &= \frac{2x}{(1-x^2)^2}.\end{aligned}$$

☞ *Silly but effective trick for remembering the Quotient Rule:*

"Lo D hi over hi D lo, over the denominator squared must go."

☞ *Alternate strategy:* Just avoid the Quotient Rule.

- In Section 3.6, we'll learn the Chain Rule.
- After you know the Chain Rule, come back to this bulleted list.
- Any quotient $\frac{f(x)}{g(x)}$ can be written as a product: $f(x) \cdot \left[g(x)\right]^{-1}$.
- Then the Product Rule and Chain Rule can be used:

$$\frac{d}{dx}\left[f(x) \cdot \left[g(x)\right]^{-1}\right] = f'(x) \cdot \left[g(x)\right]^{-1} - f(x)g(x) \cdot \left[g(x)\right]^{-2}$$

Additional exercises

Ex. 10 (§3.3—#107, 113, 115).

- $f(x) = 5x^3 - x + 1$
- $g(x) = x^2 \left(\frac{2}{x^2} + \frac{5}{x^3} \right)$
- $h(x) = \frac{4x^3 - 2x + 1}{x^2}$

Ex. 11.

- $I(x) = x^{5/2} - x^{-2}$
- $k(x) = \frac{1}{x} - \sqrt[5]{x}$
- $m(x) = \frac{5x^{3/2} + x^{5/2}}{x}$
- $j(x) = \frac{1}{x} + \frac{1}{x^2}$
- $\ell(x) = x^{2.3} + \pi^{2.3}$
- $N(x) = \frac{2w^2 - w + 4}{\sqrt{w}}$

Ex. 12. Find $\frac{dy}{dx}$ and $\frac{dy}{dt}$.

(a) $y = t^3x + tx^2$

(b) $y = \frac{t}{x} - \frac{x}{t^2}$

Ex. 13 (§3.3—#119). Find the equation of the tangent line to the curve $y = 2x^3 + 4x^2 - 5x - 3$ at the point $(-1, 4)$.

Ex. 14 (see also §3.3—#121).

The **normal line** to the graph of f at the point P is the line through the point P perpendicular to the tangent line at point P. Find the equation of the normal line to the curve

$$y = \frac{2}{x} - \frac{3}{x^2}$$

at the point $(1, -1)$.

Ex. 17 (§3.3, Example 3.31). Find the points on the curve $y = x^3 - 7x^2 + 8x + 1$ where the tangent is horizontal. (*Hint*: A horizontal line has slope $= 0$.)

Ex. 18 (§3.3—#146). The concentration of antibiotic in the bloodstream t hours after being injected is given by the function

$$C(t) = \frac{2t^2 + t}{t^3 + 50},$$

where $C(t)$ is measured in milligrams per liter of blood

(a) Find the rate of change of $C(t)$.

(b) Determine the rate of change for $t = 8, 12, 24,$ and 36.

(c) Briefly describe what seems to be occurring as the number of hours increases.

Ex. 19 (§3.3—#111, 116, 117). Differentiate.

- $f(x) = 3x\left(18x^4 + \dfrac{13}{x+1}\right)$
- $f(x) = \dfrac{x^2+4}{x^2-4}$
- $f(x) = \dfrac{x+9}{x^2-7x+1}$

Ex. 20 (§3.3—#141). Find an equation of the tangent line to the curve $y = \dfrac{6}{x-1}$ at the point $(3, 3)$.

Ex. 21. Find an equation of the normal line to the graph of $f(x) = x + x^2$ at the point $(0, 0)$. *(See above for the definition of the normal line.)*

Ex. 22. The psychologist L. L. Thurstone suggested the following relationship between learning time $T = f(n)$ and the length n of a list:

$$T = f(n) = An\sqrt{n - b},$$

where A and b are constants that depend on the person and the task.

(a) Compute $\dfrac{dT}{dn}$ and interpret your results.

(b) For a certain person, suppose $A = 4$ and $b = 4$. Compute $f'(13)$ and $f'(29)$ and interpret your results.

Differentiation Rules

In the equations below, c is a (real) constant, and $f(x)$ and $g(x)$ are functions.
Recall: $\frac{d}{dx}[\square]$ means "the derivative of $\square$ with respect to x.'

Basic formulas

1. **Derivative of a constant** $\frac{d}{dx}[c] = 0$
2. **Derivative of identity function** $\frac{d}{dx}[x] = 1$
3. **Chain Rule** $\frac{d}{dx}\big[g\big(f(x)\big)\big] = g'\big(f(x)\big) \cdot f'(x)$

Arithmetic formulas

4. **Power Rule** $\frac{d}{dx}[x^c] = cx^{c-1}$
5. **Constant Multiple Rule** $\frac{d}{dx}\big[c \cdot f(x)\big] = c \cdot f'(x)$
6. **Sum Rule** $\frac{d}{dx}\big[f(x) + g(x)\big] = f'(x) + g'(x)$
7. **Difference Rule** $\frac{d}{dx}\big[f(x) - g(x)\big] = f'(x) - g'(x)$
8. **Product Rule** $\frac{d}{dx}\big[f(x) \cdot g(x)\big] = f(x)\,g'(x) + g(x)\,f'(x)$
9. **Reciprocal Rule** $\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \frac{-g'(x)}{\big[g(x)\big]^2}$
10. **Quotient Rule** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\big[g(x)\big]^2}$

Transcendental functions

$$\frac{d}{dx}\big[\sin(x)\big] = \cos(x)$$

$$\frac{d}{dx}\big[\cos(x)\big] = -\sin x$$

$$\frac{d}{dx}\big[\csc(x)\big] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\big[\sec(x)\big] = \sec(x)\tan(x)$$

$$\frac{d}{dx}\big[\tan(x)\big] = \sec^2(x)$$

$$\frac{d}{dx}\big[\cot(x)\big] = -\csc^2(x)$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}\big[\ln(x)\big] = \frac{1}{x}$$

$$\frac{d}{dx}[b^x] = b^x \ln(b)$$

$$\frac{d}{dx}\big[\log_b(x)\big] = \frac{1}{x\ln(b)}$$

Workbook Lesson 9

§3.5, Derivatives of Trigonometric Functions

Objectives

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

Let's add two more Rules into the mix: the derivatives of $\sin$ and $\cos$.

Ex. 1. Find the derivative: $r = 4\cos(\theta) - 3\sin(\theta)$.

Solution:

Independent variable: θ
Dependent variable: r

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{d}{d\theta}\left[4\cos(\theta) - 3\sin(\theta)\right]\\ &= 4\frac{d}{d\theta}\left[\cos(\theta)\right] - 3\frac{d}{d\theta}\left[\sin(\theta)\right]\\ &= 4[-\sin(\theta)] - 3[\cos(\theta)]\\ &= -4\sin(\theta) - 3\cos(\theta).\end{aligned}$$

Ex. 2. Show that **(a)** $\frac{d}{dx}\big[\csc(x)\big] = -\csc(x)\cot(x)$ and **(b)** $\frac{d}{dx}\big[\tan(x)\big] = \sec^2(x)$.

Solution:

(a): We know $\csc(x) = 1/\sin(x)$. For the Quotient Rule:

$f(x) = 1$	$g(x) = \sin(x)$
$f'(x) = 0$	$g'(x) = \cos(x)$

$$\begin{aligned}\frac{d}{dx}\left[\csc(x)\right] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] &= \frac{\big[\sin(x)\big]\cdot[0] - [1]\cdot\big[\cos(x)\big]}{\sin^2(x)}\\ &= \frac{-\cos(x)}{\sin^2(x)}\\ &= -\frac{\cos(x)}{\sin(x)}\frac{1}{\sin(x)}\\ &= -\csc(x)\cot(x).\end{aligned}$$

(We could have used the Reciprocal Rule instead.)

(b): We know $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. For the Quotient Rule:

$f(x) = \sin(x)$	$g(x) = \cos(x)$
$f'(x) = \cos(x)$	$g'(x) = -\sin(x)$

$$\begin{aligned}
\frac{d}{dx}\big[\tan(x)\big] &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos x}\right] \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot\big[-\sin(x)\big]}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \qquad (*) \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}$$

Alternately, we could proceed (starting at $(*)$) as follows...

$$\begin{aligned}
&\frac{d}{dx}\big[\tan(x)\big] \\
&\qquad= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot\big[-\sin(x)\big]}{\cos^2(x)} \\
&\qquad= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \\
&\qquad= 1 + \tan^2(x).
\end{aligned}$$

All is well, because from the well-known identity

$$\sin^2(x) + \cos^2(x) = 1$$

we can deduce the (also well-known) identity

$$\tan^2(x) + 1 = \sec^2(x) :$$

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \qquad \text{(Divide each term by } \cos^2(x)\text{)} \qquad (\dagger) \\
1 + \tan^2(x) &= \sec^2(x)
\end{aligned}$$

By the way, the identity

$$1 + \cot^2(x) = \csc^2(x) \qquad \text{(Divide each term in } (\dagger) \text{ by } \sin^2(x)\text{)}$$

can be similarly derived.

Ex. 3. Show that

$$\frac{d^2}{dx^2}\big[\tan(x)\big] = 2\sec^2(x)\tan(x).$$

(You may use the formula for $\frac{d}{dx}\big[\tan(x)\big]$, which we derived in the previous exercise.)

Solution:

By formula,

$$\frac{d}{dx}\big[\tan(x)\big] = \sec^2(x).$$

We set up the Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)g(x) + f(x)g'(x),$$

for

$$\sec^2(x) = \sec(x)\cdot\sec(x)$$

as follows:

$f(x) = \sec(x)$	$g(x) = \sec(x)$
$f'(x) = \sec(x)\tan(x)$	$g'(x) = \sec(x)\tan(x)$

$$\begin{aligned}\frac{d^2}{dx^2}\big[\tan(x)\big] = \frac{d}{dx}\big[\sec^2(x)\big] = \frac{d}{dx}\big[\sec(x)\cdot\sec(x)\big] &= \sec(x)\cdot\sec(x)\tan(x) + \sec(x)\tan(x)\cdot\sec(x)\\ &= 2\sec^2(x)\tan(x).\end{aligned}$$

Ex. 4. Show that the 27th derivative of $\sin(x)$ is $-\cos(x)$.

Ex. 5. Differentiate $\frac{\sin(x)}{x^2}$ without using the Quotient Rule.

Solution:

Write $f(x) = \sin(x)$ and $g(x) = x^{-2}$ for the Product Rule.

$f(x) = \sin(x)$	$g(x) = x^{-2}$
$f'(x) = \cos(x)$	$g'(x) = -2x^{-3}$

$$\frac{d}{dx}\left[\frac{\sin(x)}{x^2}\right] = \frac{d}{dx}\left[\sin(x)\cdot x^{-2}\right] \overset{\text{(Prod. Rule)}}{=} \cos(x)\cdot x^{-2} + \sin(x)\cdot(-2x^{-3})$$
$$= \frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}.$$

Additional exercises

Ex. 6 (§3.5—#175, 177, 179, 181). Differentiate.

(a) $y = x^2 - \sec(x) + 1$

(b) $y = x^2\cot(x)$

(c) $y = \frac{\sec(x)}{x}$

(d) $y = (x + \cos(x))(1 - \sin(x))$

Ex. 7 (§3.5—#185). Find an equation of the tangent line to the curve $y = -\sin(x)$ at the point $(0, 0)$.

Ex. 8 (§3.5—#197). Find all values of x at which the graph of $f(x) = -3\sin(x)\cos(x)$ has a horizontal tangent line.

Ex. 9 (§3.5—#203). The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is approximately given by

$$y = 10 + 5\sin(x)$$

where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. The restaurant closes at 11 p.m.. Find y' and determine the intervals where the number of burgers being sold is increasing.

Workbook Lessons 10 and 11

§3.6, Chain Rule
§3.4, Derivatives as Rates of Change

Last revised: 2021-09-30 12:10

Objectives

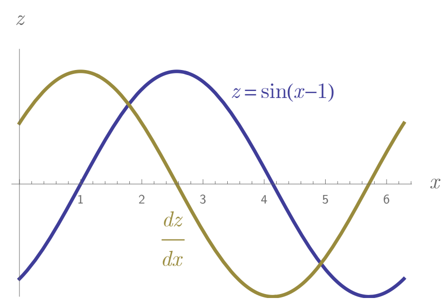
- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Find derivatives using all the Differentiation Rules learned so far.
- Solve word problems that apply differential calculus to problems in physics and business.

The Chain Rule

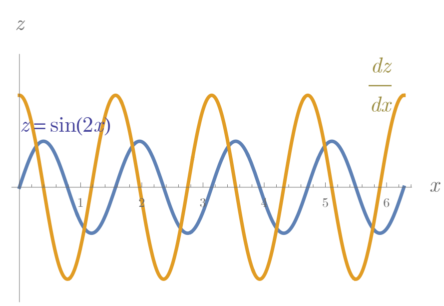
We know what the graph of (*) $z = \sin(x)$ looks like ($z =$ vertical axis, $x =$ horizontal axis). We also know (**) $\dfrac{d(\sin(x))}{dx} = \cos(x)$.

Given what we know about the graphs of (*) and (**), sketch the graph of $z = \sin(x-1)$ *and its derivative*.

(We don't have any rules that tell us what the derivative of $\sin(x-1)$ is yet. At this point, just guess what the graph of its derivative looks like.) (*Hint:* Use transformation of graphs.)



Now sketch the graph of $\sin(2x)$. Guess what the graph of its derivative must look like.

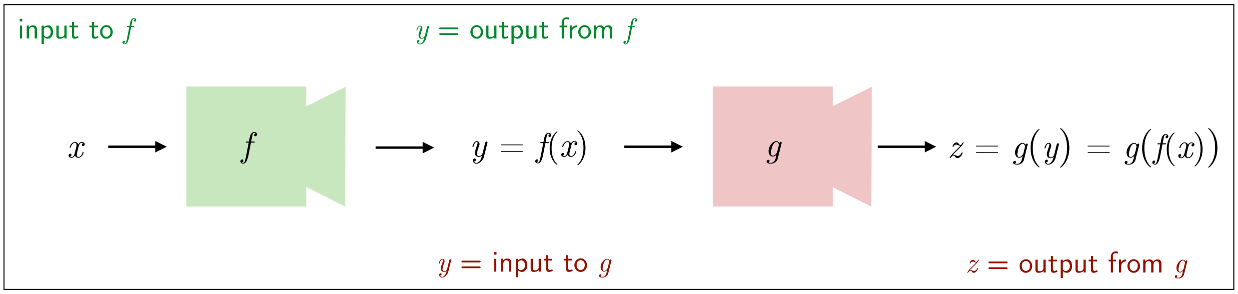


The graphs we drew suggest that $\dfrac{d}{dx}\big[\sin(x-1)\big] = \cos(x-1)$ and $\dfrac{d}{dx}\big[\sin(2x)\big] = 2\cos(2x)$. These guesses are correct.

There is a general rule we can use to find the derivative $z = \sin(f(x))$ with respect to x.

Indeed, we can replace $\sin$ by any (differentiable) function g.

Let f and g be two functions. The **Chain Rule** says that the *composite function* $g \circ f$...



...has derivative

$$\frac{d}{dx} g\big(f(x)\big) = g'\big(f(x)\big) \cdot f'(x). \qquad (\star)$$

Mnemonic: *Derivative of the outside, times derivative of the inside.*

In Leibniz notation, equation $(\star)$ is

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

(*See applet on iCollege:* "`Wheels and belts`")

Ex. 1. Find $\dfrac{d}{dx}\cos(-2x)$.

Write $y = f(x) = -2x$ and $g(y) = \cos(y)$ for the Chain Rule:

$$x \overset{f}{\mapsto} y = -2x \overset{g}{\mapsto} \cos(-2x)$$

inside	outside
$f(x) = -2x$	$g(y) = \cos(y)$
$f'(x) = -2$	$g'(y) = -\sin(y)$

$$\frac{d}{dx}\cos(-2x) \overset{\text{(C.R.)}}{=} g'(f(x)) \cdot f'(x) = -\sin(-2x) \cdot (-2) = 2\sin(-2x) \overset{(\sin \text{ odd})}{=} -2\sin(2x).$$

Setup for the same problem, using Leibniz notation:

$$x \overset{f}{\mapsto} y = -2x \overset{g}{\mapsto} z = \cos(-2x)$$

inside	outside
$y = -2x$	$z = \cos(y)$
$\dfrac{dy}{dx} = -2$	$\dfrac{dz}{dy} = -\sin(y)$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = -\sin(y) \cdot (-2) = -\sin(-2x) \cdot (-2) = \ldots$$

Ex. 2. Differentiate $f(x) = \sqrt[4]{4x^3 - 2}$.

$$x \quad \overset{f}{\mapsto} \quad y = 4x^3 - 2 \quad \overset{g}{\mapsto} \quad (4x^3 - 2)^{1/4}$$

inside	outside
$f(x) = 4x^3 - 2$	$g(y) = y^{1/4}$
$f'(x) = 12x^2$	$g'(y) = \frac{1}{4}y^{-3/4}$

$$\frac{d}{dx}(4x^3 - 2)^{1/4} = g'(f(x)) \cdot f'(x) = g'(4x^3 - 2) \cdot (12x^2) = \frac{1}{4}(4x^3 - 2)^{-3/4} \cdot (12x^2) = \frac{3x^2}{\sqrt[4]{(4x^3 - 2)^3}}.$$

Ex. 3. Differentiate $q = -\sin\left(\frac{5 - v}{v}\right)$.

$$v \quad \overset{f}{\mapsto} \quad y = \frac{5 - v}{v} \quad \overset{g}{\mapsto} \quad -\sin\left(\frac{5 - v}{v}\right)$$

$$f(v) = \frac{5 - v}{v}. \qquad f'(v) = \frac{d}{dv}\left[\frac{5 - v}{v}\right] = \frac{d}{dv}\left[5v^{-1} - 1\right] = -5v^{-2} = -\frac{5}{v^2}.$$

$$g(y) = -\sin(y). \qquad g'(y) = \frac{d}{dy}\left[-\sin(y)\right] = -\frac{d}{dy}\left[\sin(y)\right] = -\cos(y) = -\cos\left(\frac{5 - v}{v}\right).$$

$$\begin{aligned}\frac{d}{dv}\left[-\sin\left(\frac{5 - v}{v}\right)\right] &= g'(f(v)) \cdot f'(v) = -\cos\left(\frac{5 - v}{v}\right) \cdot \left(\frac{-5}{v^2}\right) \\ &= \frac{5}{v^2}\cos\left(\frac{5 - v}{v}\right).\end{aligned}$$

Ex. 4. Differentiate $f(x) = \cos^4(x)$.

Note that $\cos^4(x) = \big(\cos(x)\big)^4$.

$$x \quad \overset{f}{\mapsto} \quad y = \cos(x) \quad \overset{g}{\mapsto} \quad [\cos(x)]^4$$

$$\begin{aligned}\frac{d}{dx}\left[\big(\cos(x)\big)^4\right] &= 4\cos^3(x) \cdot \frac{d}{dx}\left[\cos(x)\right] \\ &= 4\cos^3(x)\big(-\sin(x)\big) \\ &= -4\cos^3(x)\sin(x).\end{aligned}$$

Ex. 5. Differentiate $\frac{1}{(3x^2+1)^2}$.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{(3x^2+1)^2}\right] &= \frac{d}{dx}\left[(3x^2+1)^{-2}\right]\\ &= -2(3x^2+1)^{-3}\cdot\frac{d}{dx}\left[3x^2+1\right]\\ &= -2x(3x^2+1)^{-3}(6x)\\ &= \frac{-12x}{(3x^2+1)^3}.\end{aligned}$$

Ex. 6. Differentiate $(7x-2)^3$.

$$\begin{aligned}\frac{d}{dx}\left[(7x-2)^3\right] &= 3(7x-2)^2\cdot\frac{d}{dx}\left[7x-2\right]\\ &= 21(7x-2)^2.\end{aligned}$$

Ex. 7. Differentiate $(7x-2)^3(2x-1)$.

$$\begin{aligned}\frac{d}{dx}\left[(7x-2)^3(2x-1)\right] &= \frac{d}{dx}\left[(7x-2)^3\right]\cdot(2x-1)+(7x-2)^3\cdot\frac{d}{dx}\left[2x-1\right] && \text{(Product Rule)}\\ &= 3(7x-2)^2\cdot\frac{d}{dx}\left[7x-2\right]\cdot(2x-1)+(7x-2)^3\cdot\frac{d}{dx}\left[2x-1\right] && \text{(Chain Rule)}\\ &= 21(7x-2)^2\cdot(2x-1)+2(7x-2)^3.\end{aligned}$$

Ex. 8. Differentiate $(7x-2)^3(2x-1)^5$.

$$\begin{aligned}\frac{d}{dx}\left[(7x-2)^3(2x-1)^5\right] &= \frac{d}{dx}\left[(7x-2)^3\right]\cdot(2x-1)^5+(7x-2)^3\cdot\frac{d}{dx}\left[(2x-1)^5\right] && \text{(Product Rule)}\\ &= 21(7x-2)^2(2x-1)^5+(7x-2)^3\cdot\frac{d}{dx}\left[(2x-1)^5\right] && \text{(Chain Rule)}\\ &= 21(7x-2)^2(2x-1)^5+(7x-2)^4\cdot 5(2x-1)^4\cdot\frac{d}{dx}\left[2x-1\right]\\ &= 21(7x-2)^2(2x-1)^5+10(7x-2)^4.\end{aligned}$$

Ex. 9. Prove that

(a) the derivative of an even function is odd, and

(b) the derivative of an odd function is even.

<u>Solution:</u>

f even $\implies$ $f(x) = f(-x)$.

By Chain Rule, $f'(x) = [f(-x)]' = f'(-x) \cdot (-1) = -f'(-x)$.

Thus f' is odd: $f'(x) = -f'(-x)$.

f odd $\implies$ $f(-x) = -f(x)$ $\implies$ $f(x) = -f(-x)$.

By Chain Rule, $f'(x) = [-f(-x)]' = -[f(-x)]' = -[f'(-x) \cdot (-1)] = f'(-x)$.

Thus f' is even: $f'(x) = f'(-x)$.

Ex. 10.

- Suppose the motion of a particle is described by a displacement function $s(t)$.
- As usual, let $v(t)$ be the particle's velocity, and let $a(t)$ be the particle's acceleration.
- Prove that

$$a(t) = v(t)\,\frac{dv}{ds}$$

(note the use of Lebiniz notation).

<u>Solution:</u>

First, let's clarify the meaning of each variable:

		t	time
		$s(t)$	displacement
$v(t)$	$=$	$\frac{ds}{dt}$	velocity
$a(t)$	$=$	$\frac{dv}{dt}$	acceleration

<u>Comments:</u>

By Chain Rule,

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}\,v(t).$$

$\frac{dv}{dt}$ *is acceleration, i.e. the rate of change in velocity with respect to time.*

$\frac{dv}{ds}$ *is the rate of velocity with respect to the displacement.*

Example:

$v(s) = \sin(s)$ and $s(t) = 2t$.

Then

$$\frac{dv}{ds} = \frac{d}{ds}\sin(s) = \cos(s),$$

while

$$v(t) = \frac{ds}{dt} = 2$$

and

$$\frac{d^2v}{dt^2} = \frac{dv}{ds}v(t) = \cos(s)\cdot 2 = 2\cos(2t).$$

Exercises 11–35 below are accompanied by fully worked solutions. You are not expected to practice every single exercise.

☞ But please be sure to study Exercises 24, 25, 26, 27, and 28!

☞ Business students are encouraged to also study Exercises 32–35.

Worked problems

Ex. 11. Differentiate:

(a) $\dfrac{-1}{\sqrt{(5x-1)^3}}$

(b) $t\sin\dfrac{1}{t}$

(c) $\sin(\cos(\frac{1}{3}s^3))$

Solution:

$$\begin{aligned}\frac{d}{dx}\left[\frac{-1}{\sqrt{(5x-1)^3}}\right] &= \frac{d}{dx}\left[-(5x-1)^{-3/2}\right] \\ &= \frac{3}{2}(5x-1)^{-5/2}\cdot\frac{d}{dx}[5x-1] \\ &= \frac{15}{2\sqrt{(5x-1)^5}}.\end{aligned} \tag{a}$$

$$\begin{aligned}\frac{d}{dt}\left[t\sin(t^{-1})\right] &= \frac{d}{dt}[t]\cdot\sin(t^{-1}) + t\cdot\frac{d}{dt}\left[\sin(t^{-1})\right] \\ &= \sin(t^{-1}) + t\frac{d}{dt}\left[\sin(t^{-1})\right] \\ &= \sin(t^{-1}) + t\cos(t^{-1})\cdot\frac{d}{dt}[t^{-1}] \\ &= \sin(t^{-1}) - t\cos(t^{-1})\cdot t^{-2} \\ &= \sin\frac{1}{t} - \frac{\cos\frac{1}{t}}{t}.\end{aligned} \tag{b}$$

$$\begin{aligned}\frac{d}{ds}\left[\sin\big(\cos(\tfrac{1}{3}s^3)\big)\right] &= \cos\big(\cos(\tfrac{1}{3}s^3)\big)\cdot\frac{d}{ds}\left[\cos(\tfrac{1}{3}s^3)\right] \\ &= \cos\big(\cos(\tfrac{1}{3}s^3)\big)\cdot\big(-\sin(\tfrac{1}{3}s^3)\big)\cdot\frac{d}{ds}\left[\tfrac{1}{3}s^3\right] \\ &= \cos\big(\cos(\tfrac{1}{3}s^3)\big)\cdot\big(-\sin(\tfrac{1}{3}s^3)\big)\cdot(s^2) \\ &= -s^2\cos\big(\cos(\tfrac{1}{3}s^3)\big)\sin(\tfrac{1}{3}s^3).\end{aligned} \tag{c}$$

Ex. 12. If $y(x) = 9x^3 + 3x^2 + 5$ and $x(t) = 7t^2 + 10t + 2$, find $\dfrac{dy}{dt}$.

Solution:

$$\begin{aligned}\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} &= (27x^2+6x)(14t+10) \\ &= 6x(9x+2)(7t+5) \\ &= 6(7t^2+10t+2)\big(9(7t^2+10t+2)+2\big)(7t+5) \\ &= 6(7t^2+10t+2)(63t^2+90t+20)(7t+5).\end{aligned}$$

Ex. 13. Differentiate $y = \sqrt{x}(x-1)$.

Solution:

$$\begin{aligned} y' &= \left(x^{1/2}(x-1)\right)' \\ &= \left(x^{3/2} - x^{1/2}\right)' \\ &= \frac{3}{2}\sqrt{x} - \tfrac{1}{2\sqrt{x}}. \end{aligned}$$

$$\begin{aligned} y' &= \left(x^{1/2}(x-1)\right)' \\ &= \frac{1}{2\sqrt{x}}(x-1) + x^{1/2}(1) \\ &= \frac{x-1}{2\sqrt{x}} + \tfrac{2x}{2\sqrt{x}} \\ &= \frac{3x-1}{2\sqrt{x}}. \end{aligned}$$

Ex. 14. Differentiate $g(u) = \sqrt{2}u + \sqrt{3u}$. Justify each step by giving the name of the differentiation rule you are using.

Solution:

$$\begin{aligned} g'(u) &= \left(\sqrt{2}u + \sqrt{3u}\right)' & \\ &= \left(\sqrt{2}u\right)' + \left(\sqrt{3u}\right)' & \text{Sum Rule} \\ &= \sqrt{2}\left(u\right)' + \sqrt{3}\left(\sqrt{u}\right)' & \text{Identity \& Constant Multiple} \\ &= \sqrt{2}\left(u\right)' + \sqrt{3}\left(u^{1/2}\right)' & \text{Constant Multiple} \\ &= \sqrt{2} + \tfrac{\sqrt{3}}{2}u^{-1/2} & \text{Identity \& Power Rule} \\ &= \sqrt{2} + \tfrac{\sqrt{3}}{2\sqrt{u}}. & \end{aligned}$$

Ex. 15.

(a) Differentiate $H(x) = (x + x^{-1})^3$ *without* using the Chain Rule.

(b) Then differentiate it using the Chain Rule.

Solution:

$$\begin{aligned} H'(x) &= \left((x + x^{-1})^3\right)' \\ &= \left(x^3 + 3x^2x^{-1} + 3x(x^{-1})^2 + (x^{-1})^3\right)' \\ &= \left(x^3 + 3x + 3x^{-1} + x^{-3}\right)' \\ &= 3x^2 + 3 - 3x^{-2} - 3x^{-4} \\ &= 3x^2 + 3 - \tfrac{3}{x^2} - \tfrac{3}{x^4}. \end{aligned}$$

$$
\begin{aligned}
H'(x) &= \big((x + x^{-1})^3\big)' \\
&= 3(x + x^{-1})^2 \big(x + x^{-1}\big)' \\
&= 3(x + x^{-1})^2 (1 - x^{-2}) \\
&= 3(x + \tfrac{1}{x})^2 (1 - \tfrac{1}{x^2}).
\end{aligned}
$$

Ex. 16.

(a) Differentiate $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$ using the Quotient Rule.

(b) Then differentiate it using the Product Rule.

Solution:

$$
\begin{aligned}
y' &= \frac{(x^{1/2} + 1)(x^{1/2} - 1)' - (x^{1/2} - 1)(x^{1/2} + 1)'}{(x^{1/2} + 1)^2} \\
&= \frac{(x^{1/2} + 1)(\frac{1}{2}x^{-1/2}) - (x^{1/2} - 1)(\frac{1}{2}x^{-1/2})}{(x^{1/2} + 1)^2} \\
&= \frac{\frac{1}{2}(1 + x^{-1/2}) - \frac{1}{2}(1 - x^{-1/2})}{(x^{1/2} + 1)^2} \\
&= \frac{1 + x^{-1/2} - 1 + x^{-1/2}}{2(x^{1/2} + 1)^2} \\
&= \frac{1}{2\sqrt{x}(\sqrt{x} + 1)^2}.
\end{aligned}
$$

$$
\begin{aligned}
y' &= \big((x^{1/2} - 1) \cdot (x^{1/2} + 1)^{-1}\big)' \\
&= (x^{1/2} - 1)'(x^{1/2} + 1)^{-1} + (x^{1/2} - 1)\,\big((x^{1/2} + 1)^{-1}\big)' \\
&= (\tfrac{1}{2}x^{-1/2})(x^{1/2} + 1)^{-1} + (x^{1/2} - 1)(-1)(x^{1/2} + 1)^{-2} \cdot \tfrac{1}{2}x^{-1/2} \\
&= \frac{1}{2\sqrt{x}(\sqrt{x} + 1)} - \frac{\sqrt{x} - 1}{2\sqrt{x}(\sqrt{x} + 1)^2}
\end{aligned}
$$

Ex. 17.

(a) Differentiate $y = \dfrac{t}{(t - 1)^2}$ *without* using the Chain Rule.

(b) Then differentiate it using the Chain Rule.

Solution to part (a):

$$\begin{aligned} y' &= \left(\frac{t}{(t-1)^2}\right)' \\ &= \left(\frac{t}{t^2-2t+1}\right)' \\ &= \frac{(t^2-2t+1)-t(2t-2)}{(t^2-2t+1)^2} \\ &= \frac{1-t^2}{(t^2-2t+1)^2} \\ &= \frac{(1-t)(1+t)}{(t-1)^4} \\ &= \frac{-1-t}{(t-1)^3}. \end{aligned}$$

Solution to part (b):

Take $f(t) = t-1$ and $g(z) = \dfrac{z+1}{z^2}$, so that $g\big(f(t)\big) = \dfrac{t}{(t-1)^2}$.

Then

$$f'(t) = 1$$

and

$$\begin{aligned} g'(z) &= \frac{d}{dz}\left[\frac{z+1}{z^2}\right] \\ &= \frac{d}{dz}\left[z^{-2}(z+1)\right] \\ &= \frac{d}{dz}\left[z^{-1}+z^{-2}\right] \\ &= -z^{-2}-2z^{-3}, \end{aligned}$$

so

$$\begin{aligned} y' &= g'\big(f(t)\big)\cdot f'(t) \\ &= g'(t-1)\cdot 1 \\ &= -(t-1)^{-2}-2(t-1)^{-3} \\ &= \frac{-1}{(t-1)^2}+\frac{-2}{(t-1)^3} \\ &= \frac{-1(t-1)-2}{(t-1)^3} \\ &= \frac{-t-1}{(t-1)^3}. \end{aligned}$$

Ex. 18. Differentiate $y = \dfrac{1}{(1+\sec x)^2}$ *without* using the Quotient Rule or the Product Rule.

Solution:

$$y' = \left((1+\sec x)^{-2}\right)' = -2(1+\sec x)^{-3}\sec x\tan x = \frac{-2\sec x\tan x}{(1+\sec x)^3}.$$

Ex. 19. Let $g(x) = (x^2+1)^3(x^3+2)^6$. Show that $g'(x) = 6x(x^2+1)^2(x^3+2)^5(4x^3+3x+2)$.

Solution:

$$\begin{aligned} g'(x) &= \left[(x^2+1)^3(x^3+2)^6\right]' \\ &= (x^2+1)^3\left[(x^3+2)^6\right]' + \left[(x^2+1)^3\right]'(x^3+2)^6 \\ &= 18x^2(x^2+1)^3(x^3+2)^5 + 6x(x^2+1)^2(x^3+2)^6 && \text{Collect like terms} \\ &= 6x(x^2+1)^2(x^3+2)^5\left(3x(x^2+1)+(x^3+2)\right) \\ &= 6x(x^2+1)^2(x^3+2)^5(4x^3+3x+2). \end{aligned}$$

Ex. 20. Let $f(x) = \dfrac{x}{\sqrt{7-3x}}$. Show that $f'(x) = \dfrac{14-3x}{2\sqrt{(7-3x)^3}}$.

Solution:

$$\begin{aligned} f'(x) &= \left(x(7-3x)^{-1/2}\right)' \\ &= x\left((7-3x)^{-1/2}\right)' + (7-3x)^{-1/2} \\ &= x(\tfrac{-1}{2})(7-3x)^{-3/2}(-3) + (7-3x)^{-1/2} && \text{Factor out } (7-3x)^{-3/2} \\ &= (7-3x)^{-3/2}\left(\tfrac{3}{2}x + (7-3x)\right) \\ &= \frac{-\frac{3}{2}x+7}{(7-3x)^{3/2}} = \frac{\frac{1}{2}(-3x+14)}{(7-3x)^{3/2}} = \frac{14-3x}{2\sqrt{(7-3x)^3}}. \end{aligned}$$

Ex. 21. Differentiate $y = \cos^4(\sin^3 x)$.

Solution:

$$y = \left[\cos\left((\sin x)^3\right)\right]^4.$$

$$\begin{aligned} y' &= \left\{\left[\cos\left((\sin x)^3\right)\right]^4\right\}' \\ &= 4\left[\cos\left((\sin x)^3\right)\right]^3 \cdot \left[\cos\left((\sin x)^3\right)\right]' \\ &= 4\cos^3(\sin^3 x)\cdot\left(-\sin\left((\sin x)^3\right)\right)\cdot\left((\sin x)^3\right)' \\ &= -4\cos^3(\sin^3 x)\sin(\sin^3 x)\cdot 3\left((\sin x)^2\right)\cdot(\sin x)' \\ &= -12\cos^3(\sin^3 x)\sin(\sin^3 x)\sin^2(x)\cos(x). \end{aligned}$$

Ex. 22. Let $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$. Find a formula for $f'(x)$. Does $f'(0)$ exist?

Solution:

For $x \neq 0$:

$$\begin{aligned} f'(x) &= \big(x \sin(x^{-1})\big)' \\ &= \sin(x^{-1}) + x\big(\sin(x^{-1})\big)' \\ &= \sin(x^{-1}) + x\cos(x^{-1}) \cdot \tfrac{-1}{x^2} \\ &= \sin\left(\tfrac{1}{x}\right) - \tfrac{1}{x}\cos\left(\tfrac{1}{x}\right). \end{aligned}$$

For $x \neq 0$:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} && \text{(definition of derivative)} \\ &= \lim_{x\to 0} \frac{x \sin \frac{1}{x} - 0}{x} && (f(x) = x \sin \tfrac{1}{x} \text{ for } x \neq 0) \\ &= \lim_{x\to 0} \sin \tfrac{1}{x} \text{ does not exist.} && \text{(See Section 2.5)} \end{aligned}$$

$f'(0)$ does not exist. For $x \neq 0$, $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$.

Ex. 23.

(a) Write $|x| = \sqrt{x^2}$, and use the Chain Rule to show that $\dfrac{d}{dx}|x| = \dfrac{x}{|x|}$.

(b) Let $f(x) = |\sin x|$. Find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?

Solution:

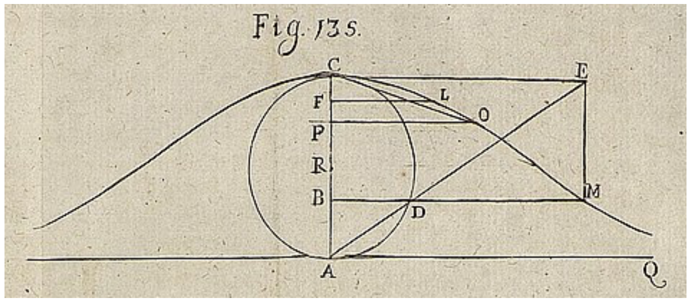
$$\frac{d}{dx}\big(|x|\big) = \frac{d}{dx}\big((x^2)^{1/2}\big) = \frac{1}{2}(x^2)^{-1/2} \cdot 2x = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|}.$$

$$\frac{d}{dx}\big(|\sin x|\big) = \frac{\sin x}{|\sin x|} \cdot \frac{d}{dx}(\sin x) = \frac{\sin x}{|\sin x|}\cos x.$$

$$\begin{aligned} \frac{d}{dx}\big(|\sin x|\big) = \frac{\sin x}{|\sin x|}\cos x \text{ is undefined} &\iff |\sin x| = 0 \\ &\iff \boxed{x = n\pi \text{ where } n \text{ is any integer.}} \end{aligned}$$

Worked problems with tangent lines

Ex. 24. Find an equation of the tangent line to the curve $y = 1/(1+x^2)$, called the **witch of Maria Agnesi**, at the point $(-1, \frac{1}{2})$.



Maria Gaetana Agnesi's construction of the curve (1748)

Solution.

General form of the equation of a tangent line to the graph of $y = y(x)$:

$$y - y(a) = y'(a) \cdot (x - a)$$

Here $\big(a, f(a)\big) = (-1, \frac{1}{2})$, and since

$$y' = -1(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2},$$

we have

$$y'(-1) = \frac{1}{2}.$$

$$\boxed{y - \tfrac{1}{2} = \tfrac{1}{2}(x+1)}$$

Ex. 25. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

Solution:

$$y' = (x^{3/2})' = \tfrac{3}{2}\sqrt{a}.$$

The slope of the line $y = 1 + 3x$ is $m = 3$.
The general form of the tangent line to $y = x\sqrt{x}$ at $x = a$ is $y - a\sqrt{a} = \frac{3}{2}\sqrt{a}(x - a)$.
Solving $\frac{3}{2}\sqrt{a} = 3$ yields $a = 4$.

$$\boxed{y - 8 = 3(x - 4)}$$

Ex. 26. Find equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$.

$$y' = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

Solution:

The slope of $x - 2y = 2$ is $m = \frac{1}{2}$:

$$\begin{aligned} x - 2 &= 2y \\ y &= \tfrac{1}{2}x - 1 \end{aligned}$$

The slope of the tangent line to $y = \dfrac{x-1}{x+1}$ at $x = a$ is $y'(a) = \dfrac{2}{(a+1)^2}$. We solve the equation $y'(a) = m$.

$$\begin{aligned} \frac{2}{(a+1)^2} &= \tfrac{1}{2} \\ (a+1)^2 &= 4 \\ a &= -1 \pm \sqrt{4} = 1 \text{ or } -3 \end{aligned}$$

Since $y(-3) = 2$ and $y(1) = 0$, the desired tangent lines are

$$\boxed{y - 2 = \tfrac{1}{2}(x+3) \text{ and } y - 0 = \tfrac{1}{2}(x-1)}.$$

Ex. 27. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

Solution:

$$y' = 18x^2 + 5$$

We solve $y'(a) = 4$:

$$\begin{aligned} y'(a) = 18a^2 + 5 &= 4 \\ 18a^2 &= -1 \\ a^2 &= \tfrac{-1}{18} < 0 \qquad \text{Impossible: the square of any real number is nonnegative.} \end{aligned}$$

We have shown that: if $y'(a) = 4$, then $a^2 < 0$.

Since $a^2 < 0$ is impossible, there is no tangent line with slope $y'(a) = 4$.

Some physical applications of basic differential calculus

We now turn our attention to a variety of applications (word problems) that can be solved using the knowledge of derivatives we have developed thus far.

Ex. 28. The position of a particle in motion is given by the equation $s = t^3 - 6t^2 + 9t$, where s is measured in meters and t is measured in seconds.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?
(*Hint*: When is $v(t) > 0$?)

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t.

(h) Find the acceleration after 4 s.

Solution.

(a) $\boxed{v(t) = s'(t) = 3t^2 - 12t + 9}$.

(b) $v(2) = -3$ and $v(4) = 9$.
$\boxed{\text{The velocity is } -3\text{m/s at time } t = 2\text{, and } 9\text{m/s at time } t = 4.}$

(c) The equation $v(t) = 0$ has solutions $t = 1$ and $t = 3$.
$\boxed{\text{The particle is at rest after } t = 1 \text{ second and after } t = 3 \text{ seconds.}}$

(d) $\boxed{\text{The particle moves forward when } t < 1 \text{ or } t > 3.}$

(f) The forward motion occurs when $t < 1$ or $t > 3$:

$$\text{(forward motion)} = |s(1) - s(0)| + |s(5) - s(3)| = |4 - 0| + |20 - 0| = 24\text{m}.$$

The backward motion occurs when $1 < t < 3$:

$$\text{(backward motion)} = |s(3) - s(1)| = |0 - 4| = 4\text{m}.$$

The total distance traveled is (forward motion) + (backward motion) = 24 + 4 = 28 m.

(g) $a(t) = v'(t) = 6t - 12$.

(h) $a(4) = 12$ m/s^2.

Ex. 29. (Calculator) If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.

(a) What is the velocity of the rock after 2 s?

(b) What is the velocity of the rock when its height is 25 m on its way up?

(c) What is the velocity of the rock when its height is 25 m on its way down?

Solution.

(a): $v(t) = h'(t) = 15 - 3.72t$, so $v(2) = \boxed{7.56 \text{ m/s}^2}$.

(b) and **(c):**

Suppose $h(t) = 25$. To solve the equation $h(t) = 25$, we use the quadratic formula:

$$1.86t^2 - 15t + 25 = 0$$

$$t = \frac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)}$$

$h(t) = 25$ when $t = t_1 \approx 2.35$ ot $t = t_2 \approx 5.71$.

What is the approximate rate of change in the height at times $t = t_1$ and $t = t_2$?

$v(t_1) \approx 6.24$ (upward)
$v(t_2) \approx -6.24$ (downward)

Answer to (b): $\boxed{6.24 \text{ m/s upward}}$

Answer to (c): $\boxed{6.24 \text{ m/s downward}}$

☞ Recall: 1 **newton** (N) $= 1 \dfrac{\text{kg} \cdot \text{m}}{\text{s}^2}$ is the force required to produce an acceleration of 1 m/s^2 in a body of mass 1 kg.

Ex. 30. Newton's Law of Gravitation says that the magnitude F of the force (in newtons N) exerted by a body of mass on a body of mass M is $F = \dfrac{GmM}{r^2}$, where G is the gravitational constant and r is the distance between the bodies.

(a) Find $\dfrac{dF}{dr}$, and explain its meaning. What does the minus sign indicate?

(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

Solution.

(a) $F(r) = \dfrac{GmM}{r^2} = \underbrace{GmM}_{\text{constants}} \cdot r^{-2}$.

$\boxed{\dfrac{dF}{dr} = -2\,GmM\,r^{-3}}$ is the rate of change in the gravitational force F between the two bodies with respect to the distance r between them. The minus sign indicates that, as the distance (independent variable) r increases, (the dependent variable) F decreases.

(When interpreting the meaning of the derivative in this and similar problems, regard the independent variable as increasing.)

(b) Given that $F'(20,000) = -2$, we want to find $F'(10,000)$. Using the fact that $GmM = 20,000^3$...

$$\begin{aligned} -2 = F'(20,000) &= \frac{-2GmM}{20,000^3} \\ GmM &= 20,000^3 \end{aligned}$$

...we find that

$$F'(10,000) = \frac{-2 \cdot 20,000^3}{10,000^3} = \frac{-16 \times 10^{12}}{1 \times 10^{12}} = -16.$$

The force decreases at 16 N/km when $r = 10,000$ km.

Ex. 31. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo **simple harmonic motion**.

(a) Find the velocity of the particle at time t.

(b) When is the velocity 0?

(a) $v(t) = s'(t) = -\omega A \sin(\omega t + \delta)$.

(b)

$$\begin{aligned} v(t) = -\omega A \sin(\omega t + \delta) &= 0 \\ \sin(\omega t + \delta) &= 0 \end{aligned}$$

Since $\sin x = 0$ only if $x = n\pi$, where n is any integer, we set $x(t) = \omega t + \delta = 0$:

$$\omega t + \delta = n\pi$$

The velocity is 0 when $t = \dfrac{n\pi - \delta}{\omega}$ where n is any integer.

☞ A frictionless pendulum undergoes simple harmonic motion—provided that the pendulum swings only through a small angle, and does not swing entirely around the pivot it hangs from! A more realistic mathematical model would include a frictional force that causes the motion to slow to an eventual halt. Such motion is called **simple harmonic motion with damping**. *For details, see any undergraduate Differential Equations textbook.*

Additional applications of basic differential calculus

Ex. 32. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) What does the statement $f'(800) = 17$ mean?

(c) Do you think $f'(x)$ will increase or decrease in the short term? What about in the long term? Explain.

(a) Rate of change in production cost with respect to number of ounces of gold produced, in dollars per ounce.

(b) After 800 oz of gold have been produced, the rate at which the production cost increases is about \$$17$ per ounce. So the cost of producing the 801st ounce is $\approx$ \$$17$.

(c) The production cost of the first ounce includes all startup costs. Initially, the rate of increase in production costs will decrease due to increasingly efficient use of startup costs. Eventually, the rate may increase due to costs specific to large-scale operations.

Ex. 33. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers as of October 1 are given.)

Year	2004	2005	2006	2007	2008
N	8569	$10,241$	$12,440$	$15,011$	$16,680$

(a) Find the average rate of growth (i) from 2006 to 2008, (ii) from 2006 to 2007, (iii) from 2005 to 2006. Include the units.

(b) Estimate the instantaneous rate of growth in 2006 by taking the average of two average rates of change. What are its units?

(a)

$$\begin{aligned}(\text{Average rate of change from } N = 2006 \text{ to } N = 2008) &= \frac{\Delta N}{\Delta t} \qquad \text{(i)}\\ &= \frac{16,680 - 12,440}{2}\\ &= 2120 \text{ locations/yr.}\end{aligned}$$

$$\begin{aligned}(\text{Average rate of change from } N = 2006 \text{ to } N = 2007) &= \frac{\Delta N}{\Delta t} \qquad \text{(ii)}\\ &= 2571 \text{ locations/yr.}\end{aligned}$$

$$\begin{aligned}(\text{Average rate of change from } N = 2006 \text{ to } N = 2007) &= \frac{\Delta N}{\Delta t} \qquad \text{(iii)}\\ &= 2199 \text{ locations/yr.}\end{aligned}$$

(b) We use the average rates of change from parts (ii) and (iii) because the instantaneous rate of change, $N'(t)$, is a two-sided limit: $\dfrac{2199 + 2571}{2} = 2385$ locations/yr.

Ex. 34. (Calculator required) The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed (i) from $x = 100$ to $x = 105$, (ii) from $x = 100$ to $x = 101$.

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the **marginal cost**.)

(a)

(i) $\dfrac{\Delta C}{\Delta x} = \$20.25$ per unit.

(ii) $\dfrac{\Delta C}{\Delta x} = \$20.05$ per unit.

(b) $C'(100) = \$20$ per unit.

☞ In the previous example,

$$x = (\text{number of units})$$

only takes integer values. It does not make literal sense to speak about the *limit*

$$C'(x) = \lim_{h\to 0} \frac{C(x+h) - C(x)}{h}, \tag{1}$$

since $C(x+h)$ is technically not defined when $h \to 0$ is so small that $x + h$ is not an integer. However, we can always replace $C(x)$ by a smooth (i.e. differentiable) approximating function.

Ex. 35. A manufacturer produces bolts of fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.

(a) What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?

(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

(a) $q = f(p) =$ quantity of the fabric sold as a function of the selling price p.

$f(20) = 10,000$ means that

$10,000$ yards are sold when the price is $\$20$/yd.

$f'(20) = -350$ means that:

As the selling price increases past $\$20$/yd, the amount of fabric sold is decreasing at a rate of 350 yards per \$1/yd increase in the price.

(b) Differentiating $R(p) = pf(p)$ yields

$$R'(p) = f(p) + pf'(p).$$

Given that $f(20) = 10,000$ and $f'(20) = -350$, we get

$$R'(20) = f(20) + 20 \cdot f'(20) = 10,000 - 7,000. \qquad (*)$$

Here, -7000 is the loss in dollars per $\$$/yd due to selling less fabric ($f'(20) < 0$).

But the fact that $\boxed{R'(20) = 3000}$ means that:

As the price of fabric per yard increases past $\$20$, the total revenue is increasing at $\$3000$ per \$/yd increase in price.

We conclude from equation (*) that the revenue $10,000$ makes up for the loss -7000 due to increasing the price.

Additional exercises

Ex. 36 (§3.6—#245). If

$$h(x) = f\big(g(x)\big), \quad f(2) = 4, \quad f'(2) = 4, \quad f'(0) = 5, \quad g(0) = 0, \quad \text{and} \quad g'(0) = 2,$$

what is $h'(0)$?

Ex. 37 (§3.6—#215, 217, 219, 223, 233, 235). Differentiate.

(a) $y = 6(7x - 4)^3$

(b) $y = \cos\left(\frac{-x}{8}\right)$

(c) $y = \sqrt{4(x^2 - 6x) + 3}$

(d) $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

(e) $y = \big(\tan(x) + \sin(x)\big)^{-3}$

(f) $y = \sin\big(\cos(7x)\big)$

Ex. 38 (§3.6—#241). Find an equation of the tangent line to the curve $y = -\sin\left(\frac{x}{2}\right)$ at the origin.

Ex. 39. (§3.6—#243). Find all points on the graph of the function $f(x) = \left(x - \frac{6}{x}\right)^8$ at which the tangent is horizontal.

Ex. 40 (§3.6—#254). A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

Find the velocity of the mass at time $t = 1.5$ seconds.

Ex. 41 (§3.4—#155). The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. When the position is 0, the back of the car is at the garage door. The car's starting position is $s(0) = -4$, that is, 4 feet inside the garage.

(a) Find the velocity of the car when $s(t) = 0$.

(b) Find the velocity of the car when $s(t) = 14$.

Ex. 42 (§3.4—#157). A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. Its height above ground level after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

(a) Find the velocity of the potato at 0.5 seconds.

(b) Find the velocity of the potato at 5.75 seconds.

(c) Find the speed of the potato at 0.5 seconds.

(d) Find the speed of the potato at 5.75 seconds.

(e) When does the potato reach its maximum height?

(f) What's the acceleration of the potato at 0.5 s and 1.5 s?

(g) How long is the potato in the air?

Ex. 43 (§3.4, Example 3.36). A particle's displacement is given by the function $s = f(t)$, where

$$f(t) = t^3 - 9t^2 + 24t + 4 \qquad (t \geq 0).$$

Here time t is measured in seconds and displacement s is measured in feet.

(a) Find the velocity at time t.

(b) When is the particle at rest?

(c) When is the particle moving in the positive direction (from left to right)? When is the particle moving in the negative direction (from right to left)?

Ex. 43. A spherical balloon is being inflated. Recall that the surface area of a sphere with radius r is given by $S = 4\pi r^3$. Find the rate of increase in the surface area when r is 1 ft, 2 ft, and 3 ft. What conclusion can you make?

Workbook Lesson 12
§3.8, Implicit Differentiation

Objectives
- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

Implicit differentiation

Consider the following equation of a circle:

$$x^2 + y^2 = 25.$$

Is y a function of x?

We can, however, solve this equation for y. For each value of x, there are two values of y:

$$y = \pm\sqrt{25 - x^2}.$$

This equation determines *two* functions of x:

$$f(x) = \sqrt{25 - x^2} \qquad\qquad g(x) = -\sqrt{25 - x^2}$$

The next equation isn't nearly as easy to solve for y.

$$x^3 + y^3 = 6xy$$

A computer algebra system gives

$$y = \frac{2^{2/3}\left(\sqrt{x^3\left(x^3 - 32\right)} - x^3\right)^{2/3} + 4\sqrt[3]{2}x}{2\sqrt[3]{\sqrt{x^3\left(x^3 - 32\right)} - x^3}}.$$

Now suppose we're asked to find a tangent line to this curve, say at the point $(3, 3)$. The equation of the tangent line would be

$$y - 3 = f'(3)(x - 3).$$

But we really don't want to take the derivative of the last equation!

Fortunately, we don't have to. We'll assume that y can be solved as one or more (differentiable) functions of x. Then both sides of the equation can be differentiated. We call this process implicit differentiation.

Ex. 1. Find the equation of the tangent line to the graph of $x^3 + y^3 = 6xy$ at $(3, 3)$,

Solution:

$$\frac{d}{dx}\left[x^3+y^3\right]=\frac{d}{dx}\left[6xy\right]$$
$$3x^2+3y^2y'=6\big[(1)(y)+(x)(y')\big]$$
$$3x^2+3y^2y'=6y+6xy'$$

We can now solve for y'.

$$3x^2-6y=6xy'-3y^2y'$$
$$x^2-2y=(2x-y^2)y'$$
$$y'=\frac{x^2-2y}{2x-y^2}$$

Notice that y' is a function that has *two* inputs: we can indicate this fact by writing $y'=y'(x,y)$.

$$y'(3,3)=\frac{9-6}{6-9}=-1.$$

The equation of the tangent line at $(3,3)$, therefore, is

$$y-3=-(x-3).$$

Ex. 2. Find the tangent line to $x^{2/3}+y^{2/3}=4$ at $\big(-3\sqrt{3},1\big)\approx(-5.196,1)$ and the tangent line at $(8,0)$.

Solution:

At the point $\big(-3\sqrt{3},1\big)$:

$$\frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}y'=0$$

$$y'=-\frac{2}{3x^{1/3}}\frac{3y^{1/3}}{2}$$

$$\boxed{y'=-\frac{y^{1/3}}{x^{1/3}}}$$

$$y'\big(-3\sqrt{3},1\big)=-\frac{1}{(-3\sqrt{3})^{1/3}}=-\frac{1}{(-\sqrt{3^2\cdot 3})^{1/3}}=-\frac{1}{(-3^{3/2})^{1/3}}=-\frac{1}{(-1)^{1/3}(3^{3/2})^{1/3}}=\frac{1}{\sqrt{3}}.$$

Tangent line:

$$y-1=\frac{1}{\sqrt{3}}(x+3\sqrt{3})$$

$$\boxed{y=\frac{1}{\sqrt{3}}x+4}$$

At the point $(8,0)$:

$$y'(8,0) = -\frac{0^{1/3}}{8^{1/3}}$$

$$y - 0 = 0(x-8)$$

$$\boxed{y = 0}$$

Ex. 3. Show that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

Solution:

Let Σ be a circle with radius r. For simplicity assume its center is $(0,0)$. Its equation is

$$x^2 + y^2 = r^2$$

Let $P = (x_0, y_0) \in \Sigma$ (that is, P is a point on the circle). By implicit differentiation,

$$2x + 2yy' = 0 \qquad \Longrightarrow \qquad y' = -\frac{x}{y},$$

so the tangent line to the circle at P is $y'(x_0, y_0) = -\frac{x_0}{y_0}$. But the slope of the radius is $\frac{y_0 - 0}{x_0 - 0}$. Since these slopes are each other's negative reciprocal, the tangent line at P is perpendicular to the radius OP.

Ex. 4. Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

Solution:

By implicit differentiation,

$$\frac{2}{a^2}x + \frac{2}{b^2}yy' = 0$$

$$y' = -\frac{2b^2 x}{2a^2 y}$$

$$y'(x_0, y_0) = -\frac{b^2 x_0}{a^2 y_0}$$

The tangent line at (x_0, y_0) is $\boxed{\frac{y_0 y}{b^2} + \frac{x_0 x}{a^2} = 1}$:

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\frac{y_0 y}{b^2} - \frac{{y_0}^2}{b^2} = -\frac{x_0 x}{a^2} + \frac{{x_0}^2}{a^2}$$

$$\frac{y_0 y}{b^2} + \frac{x_0 x}{a^2} = \frac{{x_0}^2}{a^2} + \frac{{y_0}^2}{b^2}$$

$$\frac{y_0 y}{b^2} + \frac{x_0 x}{a^2} = \frac{{x_0}^2}{a^2} + \frac{{y_0}^2}{b^2} = 1.$$

Additional exercises

Ex. 5 (§3.8—#301, 307, 309). Find $\frac{dy}{dx}$.

(a) $6x^2 + 3y^2 = 12$ **(b)** $y \sin(xy) = y^2 + 2$ **(c)** $x^3 y + xy^3 = -8$

Ex. 6. Find y'' and simplify fully.

$$\sin(y) + \cos(x) = 1$$

Ex. 7 (§3.8—#317). Find the equation of the normal line to the graph of $x^2 + 2xy - 3y^2 = 0$ at the point $(1, 1)$.

Ex. 8 (§3.8—#318). Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

Ex. 9 (§3.8—#323). The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation

$$60x^{3/4}y^{1/4} = 3240.$$

(a) Find $\frac{dy}{dx}$.

(b) Evaluate $\frac{dy}{dx}$ at the point $(81, 16)$.

(c) Interpret the result of the previous part.

Workbook Lesson 13

§3.7, Derivatives of Inverse Functions

Objectives

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

Recall:

A function f is **one-to-one** (or **invertible**) if different input values yield different output values. In symbols,

$$x_1 \neq x_2 \quad \Longrightarrow \quad f(x_1) \neq f(x_2)$$

The **inverse** of a function f—denoted f^{-1} and pronounced "f inverse"—is a function that "undoes" f. That is, the following **Cancellation Formulas** hold:

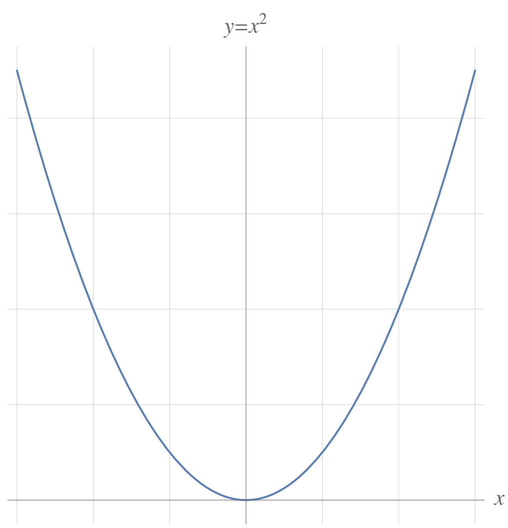
$$f^{-1}\big(f(x)\big) = x \qquad \text{for all } x \text{ in the domain of } f,$$
$$f\big(f^{-1}(y)\big) = y \qquad \text{for all } y \text{ in the range of } f,$$

Theorem. If a function f is one-to-one, then its inverse function exists.

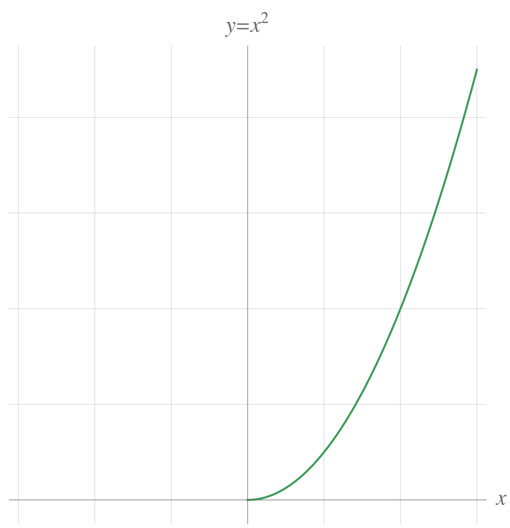
Counterexample. The function $f(x) = x^2$ with domain $(-\infty, \infty)$ is *not* one-to-one, because there exist different input values (for example, $x = 2$ and $x = -2$) that yield the same output value $f(2) = 4 = f(-2)$.

But, the function $g(x) = x^2$ *(same equation)* with domain $[0, \infty)$ *(different domain!)* IS one-to-one.

The **Horizontal Line Test** can be used to check whether a function is one-to-one: if no horizontal line meets the graph of the function in more than one point, then the function is one-to-one, and therefore has an inverse.



$f(x)$ has no inverse

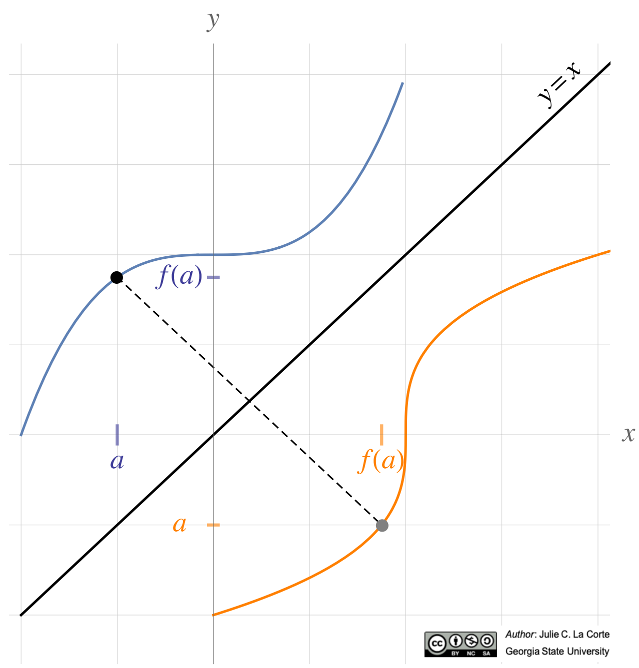


The inverse of $g(x)$ is $g^{-1}(y) = \sqrt{y}$

To construct the graph of f^{-1}, we reflect the graph of f in the line $y = x$.

- This reflection has the effect of *swapping* the roles of x and y.
- That is, for each point $\big(a, f(a)\big)$ on the graph of the original function f *(shown in blue below)*, we find the "shadow point" $\big(f(a), a\big)$ on the graph of the inverse f^{-1}*(shown in orange below)*.

(*See applet on iCollege:* "`Tangent and tangent to inverse`")



The Inverse Function Theorem

If a function f is both invertible and differentiable, it seems reasonable that its inverse f^{-1} is also differentiable.

After all, if the tangent line to the graph of f at the point $\big(a, f(a)\big)$ has slope

$$f'(a) = \frac{\Delta y}{\Delta x},$$

then shouldn't the slope of the tangent line to the graph of f^{-1} at the corresponding "shadow point" $\big(f(a), a\big)$ simply have slope

$$\big(f^{-1}\big)'\big(f(a)\big) = \frac{\Delta x}{\Delta y} = \frac{1}{f'(a)}, \qquad (\star)$$

since the roles of x and y are swapped in the graph of an inverse function?

The answer is, *yes, provided that the following conditions are met*:

- f is invertible and differentiable.
- $f'(a)$, the derivative of f at a, exists and is a real number.
- Furthermore, $f'(a)$ *cannot be equal to zero*, otherwise equation $(\star)$ will be meaningless.

Now, whenever we deal with inverse functions, it's easy to confuse the roles of x and y, since these roles swap when we move from f to f^{-1} or vice versa.

If we stick with the convention that x-values are inputs to f, and y-values are outputs of f, then y-values are inputs to f^{-1} and x-values are inputs to f, and equation $(\star)$ can be written either as

$$\left(f^{-1}\right)'\big(f(x)\big) = \frac{1}{f'(x)}.$$

or

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

However, it is more common to use x to denote the input value *no matter whether we are talking about f or f^{-1}.* If we adopt *this* convention, equation $(\star)$ becomes

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

We will follow the textbook and use the latter convention.

Inverse Function Theorem. Let f be a function. If f is invertible and differentiable, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

provided that

$$f'\big(f^{-1}(x)\big) \neq 0.$$

Ex. 1. Find $\left(f^{-1}\right)'(1)$ if $f(x) = 2x + \cos x$.

Solution:

We know f is differentiable. Is f one-to-one?

$f'(x) = 2 - \sin x > 0$, so f is increasing, so f is one-to-one.

Let's find $a = f^{-1}(1)$.

$$\begin{aligned} f^{-1}(1) &= a \\ 1 &= f(a) \\ 1 &= 2a + \cos a \\ a &= 0 && \text{(by inspection)} \end{aligned}$$

By the Inverse Function Theorem,

$$\left(f^{-1}\right)'(1) = \frac{1}{f'\big(f^{-1}(1)\big)} = \frac{1}{f'(a)} = \frac{1}{2 - \sin 0} = \boxed{\frac{1}{2}}.$$

Ex. 2. Let $f(x) = \dfrac{x+2}{x}$.

(a) Find the derivative of f.

(b) Find a formula for f^{-1}.

(c) Use the Inverse Function Theorem to find the derivative of f^{-1}.

(d) Find the derivative of f^{-1} *without* using the Inverse Function Theorem.

Solution:

(a) By the Quotient Rule, $f'(x) = \dfrac{x+2}{x} = \dfrac{(x)(1)-(x+2)(1)}{x^2} = \dfrac{2}{x^2}$.

(b) To find a formula for f^{-1}, we write $y = f(x)$ as follows:

$$y = \frac{x+2}{x}$$

Then we solve for the input x:

$$\begin{aligned} xy &= x+2 \\ xy - x &= 2 \\ (y-1)x &= 2 \\ x &= \frac{2}{y-1} \end{aligned}$$

Finally, we swap x and y to obtain a formula for $f^{-1}(x)$, bearing in mind that in this final equation, x is the OUTPUT of f:

$$f^{-1}(x) = y = \frac{2}{x-1}.$$

(c) First, let's check the conditions for the Inverse Function Theorem:

- Since
$$f(x) = \frac{x+2}{x} = 1 + 2\cdot\frac{1}{x}$$
is a transformation of the elementary function $\frac{1}{x}$, we see that f passes the Horizontal Line Test, so f is invertible.

- f is differentiable except at $x = 0$, where $f'(x) = -\dfrac{2}{x^2}$ is undefined.

We can now apply the Inverse Function Theorem:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{f'\left(\frac{2}{x-1}\right)} = \frac{1}{2/\left(\frac{2}{x-1}\right)^2} = \frac{\left(\frac{2}{x-1}\right)^2}{2} = \frac{1}{2}\left(\frac{2}{x-1}\right)^2 = \frac{2}{(x-1)^2}$$

(d) We verify our answer to **(c)** by differentiating $f^{-1}(x) = \dfrac{2}{x-1}$:

$$\begin{aligned} \frac{d}{dx}\left[\frac{2}{x-1}\right] &= \frac{(x-1)(0)-2(1)}{(x-1)^2} \\ &= \frac{2}{(x-1)^2}. \end{aligned}$$

Ex. 3. Let $g(x) = \sqrt[5]{x}$.

(a) Find a formula for g^{-1} and its derivative.

(b) Use the Inverse Function Theorem to find the derivative of g.

Solution:

(a) $g^{-1}(x) = x^5$ and $\left(g^{-1}\right)'(x) = 5x^4$.

(b) Let us write $f = g^{-1}$, noting that, since f and g are inverse functions, $g = f^{-1}$.

Now

$$\begin{aligned} g'(x) = \left(f^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)} \\ &= \frac{1}{\left(g^{-1}\right)'\left(g(x)\right)} \\ &= \frac{1}{\left(g^{-1}\right)'\left(\sqrt[5]{x}\right)} \\ &= \frac{1}{5\left(\sqrt[5]{x}\right)^4} \\ &= \frac{1}{5\left(x^{1/5}\right)^4} \\ &= \frac{1}{5}x^{-4/5}. \end{aligned}$$

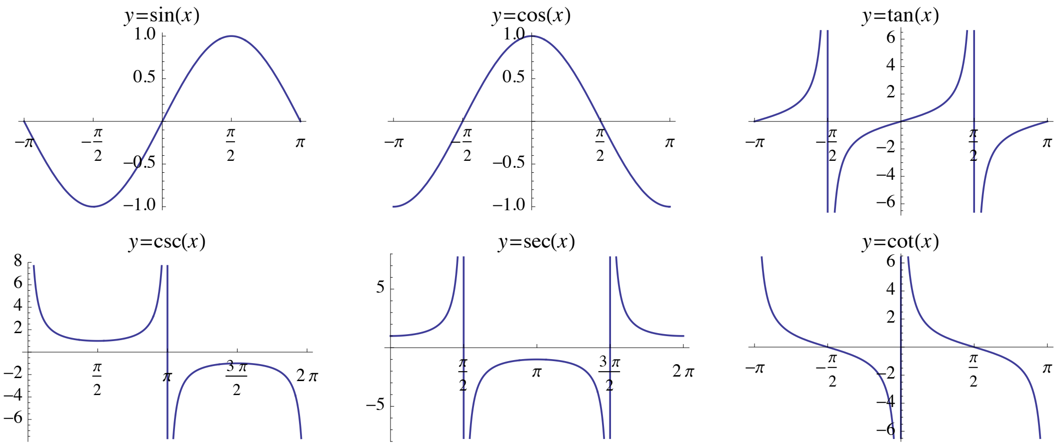
Ex. 4. Let $g(x) = \sqrt[3]{x-1}$.

(a) Find a formula for g^{-1} and its derivative.

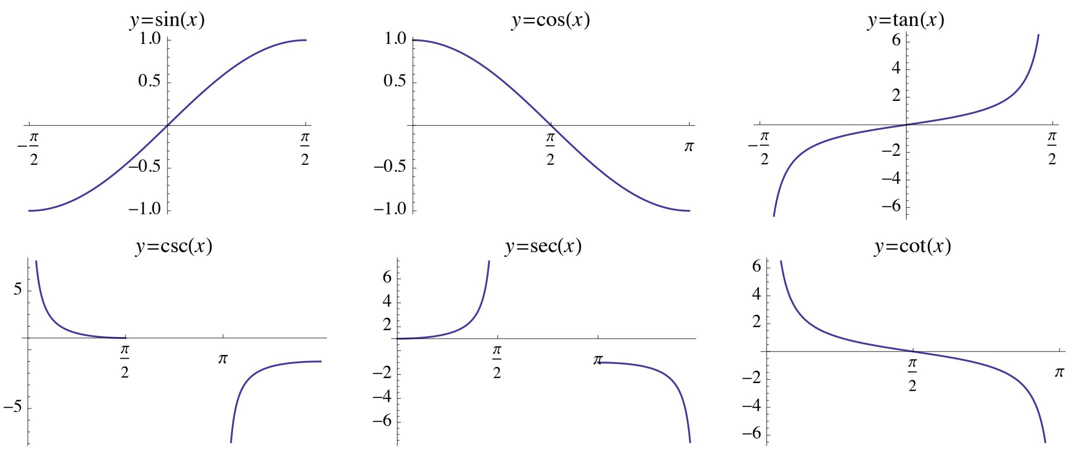
(b) Use the Inverse Function Theorem to find the derivative of g.

Derivatives of inverse trigonometric functions

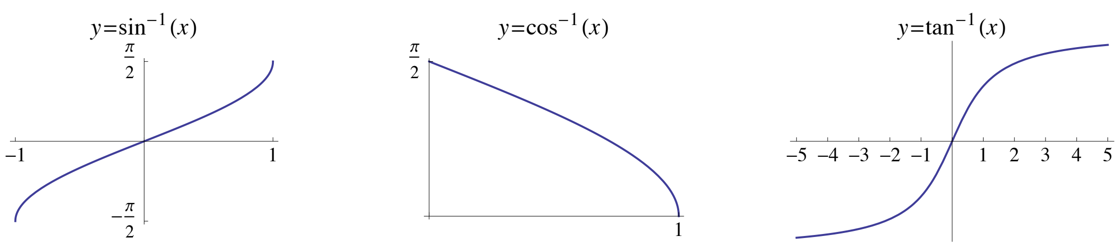
It is clear that the standard six standard trigonometric functions are not one-to-one, and thus do not have inverses.



However, by restricting their domains, we can *make* them one-to-one functions:



The inverses of these restricted functions are called the **inverse trigonometric functions**. They are denoted by $\sin^{-1}$, $\cos^{-1}$, etc.



☞ Alternate notation for the $\sin^{-1}$ function is $\arcsin$, alternate notation for $\cos^{-1}$ is $\arccos$, and so on.

☞ We will work only with the most common inverse trigonometric functions: $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually *algebraic* functions.

Ex. 5. We will find a formula for the derivative of $g(x) = \sin^{-1}(x)$.

(a) Write $\sin^{-1}(x) = \theta$, so that $\sin(\theta) = x$, and draw a picture of a right triangle in which θ and $\sin(\theta)$ are labeled. Then label the remaining sides of the triangle.

(b) The derivative of $g^{-1}(x) = \sin(x)$ is $\left(g^{-1}\right)'(x) = \cos(x)$. Use the diagram you drew in part **(a)** to find an algebraic formula for $\left(g^{-1}\right)'\left(g(x)\right)$.

(c) Use the Inverse Function Theorem to find the derivative of $g(x) = \sin^{-1}(x)$.

Solution:

(a)

Label one of the non-right angles as θ. Since $\sin(\theta) = \dfrac{\text{opp.}}{\text{hyp.}}$, we will take the length of the hypotenuse to be 1 for simplicity, and label the side opposite θ as

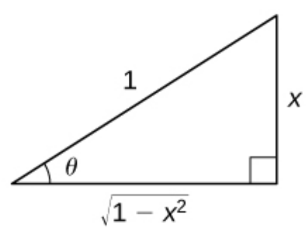
$$x = \sin(\theta).$$

Then by the Pythagorean Theorem, the length of the adjacent side satisfies

$$\text{adj.}^2 + x^2 = 1,$$

so

$$\text{adj.} = \sqrt{1 - x^2}.$$



(b)

We have

$$g(x) = \sin^{-1}(x) = \theta,$$

so

$$\left(g^{-1}\right)'\left(g(x)\right) = \cos\left(\sin^{-1}(x)\right) = \cos(\theta),$$

and from the fact that $\cos(\theta) = \dfrac{\text{adj.}}{\text{hyp.}}$ it follows that

$$\left(g^{-1}\right)'\left(g(x)\right) = \cos(\theta) = \sqrt{1 - x^2}.$$

(c)

$$\begin{aligned} g'(x) &= \frac{1}{\left(g^{-1}\right)'\left(g(x)\right)} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

Ex. 6. Using the same technique as in the previous exercise, show that

$$\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}.$$

We can also find the derivatives of inverse functions by using implicit differentiation.

Ex. 7. Find the derivative of $\tan^{-1}(x)$ *without* the Inverse Function Theorem.

Solution:

Set

$$\theta = \tan^{-1} x.$$

Then

$$\tan(\theta) = x,$$

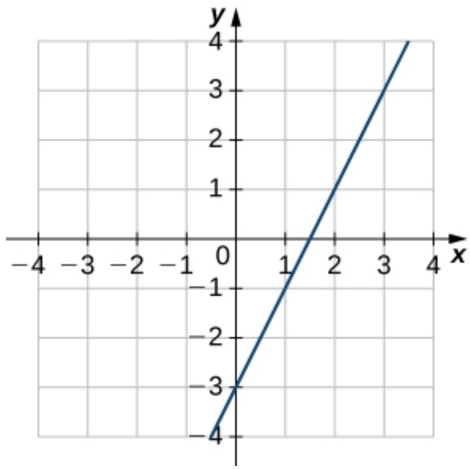
and differentiating the latter equation yields

$$\begin{aligned} \frac{d}{dx}\left[\tan(\theta)\right] &= \frac{d}{dx}\left[x\right] \\ \sec^2(\theta)\cdot\frac{d\theta}{dx} &= 1 \\ \frac{d\theta}{dx} &= \frac{1}{\sec^2(\theta)} = \frac{1}{1+\tan^2(\theta)} = \boxed{\frac{1}{1+x^2}}. \end{aligned}$$

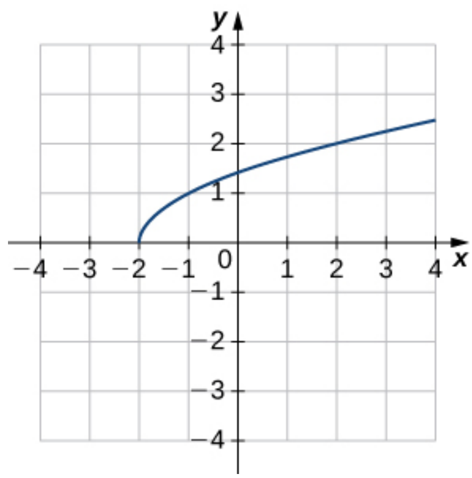
Additional exercises

Ex. 8 (§3.7—#260, 261, 262). Use the graph of $y = f(x)$ to sketch the graph of $y = f^{-1}(x)$. Then use the result to estimate $\left(f^{-1}\right)'(1)$.

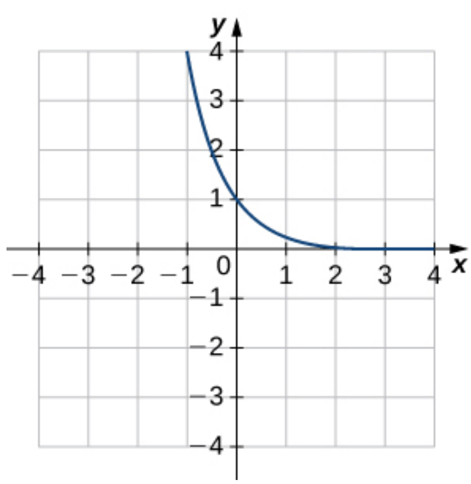
(a)



(b)



(c)



Ex. 9 (§3.7—#269). Let $f(x) = x^3 + 2x + 3$. Find $\left(f^{-1}\right)'(0)$.

Ex. 10 (§3.7—#271). Let $f(x) = x - \dfrac{2}{x}$, $x < 0$. Find $\left(f^{-1}\right)'(1)$.

Ex. 11 (§3.7—#273). Let $f(x) = \tan(x) + 3x^2$. Find $\left(f^{-1}\right)'(0)$.

Workbook Lesson 14

§3.9, Derivatives of Exponential and Logarithmic Functions

Last revised: 2021-02-18 12:45

Objectives

- Differentiate exponential functions.
- Apply Logarithm Laws.
- Differentiate logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

Question. What does it mean to raise a number to an *irrational* exponent?

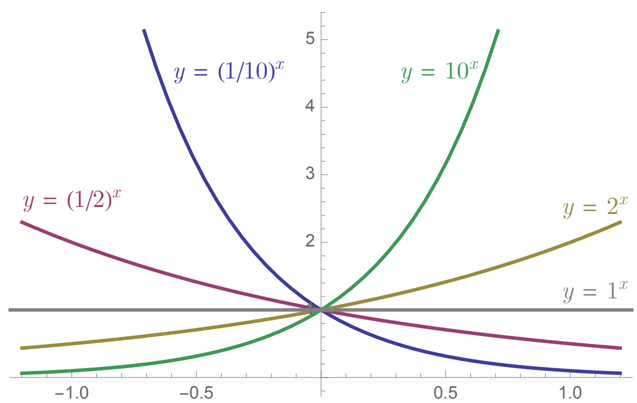
- When $n = 1, 2, 3, \ldots$, we know $3^n = \underbrace{(3)(3)\cdots(3)}_{n \text{ copies of } 3}$ by definition.
- We also know that $3^{-n} = 1/3^n$ for $n = 1, 2, 3, \ldots$.
- Also by definition, $3^{p/q} = \sqrt[q]{3^p}$ when p and q are integers, provided that $q \neq 0$.
- At this point in your mathematical career, no one has ever defined for you the meaning of an expression like 3^π (that is, an expression with an *irrational* exponent). We will do so in this lesson.

Notation: Write

$$\exp_b(x) = b^x.$$

Assumption: *(we'll prove this assumption later)* For any choice of positive real number $b > 0$, the function $\exp_b(x) = b^x$ is continuous.

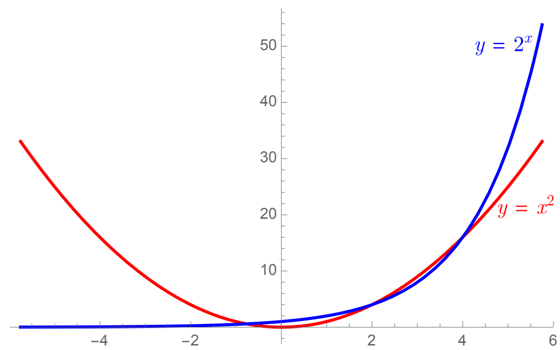
- The function $\exp_b(x) = b^x$ is called the **exponential function with base** b.
- Some authors require $b \neq 1$ in the definition of an exponential function, because the function 1^x is a bit silly—its output is a constant, 1.
- For positive $b > 1$, the exponential function models EXPONENTIAL GROWTH.
- For positive $0 < b < 1$, the exponential function models EXPONENTIAL DECAY.



- The graphs suggest it is reasonable to assume that exponential functions are continuous.

(*See applet on iCollege:* "`Graphs of exponential functions`")

☞ *Do not confuse the expression 2^x with the expression x^2.*



Informal definition of the number e

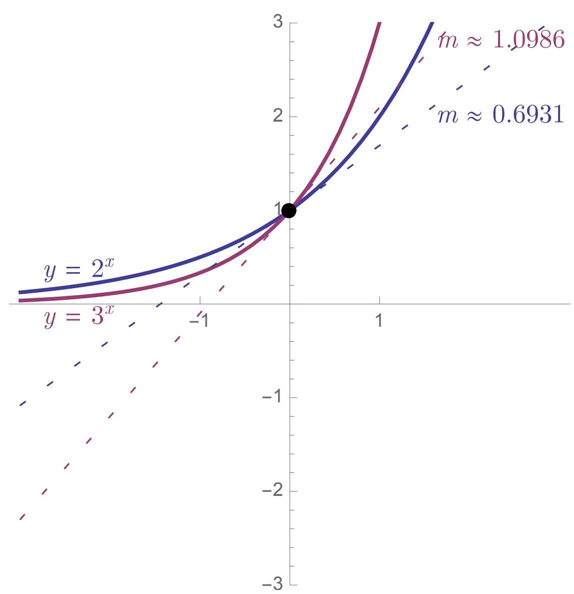
The most common definition of the irrational number $e \approx 2.7182\ldots$ is

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n.$$

However, our textbook doesn't use this definition. We'll follow the textbook's approach and define the number e informally, as follows.

Let's consider the tangent line to the exponential function $\exp_b(x) = b^x$ at the point $(0, 1)$.

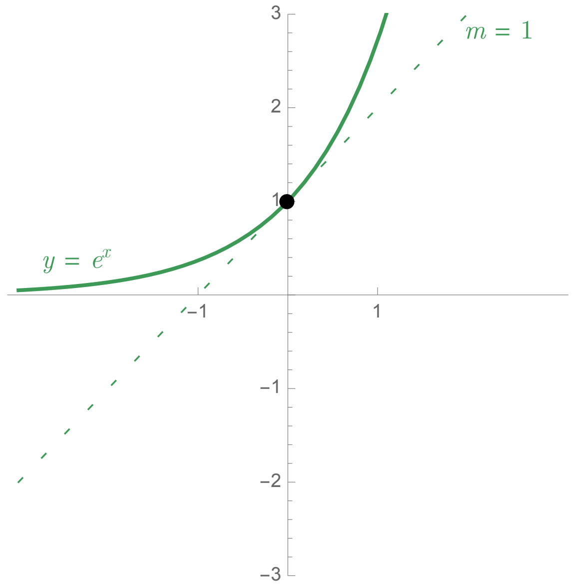
☞ Notice that the point $(0, 1)$ is on the graph of *every* exponential function, no matter what the base b is, because for any positive number b we have $b^0 = 1$.



When the base is $b = 2$, the slope of the tangent line *(dashed blue)* is a little less than 1.

When the base is $b = 3$, the slope of the tangent line *(dashed purple)* is a little greater than 1.

It stands to reason that for *some* value of b between 2 and 3, the slope of the tangent at $(0,1)$ is *exactly* 1.



We'll use this informal reasoning to define the number e. Toward the end of this lesson, we'll revisit the idea that

$$e = \lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n.$$

Definition: The number e is the real number such that the tangent line to the exponential function e^x at the point $(0,1)$ has slope $m = 1$.

☞ When the base is $b = e$, we omit the base and write

$$\exp(x) = e^x.$$

This definition of the number e only makes sense if we assume the following:

- There's only *one unique number* that satisfies our definition of the number e.
- The tangent line at $x = 0$ exists—that is, the function e^x is *differentiable* at $x = 0$.

Has our reasoning so far seemed a little shaky? Well, it is. We're making assumptions without verifying that they're true. But, as we said, we will give an alternative definition of e that doesn't rely on unproven assumptions toward the end of this lesson.

Definition of the logarithmic functions

Recall:

- The **inverse** of a function f is a function, denoted by f^{-1}, that "undoes" f.
- The fact that f^{-1} "undoes" f is expressed by the **Cancellation Formulas**:

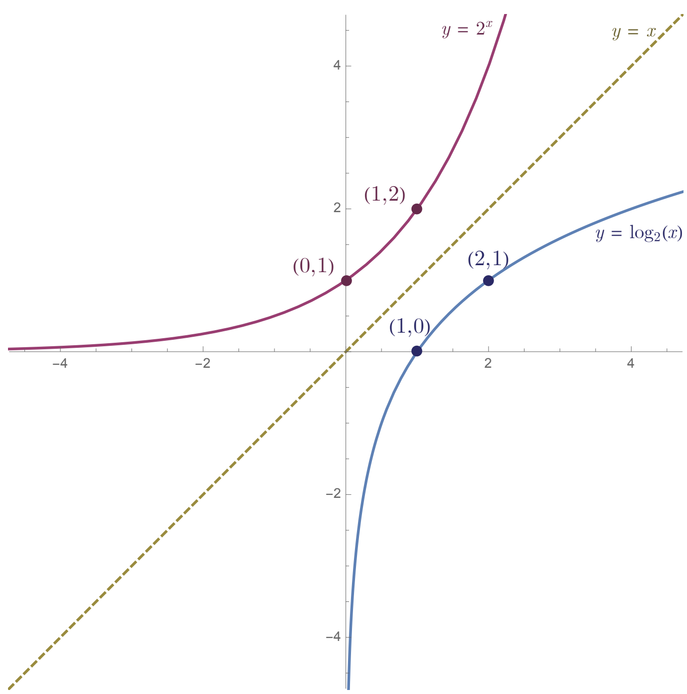
$$f^{-1}\big(f(x)\big) = x \qquad \text{for all } x \text{ in the domain of } f,$$
$$f\big(f^{-1}(y)\big) = y \qquad \text{for all } y \text{ in the range of } f.$$

- The graph of the inverse function f^{-1} is the mirror image of the graph of f reflected in the line $y = x$.

Definition. The **logarithmic function with base** b, denoted by $\log_b$, is the inverse function of the exponential function $\exp_b(x) = b^x$.

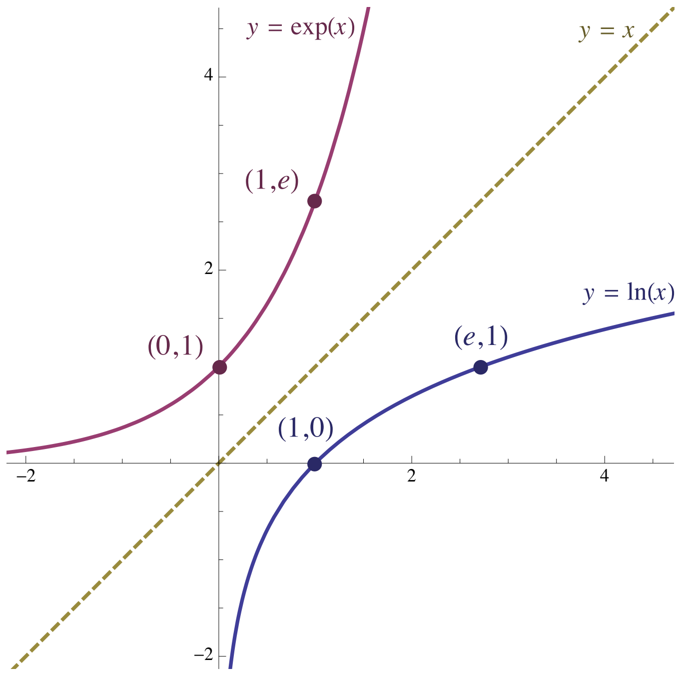
The Cancellation Formulas for an exponential function and its inverse are:

$$\log_b\Big(\exp_b(x)\Big) = \log_b\left(b^x\right) = x \qquad \text{for all real numbers } x,$$
$$b^{\log_b(y)} = y \qquad \text{for all } y > 0.$$



Some special notation:

- The logarithmic function with base 10 is denoted by $\log$ and called the **common logarithm**.
- The logarithmic function with base e is denoted by $\ln$ and called the **natural logarithm**.



Definition of b^x for all real numbers x, and continuity of b^x for any $b > 0$

As mentioned above, you have probably never been taught the precise definition of an exponential expression of the form b^x for x a *real* (possibly irrational) number. Let us now do so, using only (1) the fact that $\exp(x) = e^x$ and $\ln(x) = \log_e(x)$ are inverse functions, and (2) the rules of exponents for rational exponents.

Theorem. For $b > 0$ and rational r,

$$b^r = e^{r \ln b}.$$

Proof. By the second Cancellation Formula,

$$b = e^{\ln b} \qquad (b > 0).$$

Thus

$$\begin{aligned} b^r &= (e^{\ln b})^r \qquad (r \text{ rational}) \\ &= e^{r \ln b}. \end{aligned}$$

□

We have just proven

$$b^r = e^{r \ln b} \qquad \text{for } r \text{ rational.} \qquad (*)$$

We now extend formula (*) by defining

$$b^x = e^{x \ln b} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

The exponential function b^x has now been properly defined for any base $b > 0$. The fact that b^x is continuous for any $b > 0$ now follows by the Composite Function Theorem (Lesson 4) from the fact (which we'll assume without proof) that the natural exponential function e^x and and the function $h(x) = ax$ ($a = \text{const}$) are both continuous: $b^x = \exp\big((\ln b)x\big)$.

Derivatives of the exponential and logarithmic functions

So why do we care about e?

Here are three reasons:

- The exponential function e^x is frequently used to model real-world situations in the sciences (as you may have seen in earlier math classes).
- We can use the function e^x to define b^x for any $b > 0$ and any real number x (as seen on the previous page).
- The formula for the derivative of e^x is super easy.

The derivative of the exponential function with base e,

$$e^x,$$

is itself:

$$\frac{d}{dx}\big[e^x\big] = e^x.$$

The derivative of the exponential function with any other base is a bit more complicated:

$$\frac{d}{dx}\big[b^x\big] = b^x \ln(b).$$

We can use implicit differentiation to find the derivative of $\ln(x)$:

$$\begin{aligned}
y &= \ln(x) \\
e^y &= x && \text{(Cancellation Formula)} \\
\frac{d}{dx}\big[e^y\big] &= \frac{d}{dx}\big[x\big] \\
e^y\,\frac{dy}{dx} &= 1 \\
e^{\ln(x)}\,\frac{dy}{dx} &= 1 && \text{(Substitution: } y = \ln(x)\text{)} \\
x\,\frac{dy}{dx} &= 1 && \text{(Cancellation Formula)} \\
\frac{dy}{dx} &= \frac{1}{x} \\
\frac{d}{dx}\big[\ln(x)\big] &= \frac{1}{x}
\end{aligned}$$

We find the derivative of b^x similarly:

$$\begin{aligned} y &= \log_b(x) \\ b^y &= x && \text{(Cancellation Formula)} \\ \frac{d}{dx}\left[b^y\right] &= \frac{d}{dx}\left[x\right] \\ b^y \ln(b)\,\frac{dy}{dx} &= 1 \\ b^{\log_b(x)} \ln(b)\,\frac{dy}{dx} &= 1 && \text{(Substitution: } y = \log_b(x)\text{)} \\ x\ln(b)\,\frac{dy}{dx} &= 1 && \text{(Cancellation Formula)} \\ \frac{dy}{dx} &= \frac{1}{x\ln(b)} \\ \frac{d}{dx}\left[\log_b(x)\right] &= \frac{1}{x\ln(b)} \end{aligned}$$

To summarize:

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} \quad \text{for } x > 0$$

$$\frac{d}{dx}\left[b^x\right] = b^x \ln(b) \qquad \frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{x\ln(b)} \quad \text{for } x > 0$$

(Note that the domain of the logarithmic functions is $(0, \infty)$.)

Exercises

Ex. 1. Differentiate $y = e^{x^3+1}$.

Solution:

$$\begin{aligned} \frac{d}{dx}\left[e^{x^3+1}\right] &\overset{\text{(C.R.)}}{=} e^{x^3+1} \cdot \frac{d}{dx}\left[x^3+1\right] \\ &= \boxed{3x^2 e^{x^3+1}} \end{aligned}$$

Ex. 2. Differentiate $y = \ln(x^3+1)$.

Solution:

$$\begin{aligned} \frac{d}{dx}\left[\ln(x^3+1)\right] &\overset{\text{(C.R.)}}{=} \frac{1}{x^3+1} \cdot \frac{d}{dx}\left[\ln(x^3+1)\right] \\ &= \boxed{\frac{3x^2}{x^3+1}} \end{aligned}$$

Ex. 3. Differentiate $h(x) = xe^{2x}$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left[xe^{2x}\right] &= [x]'e^{2x} + x \cdot \left[e^{2x}\right]' \\ &= e^{2x} + x \cdot e^{2x}\,[2x]' \\ &= \boxed{e^{2x} + 2xe^{2x}}\end{aligned}$$

Ex. 4. Differentiate $f(x) = x\ln(2x)$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left[x\ln(2x)\right] &= [x]'\ln(2x) + x \cdot \big[\ln(2x)\big]' \\ &= \ln(2x) + x \cdot \frac{1}{2x}\,[2x]' \\ &= \ln(2x) + x \cdot \frac{1}{2x} \cdot (2) \\ &= \boxed{\ln(2x) + 1}\end{aligned}$$

Ex. 5. Suppose $u(x)$ is a differentiable function. Prove the formula:

$$\boxed{\frac{d}{dx}\Big[\ln\big(u(x)\big)\Big] = \frac{1}{u(x)} \cdot u'(x)}$$

Solution:

$$\frac{d}{dx}\Big[\ln\big(u(x)\big)\Big] = \frac{1}{u(x)} \cdot \frac{d}{dx}\big[u(x)\big] = \frac{1}{u(x)} \cdot u'(x).$$

Ex. 6. Compute $\dfrac{d}{dx}\left[\ln \dfrac{x+1}{\sqrt{x-2}}\right]$.

Solution:

We will apply the formula

$$\boxed{\frac{d}{dx}\big[\ln u(x)\big] = \frac{1}{u(x)} \cdot u'(x),}$$

which was proven in the previous exercise.

$$u(x) = \frac{x+1}{\sqrt{x-2}}.$$

$$\begin{aligned} u'(x) &= \frac{d}{dx}\big[(x+1)(x-2)^{-1/2}\big] \\ &= (x+1)\cdot\frac{d}{dx}\big[(x-2)^{-1/2}\big] + \frac{d}{dx}\big[(x+1)\big]\cdot(x-2)^{-1/2} \\ &= (x+1)\cdot\left(\tfrac{-1}{2}\right)(x-2)^{-3/2} + 1\cdot(x-2)^{-1/2} \\ &= -\frac{1}{2}(x+1)(x-2)^{-3/2} + 1\cdot(x-2)^{-1/2}. \\ &= -\frac{x+1}{2\sqrt{(x-2)^3}} + \frac{1}{\sqrt{x-2}}. \end{aligned}$$

$$\frac{1}{u(x)} = \frac{\sqrt{x-2}}{x+1}.$$

Applying the formula now yields

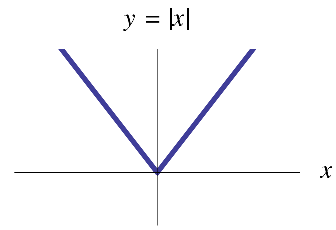
$$\begin{aligned} \frac{d}{dx}\left[\ln\big(u(x)\big)\right] &= \frac{1}{u(x)}\cdot u'(x) \\ &= \frac{(x-2)^{1/2}}{x+1}\cdot\left(-\frac{1}{2}(x+1)(x-2)^{-3/2} + (x-2)^{-1/2}\right) \\ &= \frac{1}{x+1}\cdot\left(-\frac{1}{2}(x+1)(x-2)^{-1} + 1\right) \\ &= -\frac{1}{2}(x-2)^{-1} + \frac{1}{x+1} \\ &= \frac{-1}{2(x-2)} + \frac{1}{x+1} \\ &= \frac{-(x+1)}{2(x-2)(x+1)} + \frac{2(x-2)}{2(x-2)(x+1)} \\ &= \boxed{\frac{x-5}{2(x-2)(x+1)}.} \end{aligned}$$

Ex. 7. Find $\dfrac{d}{dx}\Big[\ln|x|\Big]$.

Solution:

We will apply the formula $\boxed{\dfrac{d}{dx}\Big[\ln u(x)\Big] = \dfrac{1}{u(x)} \cdot u'(x)}$.

Note that $|x|$ is a piecewise defined function that is not differentiable at $x = 0$.
We will therefore have to calculate the derivative of $|x|$ for $x \neq 0$ piecewise.



$$u(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

$$u'(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ \text{undefined} & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

$$\frac{1}{u(x)} = \begin{cases} \dfrac{1}{x} & \text{if } x \geq 0, \\ \text{undefined} & \text{if } x = 0, \\ -\dfrac{1}{x} & \text{if } x < 0. \end{cases}$$

Applying the formula, we get

$$\frac{d}{dx}\Big[\ln|x|\Big] = \frac{1}{u(x)} \cdot u'(x) = \begin{cases} \dfrac{1}{x} \cdot 1 & \text{if } x \geq 0, \\ \text{undefined} & \text{if } x = 0, \\ -\dfrac{1}{x} \cdot (-1) & \text{if } x < 0. \end{cases}$$

$$= \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0, \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

Logarithm Laws

Recall the **Logarithm Laws**:

For $x, y > 0$ and r a real number,
(1) $\log_b(xy) = \log_b(x) + \log_b(y)$
(2) $\log_b(x/y) = \log_b(x) - \log_b(y)$
(3) $\log_b(x^r) = r\log_b(x)$

The Logarithm Laws will be needed for our next technique, called *logarithmic differentiation*.

Ex. 8. Write $\ln \dfrac{(x^2+5)^4 \sin x}{x^3+1}$ as a sum of logarithms. Cite the Logarithm Law you are using at each step.

Solution:

$$\begin{aligned}\ln \frac{(x^2+5)^4 \sin x}{x^3+1} &\overset{(2)}{=} \ln\left((x^2+5)^4 \sin x\right) - \ln(x^3+1) \\ &\overset{(1)}{=} \ln(x^2+5)^4 + \ln(\sin x) - \ln(x^3+1) \\ &\overset{(3)}{=} \boxed{4\ln(x^2+5) + \ln(\sin x) - \ln(x^3+1).}\end{aligned}$$

Ex. 9. Rewrite $\ln(a) + \frac{1}{2}\ln(b)$ as a single logarithm. Cite the Logarithm Law you are using at each step.

Solution:

$$\begin{aligned}\ln(a) + \tfrac{1}{2}\ln(b) &\overset{(3)}{=} \ln(a) + \ln\sqrt{b} \\ &\overset{(1)}{=} \boxed{\ln\left(a\sqrt{b}\right)}\end{aligned}$$

e as a limit

Theorem. $e \overset{(1)}{=} \lim_{x\to 0}(1+x)^{1/x} \overset{(2)}{=} \lim_{n\to\infty}(1+\frac{1}{n})^n$.

Proof. Let $f(x) = \ln(x)$. Then $f'(1) = \dfrac{1}{1} = 1$.

$$\begin{aligned} 1 = f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{\ln(1+h)-\cancel{\ln(1)}}{h} \\ &= \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\big[\ln(1+x)\big]^{1/x} \end{aligned}$$

Now

$$\begin{aligned} e = \exp(1) &= \exp\left(\lim_{x\to 0}\big[\ln(1+x)\big]^{1/x}\right) \\ &= \lim_{x\to 0}\left[\cancel{\exp(}\cancel{\ln}(1+x)^{1/x})\right] \\ &= \lim_{x\to 0}(1+x)^{1/x}, \end{aligned}$$

which is (1).

For (2), take $n = 1/x$. As $x \to 0^+$, we have $n \to \infty$, so

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n.$$

□

Many authors use the equation

$$e = \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n$$

as the *definition* of the number e. Then there is no need to prove the above Theorem, and:

- the natural logarithmic function
  $$\ln(x) = \log_e(x)$$
  can be defined as the inverse of the function
  $$\exp(x) = e^x,$$
- exponential functions with other bases
  $$\exp_b(x) = b^x$$
  are defined by the formula
  $$b^x = e^{b\ln(x)}, \text{ and}$$
- logarithmic functions with other bases can be defined as their inverses,
  $$\log_b(x) = \exp_b^{-1}(x).$$

Logarithmic differentiation

Consider the problem of finding the derivative of

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}.$$

Although it would be tedious, we could certainly do this using only the Power, Product, Quotient, and Chain Rules.

But there's a better way.

☞ The derivative of a function $f(x)$ that only involves products, quotients, and powers (which includes roots $\sqrt[n]{x} = x^{1/n}$) can be found using the technique of *logarithmic differentiation*.

Ex. 10. Find $\frac{dy}{dx}$ if $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution (General steps for logarithmic differentiation given in red).

(1) Take $\ln$ of both sides of the equation $y = f(x)$, and apply the Logarithm Laws.

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2).$$

(2) Find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{x^2+1}\cdot 2x\right) - 5\left(\frac{1}{3x+2}\cdot 3\right)$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

(3) Substitute y if a formula for $y = y(x)$ was given.

$$\frac{dy}{dx} = \boxed{\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)}$$

Additional exercises

Ex. 11 (§3.9—#331, 333, 337, 339, 341, 343). Differentiate.

- $f(x) = x^2 e^x$
- $f(x) = e^{x^3 \ln(x)}$
- $f(x) = 2^{4x} + 4x^2$
- $f(x) = x^\pi \cdot \pi^x$
- $f(x) = \ln \sqrt{5x - 7}$
- $f(x) = \log(\sec x)$

Ex. 12. Find the equation of the tangent line to the curve

$$y = x^4 + 2e^x$$

at the point $(0, 2)$.

Ex. 13. Find the equation of the tangent line to the curve

$$y = \ln(x)$$

at the point $(e, 1)$.

Ex. 14. Show that $\frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+1}\right)\right] = \frac{1}{\sqrt{x^2+1}}$.

Ex. 15. Use logarithmic differentiation to find the derivative of $y = x^x$.

Ex. 16. Use logarithmic differentiation to find the derivative of $y = \sqrt{\dfrac{x-1}{x^4+1}}$.

Ex. 17 (§3.9—#357a). At which points on the graph of

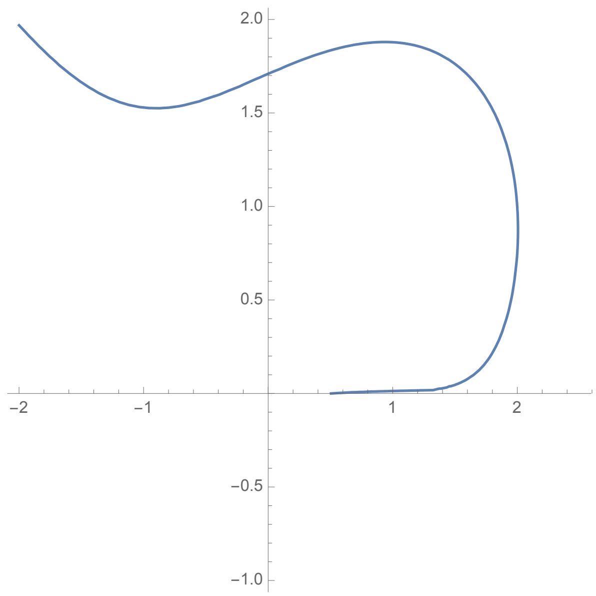
$$y = x^{1/x} \qquad (x > 0)$$

is the tangent line horizontal?

Ex. 18 (§3.9—#356). Find the equation of the tangent line to the curve

$$x^3 - x\ln(y) + y^3 = 2x + 5$$

at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\dfrac{dy}{dx}$.)



Workbook Lesson 15

§4.1, Related Rates

Last revised: 2021-06-03 14:09

Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the Chain Rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

Strategy for Solving a Related Rates Problem

1. Write a **Legend** that assigns symbols to all variables involved in the problem. The Legend should also state the meaning of each variable. Draw a picture, if applicable.

2. Identify the independent variable. Then state, in terms of the variables, the information that is given and the rate to be determined.

3. Find an equation relating the variables in the **Legend**.

4. Differentiate both sides of the equation found in Step 3 with respect to the independent variable.

5. Substitute all known values from Step 2 into the equation from Step 4, then solve for the unknown rate of change.

Worked example

Ex. 1. Consider a 10 foot ladder that is leaning against a wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Recall that the velocity of a object that is moving over time t is given by the formula

$$\text{velocity} = \frac{d}{dt}\big[\text{position function}\big].$$

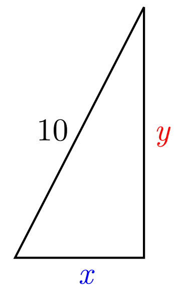
So, if y is the distance from the top of the ladder to the ground, then the speed at which the top of the ladder slides down the wall is $\frac{dy}{dt}$.

Similarly, the speed at which the bottom of the ladder slides away from the wall is $\frac{dx}{dt}$.

Step 1: Write a **Legend**. Draw a picture, if applicable.

Legend:

$$x = \text{distance from bottom of ladder to wall}$$
$$y = \text{distance from top of ladder to ground}$$
$$t = \text{time}$$
$$\frac{dy}{dt} = \text{velocity of top of ladder, sliding down wall}$$
$$\frac{dx}{dt} = \text{velocity of bottom of ladder, sliding away from wall}$$



Step 2: Identify the independent variable. Then state, in terms of the variables, the information that is given and the rate to be determined.

- The independent variable is time t.
- The desired rate of change is $\frac{dy}{dt}$.
- The given information is:

$$\text{length of ladder} = 10$$
$$\frac{dx}{dt} = \text{velocity of bottom of ladder} = 1$$
$$x = \text{distance of bottom of ladder from wall} = 6$$

Step 3: Find an equation relating the variables in the **Legend**.

By Pythagorean Theorem,

$$x^2 + y^2 = 10^2. \qquad (\star)$$

Step 4: Differentiate both sides of the equation found in Step 3 with respect to the independent variable.

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[10^2\right]$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \text{(Chain Rule)}$$

Step 5: Substitute all known values from Step 2 into the equation from Step 4, then solve for the unknown rate of change.

$$2x\frac{dx}{dt}\bigg|_{x=6} + 2y\frac{dy}{dt}\bigg|_{x=6} = 0$$
$$2(6)(1) + 2y\frac{dy}{dt} = 0$$
$$y\frac{dy}{dt} = -6$$

What is y when $x = 6$? We can use equation ($\star$) to find out:

$$\begin{aligned} 36 + y^2 &= 100 \\ y &= 8 \end{aligned}$$

Now we can solve for the desired rate of change, $\dfrac{dy}{dt}$:

$$\begin{aligned} 8\frac{dy}{dt} &= -6 \\ \frac{dy}{dt} &= -\frac{6}{8} = -\frac{3}{4} \end{aligned}$$

The top of the ladder is sliding down the wall at a rate of 9 inches ($= \frac{3}{4}$ feet) per second.

An activity is presented in the remainder of this document. It may be done in groups of students, or on your own.

- On the next page, you'll find eight exercises. (We've already done **Ex. 1**.)
- For each exercise, write a Legend (and draw a picture, if applicable).
- The subsequent page gives the instructor's Legends for each problem.
  - *Do not look at the Legends until you've made a real effort to come up with your own.*
  - Compare your Legend with the instructor's Legend.
  - It's okay if they're not exactly the same—we may use different words and variable names to express the same relationships.
- The remainder of this document presents the instructor's solutions for each of the exercises.
  - *Do not look at the instructor's solution until you've made your best effort at figuring out your own solution.*

Additional practice exercises can be found in the Final Exam Review on iCollege (under Section 4.1).

Related rates—Exercises

Ex. 1. Consider a 10 foot ladder that is leaning against a wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Ex. 2. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that a certain instant the volume is 600 cm^3, the pressure is 150 kilopascals (kPa), and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

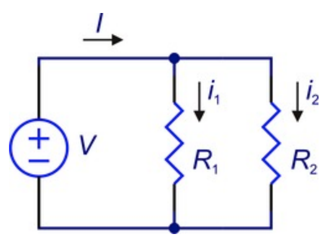
Ex. 3. The length of a rectangle is increasing at a rate of 8 cm/s. Its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Ex. 4. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

(a) At what rate is his distance to second base decreasing when he is halfway to first base?

(b) At what rate is his distance to third base increasing when he is halfway to first base?

Ex. 5. In an electrical circuit, two resistors with resistances R_1 and R_2 are connected in parallel as shown.



The total resistance R in ohms (Ω) satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Suppose R_1 and R_2 are increasing at rates of 0.3 Ω/sec and 0.2 Ω/sec respectively. How fast is R changing when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$?

Ex. 6. A plane flying at a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30°$. At what rate is the distance from the plane to the radar station increasing one minute later? *(Hint: Draw a picture, then use the Law of Cosines.)*

Ex. 7. A sprinter runs away from a tall stadium light. The light is 30 feet from the ground. If the sprinter is 6 feet tall, and runs at 24 feet per second, at what rate does her shadow grow longer? *(Hint: Use similar triangles.)*

Ex. 8. Water is leaking out of an inverted (i.e. upside-down) conical tank at a rate of $10,000$ cm^3/min. At the same time, water is pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. When the height of the water is 2 m, the water level is rising at a rate of 20 cm/min. Find the rate at which water is being pumped into the tank. *(This exercise requires the formula for the volume of a right circular cone, $V = \frac{1}{3} \times (\text{AREA OF BASE}) \times (\text{HEIGHT}) = \frac{1}{3}\pi r^2 h$.)*

Related rates—Legends

Ex. 2.

$P =$ pressure

$V =$ volume

$t =$ time (independent variable)

$\frac{dP}{dt} =$ change in pressure

$\frac{dV}{dt} =$ change in volume

Ex. 3.

$\ell =$ length

$w =$ width

$A =$ area

$t =$ time (independent variable)

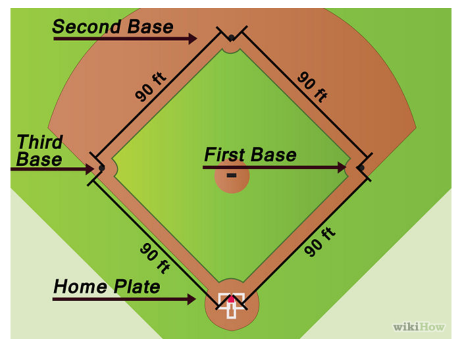
Ex. 4.

$x =$ runner's distance to first base

$y =$ runner's distance to second base

$z =$ runner's distance to third base

$t =$ time (independent variable)



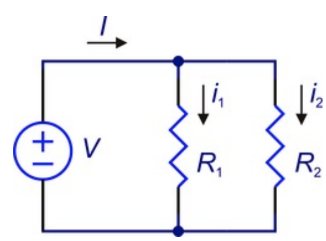
Ex. 5.

$R_1 =$ resistance in resistor #1

$R_2 =$ resistance in resistor #2

$R =$ total resistance

$t =$ time (independent variable)



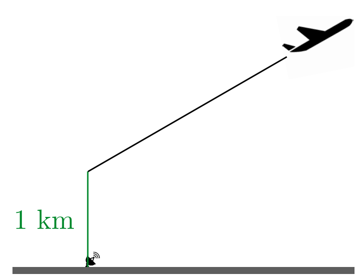
Ex. 6.

$y =$ *initial* height of plane (constant)

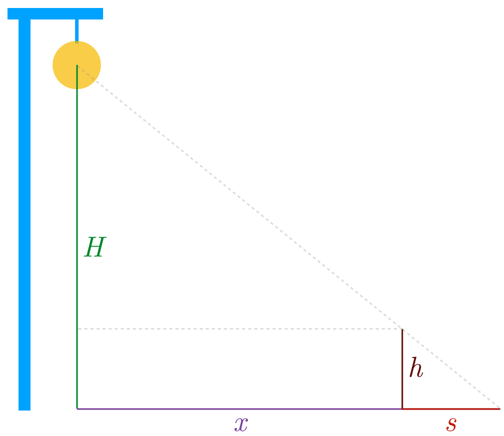
$x =$ distance traveled by the plane (variable)

$D =$ distance from plane to radar station (variable)

$t =$ time (independent variable)



Ex. 7.



$H =$ height of light (constant)

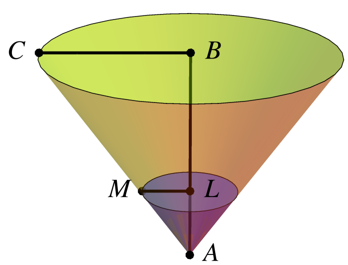
$h =$ height of sprinter (constant)

$x =$ distance from sprinter to base of light (increasing)

$s =$ length of shadow (increasing)

$t =$ time (independent variable)

Ex. 8.



C: rate at which water is pumped into tank (constant)

V: volume of water in the tank (increasing)

t: time (independent variable)

$\dfrac{dV}{dt}$: rate of change in volume of water in the tank

$h = AL$: height of water (rising)

$r = LM$: radius of surface of water (increasing)

$6 = AB$: height of cone (constant)

$2 = BC$: radius of cone's base (constant)

Related rates—Solutions

Ex. 2. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that a certain instant the volume is 600 cm^3, the pressure is 150 kilopascals (kPa), and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

Step 1 (Legend):

$P =$ pressure

$V =$ volume

$t =$ time (independent variable)

$\frac{dP}{dt} =$ change in pressure

$\frac{dV}{dt} =$ change in volume

Step 2 (State what's known and what's asked for, in terms of the variables):

$$\frac{dP}{dt} = 20$$

$\frac{dV}{dt} = \boxed{?}$ when $V = 600$ and $P = 150$

Step 3 (Equation relating the variables): $PV = C$ $\quad$ ($C =$ const)

Step 4 (Differentiate with respect to the independent variable):

$$\frac{dP}{dt}V + P\frac{dV}{dt} = 0 \quad \text{(Product \& Chain Rules)}$$

Step 5 (Substitute what's known and find the rate that was asked for):

$$\left.\frac{dV}{dt}\right|_{\substack{V=600\\P=150}} = -\frac{600}{150} \cdot 20 = 80$$

The volume is decreasing at 80 cm^3/min.

Ex. 3. The length of a rectangle is increasing at a rate of 8 cm/s. Its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Step 1 (Legend):

$$\ell = \text{length}$$
$$w = \text{width}$$
$$A = \text{area}$$
$$t = \text{time (independent variable)}$$

Step 2 (State what's known and what's asked for, in terms of the variables):

$$\frac{d\ell}{dt} = 8$$
$$\frac{dw}{dt} = 3$$
$$\frac{dA}{dt} = \boxed{?} \text{ when } \ell = 20 \text{ and } w = 10$$

Step 3 (Equation relating the variables):

$$A = \ell w$$

Step 4 (Differentiate with respect to the independent variable):

$$\frac{dA}{dt} = \frac{d}{dt}[\ell w]$$
$$\frac{dA}{dt} = w\frac{d\ell}{dt} + \ell\frac{dw}{dt}$$

Step 5 (Substitute what's known and find the rate that was asked for):

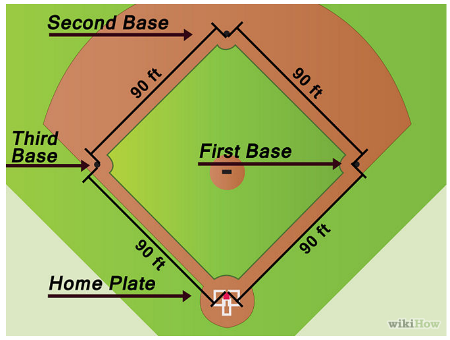
$$\left.\frac{dA}{dt}\right|_{\substack{\ell=20\\w=10}} = 10\left.\frac{d\ell}{dt}\right|_{\substack{\ell=20\\w=10}} + 20\cdot\left.\frac{dw}{dt}\right|_{\substack{\ell=20\\w=10}}$$
$$\left.\frac{dA}{dt}\right|_{\substack{\ell=20\\w=10}} = 20(3) + 3(8) = 140$$

The area of the rectangle is increasing at 140 cm per second.

Ex. 4. A baseball diamond is a square with side 90 ft. *(See figure below.)* A batter hits the ball and, starting at home plate, runs toward first base with a speed of 24 ft/s.

(a) At what rate is his distance to second base decreasing when he is halfway to first base?

(b) At what rate is his distance to third base increasing when he is halfway to first base?



Legend:

$$\begin{aligned} x &= \text{runner's distance to first base} \\ y &= \text{runner's distance to second base} \\ z &= \text{runner's distance to third base} \\ t &= \text{time (independent variable)} \end{aligned}$$

(a)

$$\begin{aligned} x^2 + 90^2 &= y^2 \\ 2xx' &= 2yy' \\ y' &= \frac{xx'}{y} \end{aligned}$$

$$\begin{aligned} y\Big|_{x=45} &= \sqrt{45^2 + 90^2} = \sqrt{45^2 + 4 \cdot 45^2} = 45\sqrt{5} \\ y'\Big|_{x=45} &= \frac{45(-24)}{45\sqrt{5}} = \frac{-24}{\sqrt{5}} \end{aligned}$$

The runner's distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}}$ ft/s.

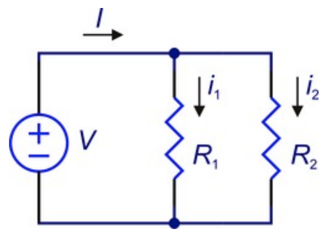
(b) For this part of the problem, we'll let $w =$ runner's distance from home plate.

$$\begin{aligned} w^2 + 90^2 &= z^2 \\ z' &= \frac{ww'}{z} \end{aligned}$$

$$\begin{aligned} z\Big|_{w=45} &= 45\sqrt{5} \\ y'\Big|_{w=45} &= \frac{45(24)}{45\sqrt{5}} = \frac{24}{\sqrt{5}} \end{aligned}$$

The runner's distance from third base is increasing at a rate of $\frac{24}{\sqrt{5}}$ ft/s.

Ex. 5. If two resistors with resistances R_1 and R_2 are connected in parallel as shown,



then the total resistance R in ohms (Ω) satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Suppose R_1 and R_2 are increasing at rates of $0.3\ \Omega$/sec and $0.2\ \Omega$/sec respectively. How fast is R changing when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$?

Legend:

$R_1 =$ resistance in resistor #1
$R_2 =$ resistance in resistor #2
$R =$ total resistance
$t =$ time (independent variable)

We find the relation between $R_1, R_2, \dfrac{dR_1}{dt}, \dfrac{dR_2}{dt}$, and $\dfrac{dR}{dt}$.

$$\frac{d}{dt}\left[R^{-1}\right] = \frac{d}{dt}\left[{R_1}^{-1} + {R_2}^{-1}\right]$$

$$-R^{-2}\frac{dR}{dt} = -{R_1}^{-2}\frac{dR_1}{dt} - {R_2}^{-2}\frac{dR_2}{dt}$$

Solve for $\dfrac{dR}{dt}$ by multiplying both sides in the previous equation by $-R^2$.

$$\frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}\right) \qquad (***)$$

When $R_1 = 80$ and $R_2 = 100$, the value of R is

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100} = \frac{9}{400}.$$

Now substitute $R_1 = 80$, $R_2 = 100$, $dR_1/dt = 0.3$, and $dR_2/dt = 0.2$ into (***).

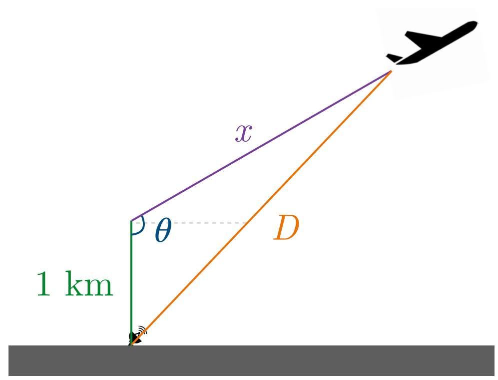
$$\frac{dR}{dt} = \left(\frac{400}{9}\right)^2\left(\frac{1}{80^2}\frac{3}{10} + \frac{1}{100^2}\frac{2}{10}\right) = \frac{107}{810}$$

R is changing at $\dfrac{107}{810} \approx 0.132099\ \Omega$/sec.

Ex. 6. A plane flying at a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing one minute later?

Legend:

$x =$ distance traveled by the plane (variable)
$y =$ *initial* height of plane (constant)
$D =$ distance from plane to radar station (variable)
$t =$ time (independent variable)



By the Law of Cosines, writing $\theta =$ (angle opposite D) $= 120° = 2\pi/3$,

$$D^2 = x^2 + y^2 - 2xy\cos\theta$$

$$D^2 = x^2 + 1 - 2x\left(-\tfrac{1}{2}\right) \qquad (y = 1,\ \theta = 2\pi/3)$$

$$D^2 = x^2 + x + 1$$

By implicit differentiation,

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{2x+1}{2D}\frac{dx}{dt}$$

After 1 minute,

$$x = \frac{300 \text{ km}}{60 \text{ min}} = 5 \text{ km},$$

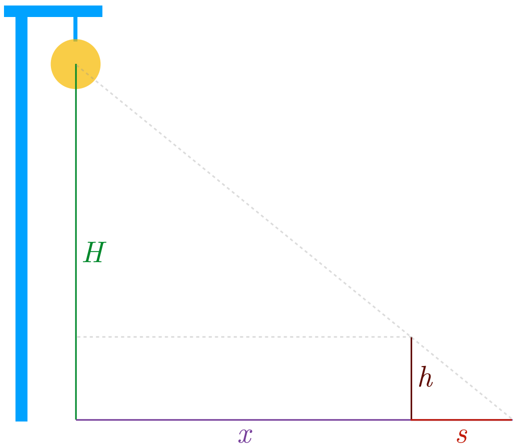
so

$$D = D(5) = \sqrt{5^2 + 5 + 1} = \sqrt{31},$$

giving

$$\frac{dD}{dt} = \frac{2(5)+1}{2\sqrt{31}} \cdot 300 = \frac{3300}{2\sqrt{31}} \approx \boxed{296.349 \text{ km/h}}.$$

Ex. 7. A sprinter runs away from a tall stadium light. The light is 30 feet from the ground. If the sprinter is 6 feet tall, and runs at 24 feet per second, at what rate does her shadow grow longer?



Legend:

$H =$ height of light (constant)
$h =$ height of sprinter (constant)
$x =$ distance from sprinter to base of light (increasing)
$s =$ length of shadow (increasing)
$t =$ time (independent variable)

By similar triangles,

$$\frac{H-h}{x} = \frac{h}{s}.$$

We solve for s:

$$s = \frac{h}{H-h}x.$$

Then

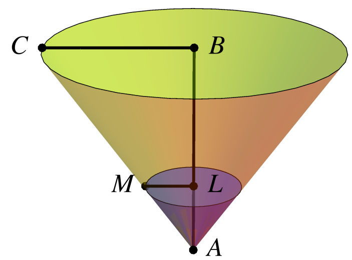
$$\frac{ds}{dt} = \frac{h}{H-h}x'$$

and when $h = 6$, $H = 30$, and $x' = 24$, we get

$$\frac{ds}{dt} = \frac{6}{30-6} \cdot 24 = \boxed{6 \text{ ft/sec}}.$$

Ex. 8. Water is leaking out of an inverted (i.e. upside-down) conical tank at a rate of $10,000$ cm^3/min. At the same time, water is pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. When the height of the water is 2 m, the water level is rising at a rate of 20 cm/min. Find the rate at which water is being pumped into the tank.

Given the points labeled as shown, we set up the legend.



Legend:

C:	rate at which water is pumped into tank (constant)
V:	volume of water in the tank (increasing)
t:	time (independent variable)
$\frac{dV}{dt}$:	rate of change in volume of water in the tank

$h = AL$:	height of water (rising)
$r = LM$:	radius of surface of water (increasing)
$6 = AB$:	height of cone (constant)
$2 = BC$:	radius of cone's base (constant)

We have

$$\text{(rate of change in volume of water)} = \text{(speed of inflow)} - \text{(speed of outflow)}$$

that is,

$$\frac{dV}{dt} = C - 10,000.$$

The formula for the volume of a cone tells us

$$V = \frac{1}{3}(\text{area of base})(\text{height}) = \frac{1}{3}\pi(LM)^2(AL) = \frac{1}{3}\pi r^2 h.$$

Is r a function of h? Yes: by similar triangles,

$$\frac{LM}{AL} = \frac{BC}{AB},$$

so

$$r = LM = \frac{BC}{AB}AL = \frac{1}{3}h.$$

Thus

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3^3}\pi h^3, \qquad \frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}.$$

Noting that $h = 2$ m $= 200$ cm (must make length units of h and dh/dt match),

$$\left.\frac{dV}{dt}\right|_{\substack{h=200 \\ \frac{dh}{dt}=20}} = \frac{1}{9}\pi(200)^2 20 = \frac{800,000\pi}{9}.$$

Water is being pumped into the tank at a rate of $C = 10,000 + \frac{800,000\pi}{9} \approx 289,253$ cm^3/min.

Additional exercises

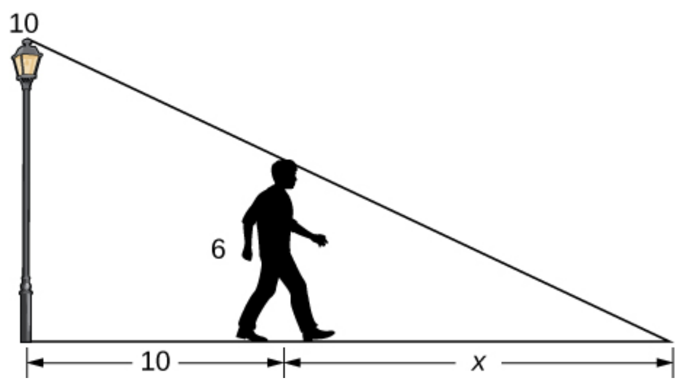
Ex. 9 (§4.1—#17). The volume of a cube decreases at a rate of 10 m3/s. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Ex. 10 (§4.1—#19). Recall that, in general, the surface area of a sphere with radius r is

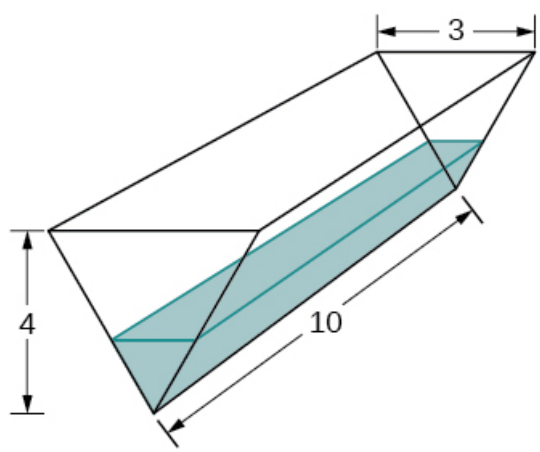
$$A = 4\pi r^2.$$

The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

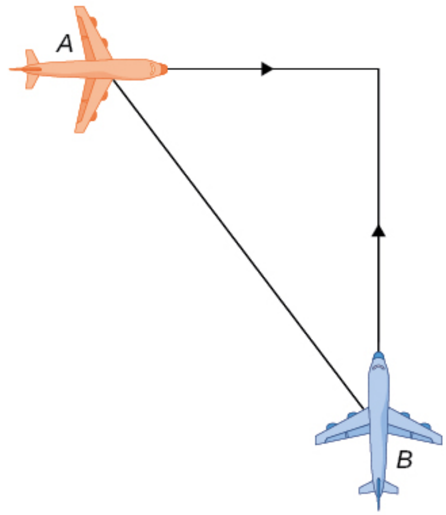
Ex. 13 (§4.1—#11). A 6-foot-tall person walks away from a 10-ft. lamppost at a constant rate of 3 ft./sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft. away from the pole?



Ex. 11 (§4.1—#30). A trough has ends shaped like isosceles triangles with width 3 m and height 4 m. The trough is 10 m long. Water is being pumped into the trough at a rate of 5 m^3/min. At what rate does the height of the water change when the water is 1 m deep?

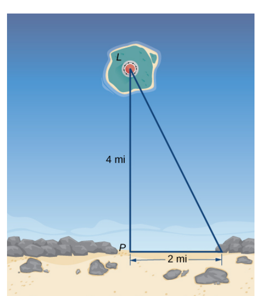


Ex. 12 (§4.1—#7). Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



Ex. 14 (§4.1—#37). You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

Ex. 14 (§4.1—#39). A lighthouse (L) is on an island 4 mi away from the closest point, P, on the beach *(see image)*. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



Workbook Lesson 16

§4.2, Linear approximations and differentials

Last revised: 2020-09-29 12:44

Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

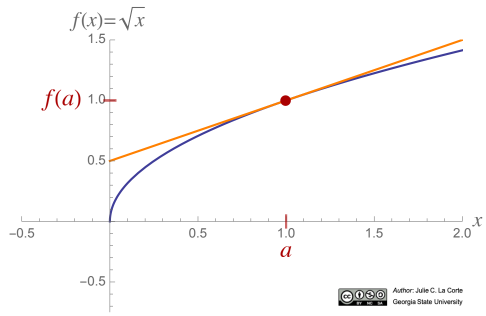
Linear approximation to a differentiable function

Linear functions are simpler and easier to work with than nonlinear functions. It therefore makes sense to want to replace a nonlinear function by a linear approximation.

For a DIFFERENTIABLE function f, the linear approximation is guaranteed to be a good approximation *locally*—that is, for x NEAR some fixed input value a.

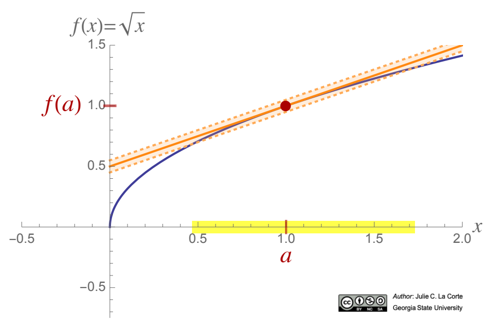
What exactly does this mean?

- The linear approximation to a function $f(x)$ AT THE POINT $x = a$ is given by the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

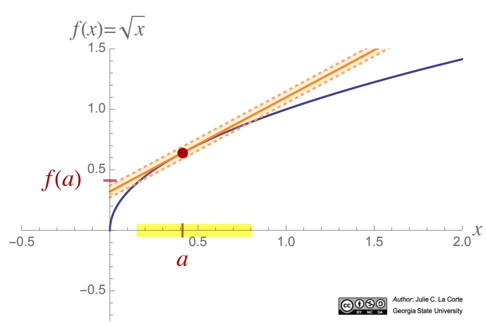


- Suppose we need our approximation to be within a certain tolerance—say, $\varepsilon = 0.05$—of $f(x)$.

- For x "sufficiently near" a, the linear approximation (height of the tangent line) is within ε of the value of f (height of the graph of $y = f(x)$).



- How near to a must x be for the linear approximation to be within the specified tolerance ε? That depends on the tangent point. That is, if we change a, we may need x to be closer to a in order for the linear approximation to be "close enough."



(*See applet on iCollege:* "`Linear approximation`")

Recall:

The equation of the **tangent line** to the graph of f at the point $\big(a, f(a)\big)$ is:

$$y = f'(a)(x - a) + f(a)$$

Let's rewrite this in function notation, and call it the **linear approximation to** (or **linearization**, or **tangent line approximation of**) f **near** a:

$$L(x) = f'(a)(x - a) + f(a)$$

When we say that "the linear approximation is CLOSE to f for x near a," what we are saying is that

$$|f(x) - L(x)| < \varepsilon$$

for all x sufficiently near a.

Ex. 1.

(a) Find the linear approximation $L(x)$ to the function $f(x) = \sqrt{x}$ near 1.

(That is, find the equation of the tangent line to the graph of f at $\big(1, f(1)\big)$ and write it in function notation $L(x) = \cdots$.)

(b) Using a calculator to evaluate $L(9.1)$, approximate the value of $f(9.1) = \sqrt{9.1}$.

You may say, why are we bothering with this linear approximation stuff, if in the end we're just going to use a calculator to evaluate $\sqrt{9.1}$?

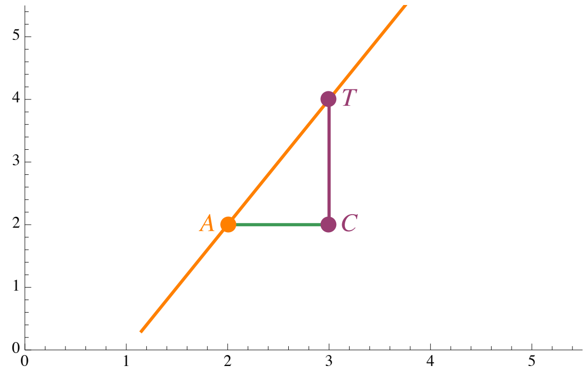
That's a fair question. The answer is that, when we are dealing with a complicated function f, it is often the case that evaluating the linear approximation L is *faster*, *easier*, and yields an approximation that is *close enough for practical purposes* to the true value.

Here are some examples of situations in which a linear approximation is used instead of an exact calculation:

- In computer animation, when the position of many moving objects must be evaluated many times per second (nonlinear functions require faster, more expensive graphics cards and processors)
- In engineering, when an irregular shape can be approximated by a flat shape (after all, there are no *perfectly* smooth surfaces in the real world)
- In physics, when a theoretical calculation by hand becomes much, much easier if a nonlinear function is replaced by a linear function (e.g., $\sin(x) \approx x$ near $x = 0$)
- In statistics, when a simple description of a general trend is desired (a line is easier to intuitively understand than a complicated curve)
- In economics, when an observed trend is "jittery" due to frequent fluctuations (think of the stock market) and a short-term prediction is sought

The differential of a function

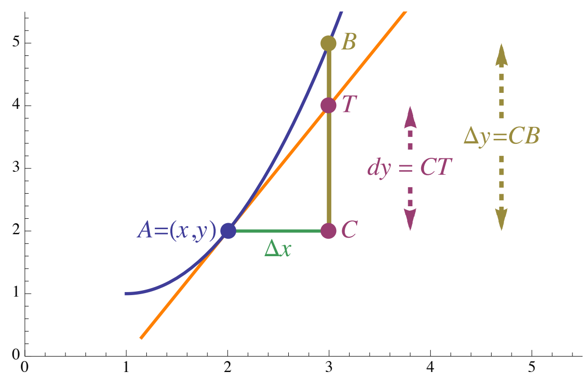
Recall: In the following figure, the slope of the orange line is $\dfrac{CT}{AC}$.



Consider the differentiable function $y = f(x)$ whose graph is shown below.

Starting at any point $A = (x, y)$ on the graph, let x increase (or decrease) by adding a small number $\Delta x \neq 0$ (positive or negative) to x. What are the coordinates of the point B whose horizontal coordinate is Δx larger than x?

Let $\Delta y = CB$ be the corresponding change in y. That is, $\Delta y = f(x + \Delta x) - f(x)$.



The slope of the tangent line $\overleftrightarrow{AT}$ to f at $A = (x, y)$ is

$$\frac{CT}{\Delta x} = f'(x).$$

Since, near x, the graph of $y = f(x)$ is close to the tangent line $\overleftrightarrow{AT}$ of f at x, we see that when Δx is small, we have

$$\Big(\text{actual change in } y = f(x)\Big) = \Delta y \approx CT = \Big(\text{change in height of tangent line of } f \text{ at } x\Big),$$

so

$$\Delta y \approx f'(x) \cdot \Delta x. \qquad (\star)$$

Definition. The **differential** of the function $y = f(x)$ is: $dy \overset{\text{def}}{=} f'(x) \cdot \Delta x$.

☞ Since the differential of the function $y = x$ is

$$dx = 1 \cdot \Delta x = \Delta x,$$

we often write the definition of the differential as

$$\boxed{dy = f'(x)\,dx.}$$

☞ It should be kept in mind that the variable dy depends on (the independent variables) x and dx. So $dy = dy(x, dx)$, that is, dy is really a function of two variables.

Ex. 2.

- $d(\sin x) = \cos(x)\,\Delta x = \cos(x)\,dx$.
- If $y = 6x^2 + 3$, then $dy = 12x\,dx$.
- If $A(r) = \pi r^2$, then $dA = 2\pi r\,dr$.

☞ Until this point, we have not regarded the symbol $\dfrac{dy}{dx}$ as a fraction. Now that we have defined the differential, it is clear that we can think of $\dfrac{dy}{dx} = f'(x)$ as an ordinary fraction.

Ex. 3. Compute approximately the volume of metal in a hollow spherical shell of thickness 0.05 in., with inside radius 5 in.

Solution. The volume of the metal in the shell is the amount by which the volume of a sphere increases when its radius changes from 5 to 5.05 in.

Using the formula for the volume of a sphere,

$$V = \tfrac{4}{3}\pi r^3,$$

we find that

$$dV = 4\pi r^2\,dr.$$

Taking $r = 5$ and $dr = 0.05$, we find

$$dV = 4\pi(25)(0.05) \approx 4 \times 3.14 \times 25 \times 0.05 = \boxed{15.7 \text{ in}^3}$$

The exact value of the volume of the shell is

$$\int_5^{5.05} dV = V(5.05) - V(5) = 15.865566418484217\ldots$$

cubic inches. (The meaning of the symbol $\int$ will be explained in Chapter 5—for now, let's just accept this calculation on faith.) The error in our estimate was < 0.17 in^3.

Ex. 4. Approximate $(1.98)^5$ by hand.

Solution.

Take

$$y = x^5.$$

Then

$$dy = 5x^4\, dx.$$

Taking $x = 2$ and $dx = -0.02$, we find

$$dy = 5(16)(-0.02) = -\frac{80}{50} = -\frac{8}{5} = -1.6.$$

This means that $y = x^5$ decreases by 1.6 when x decreases from 2 to 1.98. Hence

$$(1.98)^5 \approx 32 - 1.6 = \boxed{30.4}.$$

The actual value is $30.431681596799997\ldots$.

Calculating the amount of error

Consider a function f with an input that is a measured quantity.

Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the **measurement error** is dx (or Δx).

As a result of the error in measurement, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly.

However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is differentiable at a, then the propagated error is

$$\Delta y \approx dy = f'(a)\, dx.$$

We do not know what a is—we only know the measured value $a + dx$. However, provided that $a + dx \approx a$ (that is, the measurement error dx is small), we have

$$\Delta y \approx dy \approx f'(a + dx)\, dx.$$

Ex. 5. Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

(a) Use differentials to estimate the error in the computed volume of the cube.

(b) Compute the volume of the cube if the side length is 4.9 cm to compare the estimated error with the actual potential error.

Relative error and percentage error

The measurement error $dx = \Delta x$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error *relative to the size of the quantity being measured or calculated*.

In general, if a measured quantity q has an absolute error Δq, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the quantity's true value.

The **percentage error** is the relative error expressed as a percentage.

- For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is $63 - 62 = 1$ in., but the relative error is

$$\frac{1}{62} = 0.016,$$

  or 1.6%.

- By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is

$$0.258 = \frac{1}{32},$$

  or 3.1%.

- Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Ex. 6. An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ±80 mi. Use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution:

If the measurement of the radius is accurate to within ±80, we have

$$-80 \leq dr \leq 80.$$

We know from a previous exercise that the differential of the volume V of a sphere is

$$dV = 4\pi r^2\, dr.$$

Using the measured radius of 4000 mi, we can estimate bounds on the propagated error dV:

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80)$$

To estimate the relative error $\frac{dV}{V}$, use the measured radius $r = 4000$ mi. to estimate V:

$$V \approx \frac{4}{3}\pi(4000)^3.$$

Therefore,

$$-.06 = \frac{-4\pi(4000)^2(80)}{\frac{4}{3}\pi(4000)^3} \le \frac{dV}{V} \le \frac{4\pi(4000)^2(80)}{\frac{4}{3}\pi(4000)^3} = .06.$$

The relative error is $.06$ and the percentage error is 6%.

Additional exercises

Ex. 7 (§4.2—#51). Find the linear approximation $L(x)$ to $f(x) = \dfrac{1}{x}$ at $a = 2$.

Ex. 8 (§4.2—#53). Find the linear approximation $L(x)$ of $f(x) = \sin(x)$ at $a = \frac{\pi}{2}$.

Ex. 9 (§4.2—#55). Find the linear approximation $L(x)$ of $f(x) = \sin^2(x)$ at $a = 0$.

Ex. 10 (§4.2—#69, 71). Find the differential of the function.

(a) $y = \cos(x)$

(b) $y = \dfrac{x^2 + 2}{x - 1}$

Ex. 11. Find the differential of the function $y = \ln(\cos(\theta))$.

Ex. 12 (§4.2—#73). Find the differential of $y = \dfrac{1}{x+1}$ and evaluate at $x = 1$ and $dx = 0.25$.

Ex. 13 (§4.2—#79). Find the change in volume, dV, if the sides of a cube change from x to $x + dx$.

Ex. 14 (§4.2—#81). Find the change in volume, dV, if the radius of a sphere changes from r to $r + dr$.

Ex. 15 (§4.2—#84). A spherical ball is measured to have a radius of 5mm, with a possible measurement error of 0.1mm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing the volume of the ball.

Ex. 16 (§4.2—#85). A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Workbook Lesson 17

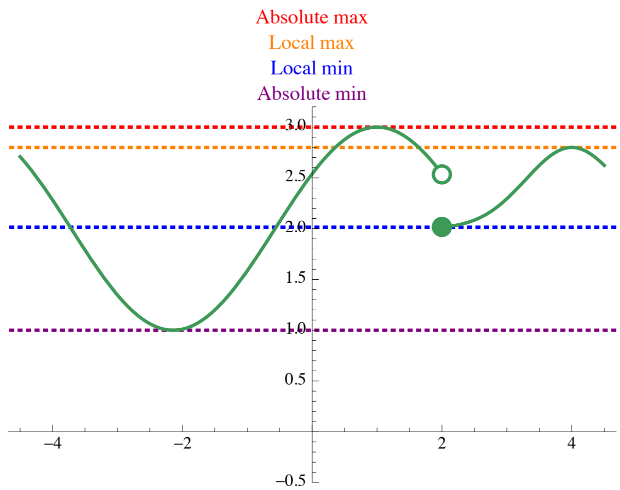
§4.3, Maxima and Minima

Last revised: 2021-03-02 11:34

Objectives

- Define absolute extrema and local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Extreme values



Let $y = f(x)$ be a function. Let c be a number in the domain of f. The value $f(c)$ is

- a **global** (or **absolute**) **maximum value** of f if

$$f(x) \leq f(c) \text{ for all } x \text{ in the domain of } f.$$

- a **global** (or **absolute**) **minimum value** of f if

$$f(x) \geq f(c) \text{ for all } x \text{ in the domain of } f.$$

- a **local** (or **relative**) **maximum value** of f if for some $d > 0$,

$$f(x) \leq f(c) \text{ for all } x \text{ in the domain of } f \text{ such that } |x - c| < d.$$

- a **local** (or **relative**) **minimum value** of f if for some $d > 0$,

$$f(x) \geq f(c) \text{ for all } x \text{ in the domain of } f \text{ such that } |x - c| < d.$$

The global and local maxima and minima are the **extreme values** (or **extrema**) of f.

Note: The conditions in (3) and (4) are sometimes stated "…for all x in the domain of f **near** c,' or "for all x in an interval containing c." (See Lesson 2.5, first Objective.)

Fact. Every global maximum of f is also a local maximum of f.

☞ Note the difference between an *extreme value* (output of a function—say, $f(c)$) and the input value (say, c) that corresponds to that value.

Ex. 1. What are the *extreme values* (that is, the global and local maxima and minima) of these functions?

- $h(x) = 3(x-2)^2 + 1$
- $f(x) = \cos(x)$
- $g(x) = 4/x$
- $j(x) = 0$

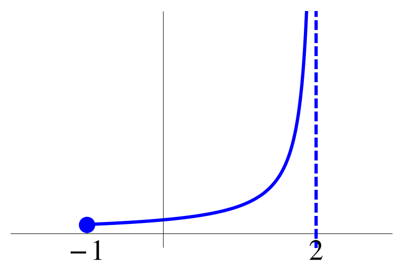
The Extreme Value Theorem, Fermat's Theorem, and critical numbers

Extreme Value Theorem (EVT). If f is a function that is continuous on $[a, b]$ for some $a < b$, then

- for some $c \in [a, b]$, $f(c)$ is a global maximum of f on $[a, b]$, and
- for some $d \in [a, b]$, $f(d)$ is a global minimum of f on $[a, b]$.

Ex. 2. The function $y = g(x)$ pictured is continuous on its domain, but does not have a global maximum. Why doesn't this contradict the EVT?

Scratchwork:



Answer: Left as exercise.

Commentary: If we restrict this function so that its domain is $[-1, 1]$, the restricted version of g does have a global maximum, namely $g(1)$ (since g is increasing).

Ex. 3. Define

$$f(x) = \frac{1+x}{1+x^2}, \qquad -5 \le x \le 5.$$

The notation "$\ldots, -5 \le x \le 5$" means f is a function with domain $[-5, 5]$.

Prove that there exists a number c in the interval $[-5, 5]$ such that f has a global maximum $f(c)$ on $[-5, 5]$ by verifying the hypotheses of the EVT.

Solution:

The Extreme Value Theorem applies, because f is continuous on $[-5, 5]$. Justification:

- $\dfrac{1+x}{1+x^2}$ is a rational function, so it is continuous at every point in its domain.
- Every number in $[-5, 5]$ is in the domain of f.

By EVT, there exists a number c in the interval $[-5, 5]$ such that $f(c)$ is an global maximum of f on $[-5, 5]$.

(There's a d in the domain of f such that $f(d)$ is a global minimum, too.)

Fermat's Theorem. If $f(c)$ is a local maximum or local minimum value of f, and $f'(c)$ exists, then $f'(c) = 0$.

Definition. A **critical number** (or **critical point in the domain**) of a function f is a number c in the domain of f such that

(i) $f'(c)$ does not exist, or

(ii) $f'(c) = 0$.

Ex. 4. Find the critical numbers of the function

$$f(x) = x^3(x^2 - 16), \qquad -\infty < x < \infty.$$

Solution:

f is differentiable everywhere, so we only need to check (ii) in the definition of a critical number.

$f(x) = x^5 - 16x^3$

$f'(x) = 5x^4 - 48x^2 = x^2(5x^2 - 48)$

Set $f'(x) = 0$:

$$x^2(5x^2 - 48) = 0$$

has solutions $x = 0$ and $x = \pm\sqrt{\frac{48}{5}} = \pm 4\sqrt{\frac{3}{5}}$.

Answer: The critical numbers of f are 0 and $\pm 4\sqrt{\frac{3}{5}}$.

Question: Fermat's Theorem says

$$f(c) \text{ extreme value of } f \Longrightarrow c \text{ critical number of } f$$

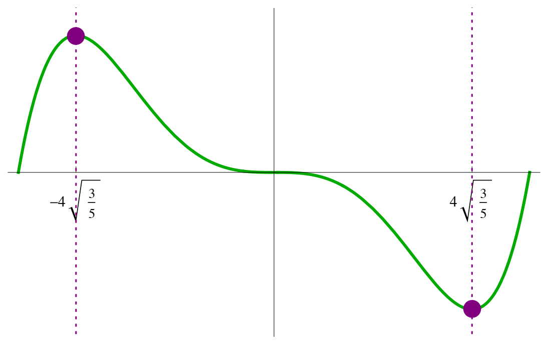
Does the converse implication,

$$f(c) \text{ extreme value of } f \Longleftarrow c \text{ critical number of } f$$

hold?

Answer: *No.* In the previous exercise *(graph shown below)*, $f(0) = 0$ is not an extreme value, but $c = 0$ is a critical number.

$f(x) = x^3(x-4)(x+4)$



☞ Can you think of another function f such that $f'(c) = 0$ for some number c in its domain, but $f(c)$ is not an extreme value of f?

Ex. 5. Find all critical numbers of $f(x) = x^3 + 6x^2 - 15x$.

Solution:

$f'(x) = 3x^2 + 12x - 15$

Solve $f'(x) = 0$:

$$\begin{aligned} 3x^2 + 12x - 15 &= 0 \\ x^2 + 4x - 5 &= 0 \\ (x+5)(x-1) &= 0 \\ x = -5 \text{ or } x &= 1 \end{aligned}$$

$f'(x)$ does not exist:

Never.

$\boxed{-5, 1}$

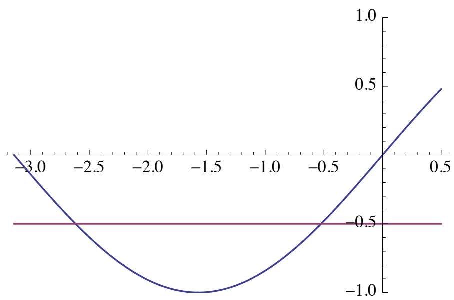
Ex. 6. Find all critical numbers of $f(x) = x - 2\cos x$ $(-\frac{\pi}{2} \leq x \leq 0)$.

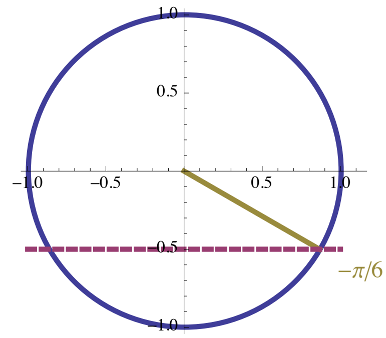
Solution:

$f'(x) = 1 + 2\sin x$

Solve $f'(x) = 0$:

$$1 + 2\sin x = 0$$
$$\sin x = -\frac{1}{2}$$





$f'(x)$ does not exist:

Never.

$\boxed{-\frac{\pi}{6}}$

Ex. 7. Find all critical numbers of $h(p) = \dfrac{p-1}{p^2+4}$.

Solution:

$h'(p) = \dfrac{(p^2+4)(1) - (p-1)(2p)}{(p^2+4)^2} = \dfrac{-p^2+2p+4}{(p^2+4)^2}$

Solve $h'(p) = 0$:

$$\frac{-p^2+2p+4}{(p^2+4)^2} = 0$$
$$-p^2 + 2p + 4 = 0$$
$$p = \frac{-2 \pm \sqrt{2^2 - 4(-1)(4)}}{2(-1)} = 1 \pm \tfrac{1}{2}\sqrt{20} = 1 \pm \sqrt{5}$$

$h'(p)$ does not exist: $(p^2+4)^2 = 0$

Never.

$\boxed{1 \pm \sqrt{5}}$

Notice that in the previous three exercises, the instructions were the same—"find the critical numbers"—*but the methods used to actually* do *this were entirely different.* (Factoring in Exercise 5, graphing in Exercise 6, and the Quadratic Formula in Exercise 7.)

- Get used to this. In calculus and later classes, the techniques we've learned are tools in our toolbox: we are asked to solve problems using any tool we can, *not* to mechanically run through the same step-by-step procedure over and over like robots.

Ex. 8. Find all critical numbers of $F(x) = x^{4/5}(x-4)^2$.

Solution:

$$\begin{aligned} F'(x) &= \frac{4}{5}x^{-1/5}(x-4)^2 + 2x^{4/5}(x-4) \\ &= \frac{1}{5}x^{-1/5}(x-4)\big(4(x-4)+10x\big) \\ &= \frac{1}{5}x^{-1/5}(x-4)(14x-16) \\ &= \frac{2}{5\sqrt[5]{x}}(x-4)(7x-8) \end{aligned}$$

Solve $F'(x) = 0$:

$$\frac{2}{5\sqrt[5]{x}}(x-4)(7x-8) = 0 \qquad \rightsquigarrow \qquad x = 4 \text{ or } x = \frac{8}{7}$$

$F'(x)$ does not exist: $x = 0$

$\boxed{0,\ \frac{8}{7},\ 4}$

Finding global extreme values with the Closed Interval Theorem

Closed Interval Theorem. To find the global maximum and global minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find $f(c)$ for all critical numbers c in the domain of $[a, b]$.
2. Find $f(a)$ and $f(b)$.
3. The largest (smallest) of the numbers you found is the global maximum (minimum).

Ex. 9. Find the global maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad\qquad (-\tfrac{1}{2} \le x \le 1)$$

Solution:

Critical numbers:

$f'(x)$ exists for every x in the domain $[-\frac{1}{2}, 1]$ of f, so condition (i) in the definition of a critical number is not true for any c in the interval $[-\frac{1}{2}, 1]$.

$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$

$x = 0$: in domain ✓

$x = 2$: not in domain *(reject)*

Critical numbers: 0

Extreme values:

Endpoints:

$f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$

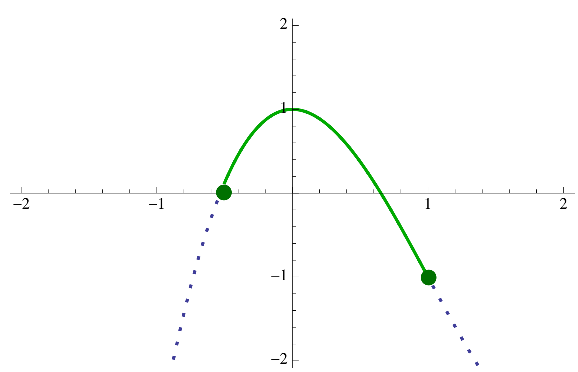
$f(1) = 1 - 3 + 1 = -1$

Critical points:

$f(0) = 1$

Global minimum value: -1

Global maximum value: 1



Ex. 10. Find all absolute maxima and minima of $f(x) = \dfrac{x}{x^2 - x + 1}$ on $[0, 3]$.

Solution:

$$f'(x) = \left[x(x^2 - x + 1)^{-1}\right]'$$
$$f'(x) = (x^2 - x + 1)^{-1} - x(x^2 - x + 1)^{-2}(2x - 1)$$
$$f'(x) = \frac{1}{x^2 - x + 1} - \frac{2x^2 - x}{(x^2 - x + 1)^2}$$

$$\frac{1}{x^2 - x + 1} - \frac{2x^2 - x}{(x^2 - x + 1)^2} = 0$$
$$\frac{-x^2 + 1}{(x^2 - x + 1)^2} = 0$$
$$x = \pm 1$$

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(3) &= \tfrac{3}{7} \end{aligned}$$

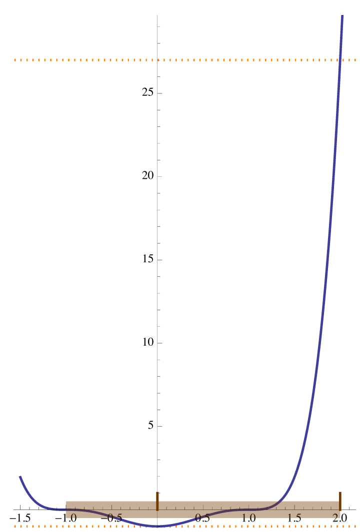
Absolute minimum value: $f(0) = 0$
Absolute maximum value: $f(1) = 1$

Ex. 11. Find all absolute maxima and minima of $f(x) = (x^2 - 1)^3$ on $[-1, 2]$.

Solution:

$$f'(x) = 3(x^2 - 1)^2(2x) = 0$$
$$x = 0 \text{ or } x = \pm 1$$

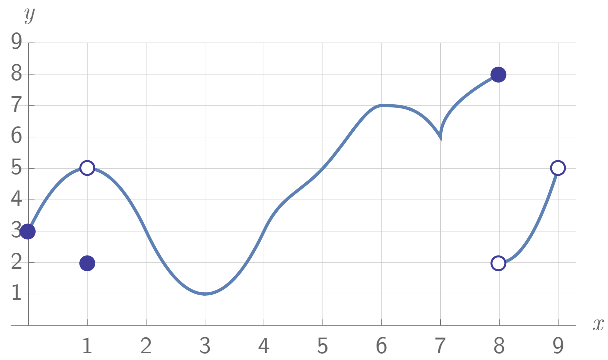
$$\begin{aligned} f(-1) &= 0 \\ f(0) &= -1 \\ f(1) &= 0 \\ f(2) &= 27 \end{aligned}$$



Absolute minimum value: $f(0) = -1$
Absolute maximum value: $f(2) = 27$

Additional exercises

Ex. 12. The graph of a function is shown. State the absolute and local maximum and minimum values of the function.



Ex. 13 (§4.3—#107). Sketch the graph of a function that is continuous on $[-4, 4]$ with absolute maximum values at $x = 2$ and $x = -3$, a local minimum value at $x = 1$, and an absolute minimum value at $x = 4$.

Ex. 16 (§4.3—#109, 111, 113, 115, 116, 117). Find the critical numbers of the function.

(a) $f(x) = 4\sqrt{x} - x^2$

(b) $f(x) = \ln(x-2)$

(c) $f(x) = \sqrt{4-x^2}$

(d) $f(x) = \dfrac{x^2-1}{x^2+2x-3}$

(e) $f(x) = \sin^2(x)$

(f) $f(x) = x + \dfrac{1}{x}$

Ex. 17 (§4.3—#90). Recall: the maximum or minimum value of a quadratic function $f(x) = Ax^2 + Bx + C$ is given by the formula $f\big(-\frac{B}{2A}\big)$. Prove this formula using calculus.

Ex. 18 (§4.3—#119, 121, 123, 127, 129, 133). Find the local and absolute maximum values and the local and absolute minimum values of the function over the given interval.

(a) $f(x) = x^2 + \dfrac{2}{x}$ over $[1, 4]$

(b) $f(x) = \dfrac{1}{x - x^2}$ over $(0, 1)$

(c) $f(x) = x + \sin(x)$ over $[0, 2\pi]$

(d) $f(x) = \sin(x) + \cos(x)$ over $[0, 2\pi]$

(e) $f(t) = x^2 + 4x + 5$ over $(-\infty, \infty)$

(f) $f(t) = \dfrac{x^2 + x + 6}{x - 1}$ over $(-\infty, \infty)$

Ex. 19 (§4.3—#135, 139). Technology required. Use a graph to estimate the absolute maximum and minimum values of the function. Then use calculus to find the exact maximum and minimum values.

(a) $f(x) = 3x\sqrt{1 - x^2}$

(b) $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}$

Ex. 20 (§4.3—#141). A ball is thrown into the air and its height (in meters) is given by

$$h(t) = -4.9t^2 + 60t + 5.$$

(a) Find the height at which the ball stops ascending.

(b) How long after it is thrown does this happen?

Workbook Lesson 18

§4.4, The Mean Value Theorem

Last revised: 2021-09-30 12:08

Objectives

- Explain the conclusion of Rolle's Theorem in plain English.
- Explain the conclusion of the Mean Value Theorem in plain English.
- Explain three consequences of the Mean Value Theorem (Corollaries 1–3 below).

Rolle's Theorem

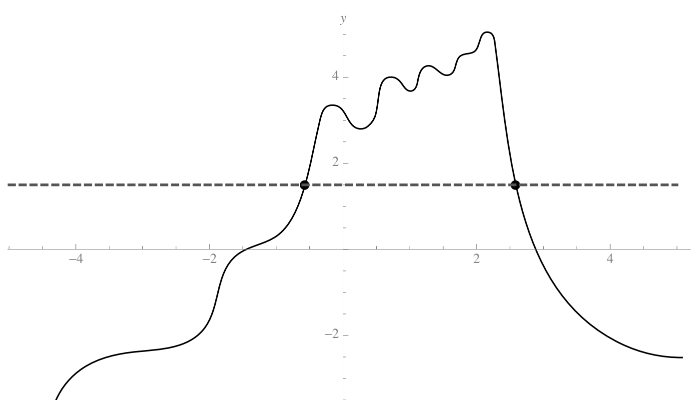
Rolle's Theorem. Let f be a function. There is some number c in the interval (a, b) such that $f'(c) = 0$ if:

- f is continuous on $[a, b]$,
- f is differentiable on (a, b), and
- $f(a) = f(b)$.

To put it in other words, Rolle's Theorem says that:

However complicated the graph of a function may be,
there must be some point at which the tangent is horizontal
if the conditions of Rolle's Theorem are satisfied.

Consider, for example, the function whose graph is shown below.



We check the conditions of Rolle's Theorem:

- The function values are equal at the two marked points *(dashed horizontal line)*.
- The function is continuous from one marked point to the other *(the graph can be drawn without lifting one's pencil)*.
- The function is differentiable in-between the two marked points *(there are no sharp corners or vertical tangents)*.

Rolle's Theorem now guarantees that at some point between the two marked points, the graph has a horizontal tangent. (Indeed, we can plainly see this is true at several points on the graph.)

☞ *Use the applet "Rolle's Theorem" on iCollege to explore the meaning of Rolle's Theorem with several different functions.*

Proof of Rolle's Theorem:

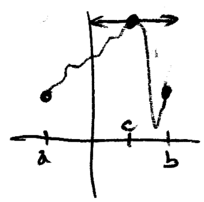
Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

Suppose $f(a) = f(b)$.

Case 1. $f(x) =$ const.

Then $f'(x) = 0$ for all x, so any c in the interval (a, b) works.

Case 2. $f(x) > f(a)$ for some x in the interval (a, b).



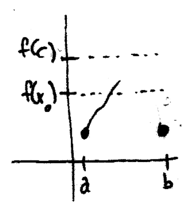
If there is some function value $f(x)$ greater than $f(a) = f(b)$,
where x is in-between a and b, then there must be some c
in-between a and b at which the tangent is horizontal.

The EVT can be applied: f is continuous on $[a, b]$ ✓

By EVT, there is some c in the interval $[a, b]$ s.t. $f(c)$ is a local max for f on $[a, b]$.

Since $f(b) = f(a)$, the local max must occur at some c in the interval (a, b).

(Why? Because we know $f(a) = f(b)$ can't be a local max by the hypothesis of this case.)



Fermat's Theorem can be applied: $f(c)$ is a local max ✓ $f'(c)$ exists ✓ Thus $f'(c) = 0$.

Case 3. $f(x) < f(a)$ for some x in the interval (a, b).

(Similar to Case 2)

EVT $\Longrightarrow$ there exists a number c in the interval $[a, b]$ such that $f(c)$ is a local min.

$f(b) = f(a) \Longrightarrow a \neq c \neq b$.

Fermat $\Longrightarrow f'(c) = 0$. $\square$

Ex. 1. Show that $x^3 + x - 1 = 0$ has exactly one (real) root.

Solution:

Claim: $f(x) = x^3 + x - 1$ has at least one root.

$f(0) = 0 + 0 - 1 = -1 < 0$
$f(1) = 1 + 1 - 1 = 1 > 0$

So by $\underbrace{\qquad\qquad}_{\text{Which Theorem?}}$, there is some number x_0 in the interval $(0, 1)$ such that $f(x_0) = 0$.

Claim: It is impossible for $f(x)$ to have more than one root.

Assume for a contradiction that f has at least *two* roots, say $f(a) = 0 = f(b)$.

Then Rolle's Theorem applies:
- f differentiable on (a, b) ✓
- f continuous on $[a, b]$ ✓
- $f(a) = f(b)$ ✓

By Rolle's Theorem, there is some number c in the interval (a, b) s.t. $f'(c) = 0$.
However, $f'(x) = 3x^2 + 1 > 0$, so it is impossible for $f'(x) = 0$.
We have a contradiction: $f'(c) = 0$ and $f'(c) \neq 0$.
Therefore, our assumption must have been false: f does not have at least two roots.

Mean Value Theorem

Rolle's Theorem is used to prove a more general theorem: the Mean Value Theorem (see textbook for proof).

Mean Value Theorem on a pedestrian bridge across East Zhushikou Avenue in Beijing

Mean Value Theorem. If f is differentiable on (a, b) and f is continuous on $[a, b]$, then there is a number c in the interval (a, b) such that

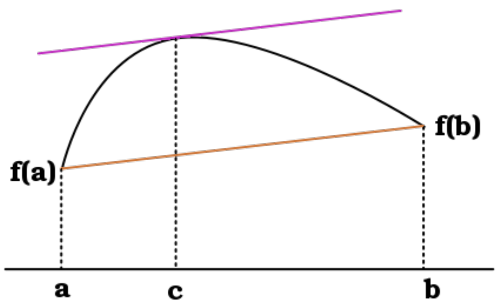
$$f(b) - f(a) = f'(c) \cdot (b - a). \tag{*}$$

Note that (*) can be rewritten

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

So the Theorem says that:

$$\begin{pmatrix} \text{the average} \\ \text{rate of change} \\ \text{over } [a, b] \end{pmatrix} = \begin{pmatrix} \text{the instantaneous} \\ \text{rate of change} \\ \text{at } c \end{pmatrix} \quad \text{for some } c \text{ in the interval } (a, b).$$



The slope of the line through $(a, f(a))$ and $(b, f(b))$
is $\dfrac{\Delta y}{\Delta x}$ = the average rate of change over $[a, b]$.

The slope of the tangent line through $(c, f(c))$ is $f'(c)$.

According to the Mean Value Theorem, these two slopes are equal
for some choice of c between a and b.

The Mean Value Theorem (and its special case, Rolle's Theorem) just says there *exists* some c between a and b.

Like some other theorems we have seen, the Mean Value Theorem does *not* tell you what the value of c is. *(Recall:)* We call such a theorem an **existence theorem**.

Some other existence theorems:

- Intermediate Value Theorem
- Extreme Value Theorem

Definition. A constant M is an **upper bound** on a quantity y if $y \leq M$ for all possible values of y, and a **lower bound** if $y \geq M$ for all possible values of y.

- For example, in high school, the constant 100 was an upper bound on your final grade y in math class, because 100 was the highest score allowed on your report card—no matter how much extra credit you might have earned.

Ex. 2. Suppose f is a differentiable function such that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. Use the Mean Value Theorem to find an upper bound on $f(2)$.

The MVT applies: f differentiable on $(0, 1)$ ✓
f continuous on $[0, 1]$ ✓

By MVT, there is some $c \in (0, 2)$ s.t.

$$\begin{aligned} f(2) - f(0) &= f'(c)(2-0) \\ f(2) + 3 &= 2f'(c) \\ f(2) + 3 &= 2f'(c) \leq 10 && \text{since } f'(c) \leq 5 \\ f(2) &\leq 7 \end{aligned}$$

$\boxed{f(2) \leq 7.}$

Ex. 3. Find a number c that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$.

The MVT applies: f differentiable on $(-2, 2)$ ✓
f continuous on $[-2, 2]$ ✓

Use MVT:

$$\begin{aligned} f(2) - f(-2) &= f'(c)(2-(-2)) && \text{(by MVT)} \\ 4 = (8-2+2) - (-8+6+2) &= 4f'(c) \\ f'(c) &= 1 \end{aligned}$$

Now we know the value of the f' of c. But what's the value of c? We can solve for it:

$$\begin{aligned} f'(x) &= 3x^2 - 3 && \text{(get a formula for } f' \ldots) \\ 1 = f'(c) &= 3c^2 - 3 && (\ldots \text{then plug in } c\text{, and use the fact that } f'(c) = 1) \\ 4 &= 3c^2 \\ c &= \pm\sqrt{\frac{4}{3}} \end{aligned}$$

Both of $\pm\sqrt{\frac{4}{3}}$ are in $[-2, 2]$.

$\boxed{c = \pm\sqrt{\frac{4}{3}}.}$

Ex. 4. Before reading the proof to follow, explain why the following corollary to the Mean Value Theorem is true.

> **Corollary 1.** If $f'(x) = 0$ for all x in the interval (a, b), then f is constant on (a, b).

Proof:

For any x_1, x_2 in the interval (a, b) such that $x_1 < x_2$, by the Mean Value Theorem, there is some c in the interval (x_1, x_2) such that

$$\begin{aligned} f(x_2) - f(x_1) &= f'(c)(x_2 - x_1) \\ f(x_2) - f(x_1) &= 0 \\ f(x_2) &= f(x_1) \end{aligned}$$

(Why does this prove the theorem?)

□

Ex. 5. Complete the proof of the following corollary to the Mean Value Theorem.

> **Corollary 2.** Suppose f and g are each differentiable over an interval (a, b). If $f'(x) = g'(x)$ for all x in the interval (a, b), then there exists some constant C such that $f(x) = g(x) + C$ for every x in (a, b).

Proof:

Let $h(x) = f(x) - g(x)$. Then

$$h'(x) =$$

for all x in the interval (a, b).

It follows that there exists some constant C such that $h(x) = C$ for all x in (a, b)

______________________________.
Why?

□

We will present one more consequence of the Mean Value Theorem.

But first, let's review two definitions you may have seen in earlier classes.

Definition: Let f be a function. Let I be an interval.

- f is **increasing** if, for every x_1 and x_2 in the interval I such that $x_1 < x_2$,
$$f(x_1) < f(x_2).$$
- f is **decreasing** if, for every x_1 and x_2 in the interval I such that $x_1 < x_2$,
$$f(x_1) > f(x_2).$$

Ex. 6. State what it would mean for a function to be NOT decreasing.

(*Hint: See applet on iCollege:* "`Not increasing`")

Ex. 7. On what interval(s) is $f(x) = \dfrac{1}{x}$ increasing? On what interval(s) is it decreasing?

Corollary 3 (Increasing/Decreasing Test). Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(i) If $f'(x) > 0$ for all x in the interval (a, b), then f is increasing on $[a, b]$.

(ii) If $f'(x) < 0$ for all x in the interval (a, b), then f is decreasing on $[a, b]$.

Proof:

We will prove (i); the proof of (ii) is similar.

Assume for a contradiction that $f'(x) > 0$ for all x on the interval (a, b), but f is NOT an increasing function on $[a, b]$.

Since f is not increasing on $[a, b]$, there exist numbers x_1 and x_2 in the interval $[a, b]$ such that

$$f(x_1) \geq f(x_2) \qquad \text{and} \qquad x_1 < x_2. \tag{*}$$

Since f is differentiable on the interval (a, b) and continuous on $[a, b]$, by the Mean Value Theorem there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Subtracting $f(x_1)$ from both sides of (*) yields

$$f(x_2) - f(x_1) \leq 0,$$

and we know that $b - a > 0$ since $a < b$.

Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

But contradicts that $f'(x) > 0$ for all x in (a, b).

□

Exam questions for the theorems of Lessons 17 & 18 in previous semesters

Ex. 6.

(a) Complete the statement of the theorem.

> **Mean Value Theorem.** There is a number c in the interval (a, b) such that
>
> ____________________
>
> if:
>
> - f is __________ on (a, b), and
> - f is __________ on $[a, b]$.

(b) Suppose f is a differentiable function such that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. Use the Mean Value Theorem to find the largest possible value for $f(2)$.

(c) Find a number c that satisfies the conclusion of the Mean Value Theorem for $g(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$.

Sample answer:

See **Ex. 2** and **Ex. 3** above.

Ex. 7.

> **Fermat's Theorem.** If $f(c)$ is a local maximum or local minimum value of f, and $f'(c)$ exists, then $f'(c) = 0$.

A student argues that 0 is an extreme value of the function $f(x) = x^3$ as follows:

(1) *The function $f(x) = x^3$ has derivative $f'(x) = 3x^2$.*

(2) *The domain of f' is $(-\infty, \infty)$. In particular, $f'(0)$ exists.*

(3) $f(0) = 0$.

(4) *Since $f'(0) = 0$, by Fermat's Theorem $f(0) = 0$ is either a local maximum or a local minimum value of f.*

Which line contains an error? Explain what is wrong with the student's reasoning.

Sample answer:

Line (4) is incorrect. The statement "if $f'(c) = 0$, then $f(c)$ is a local min. or max." is false.

Ex. 8.

> **Extreme Value Theorem (EVT).** Suppose $a < b$. If f is a function that is continuous on $[a, b]$, then
> - for some c in the interval $[a, b]$, $f(c)$ is a global maximum of f on $[a, b]$
> - for some d in the interval $[a, b]$, $f(d)$ is a global minimum of f on $[a, b]$

Why can't the Extreme Value Theorem be applied to find a global maximum of the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

on the interval $[-1, 1]$?

Sample answer:

The Extreme Value Theorem does not apply to the function $H(x)$ on the interval $[-1, 1]$ because H is not continuous on $[-1, 1]$.

☞ *Note that the Extreme Value Theorem will* NOT *be reprinted on exams or quizzes.*

Ex. 9.

Use the Extreme Value Theorem to find the global maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad (-\tfrac{1}{2} \leq x \leq 1).$$

Ex. 10.

Prove that $f(x) = x^3 + x - 1$ has at least one root by following these steps.

(a) Find $f(0)$ and $f(1)$.

(b) What Theorem guarantees that there is some number x_0 in the interval $(0, 1)$ such that $f(x_0) = 0$?

(c) What facts must be known for the Theorem to apply?

Sample answer:

(a) $f(0) = -1$ and $f(1) = 1$

(b) Intermediate Value Theorem

(c) f is continuous, $f(0) < 0$, and $f(1) > 0$.

Ex. 11.

> **Rolle's Theorem.** Let f be a function. There is some number c in the interval (a, b) such that $f'(c) = 0$ if:
> - f is continuous on $[a, b]$,
> - f is differentiable on (a, b), and
> - $f(a) = f(b)$.

Assume that the function

$$f(x) = x^3 + x - 1$$

has at least two roots a and b: that is,

$$f(a) = 0 = f(b).$$

Show that this assumption leads to a contradiction by following these steps.

(a) Check that Rolle's Theorem applies to f on the interval $[a, b]$ by verifying that

- f is differentiable on (a, b)
- f is continuous on $[a, b]$

(b) Since Rolle's Theorem applies, we must conclude there is some number c in the interval (a, b) such that $f'(c) = 0$.
Prove this is impossible by solving the equation $f'(x) = 0$, and say why are there no (real) solutions.

Sample answer:

(a) f is a polynomial, and any polynomial is continuous and differentiable on $(-\infty, \infty)$.

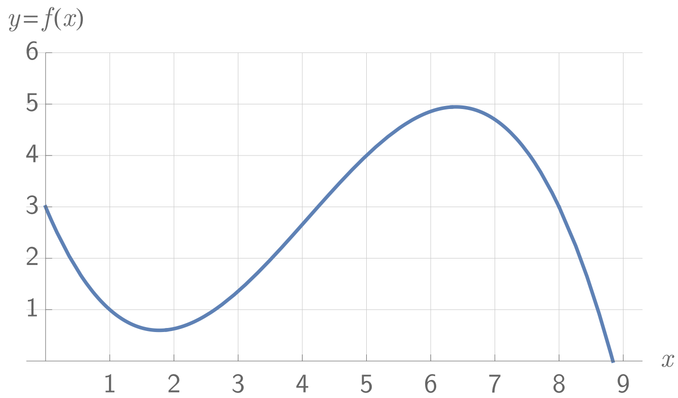
(b)

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 + 1 &= 0 \\ x^2 &= -\frac{1}{3} \end{aligned}$$

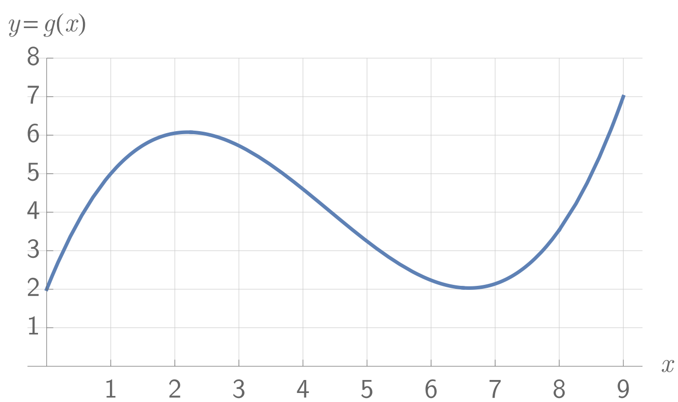
The equation has no solutions because x^2 cannot be negative.

Additional exercises

Ex. 12. The graph of a function f is shown. Verify that f satisfies the three hypotheses of Rolle's Theorem on the interval $[0, 8]$. Then estimate the value(s) of c that satisfy the conclusion of Rolle's Theorem on that interval.



Ex. 13. The graph of a function g is shown.



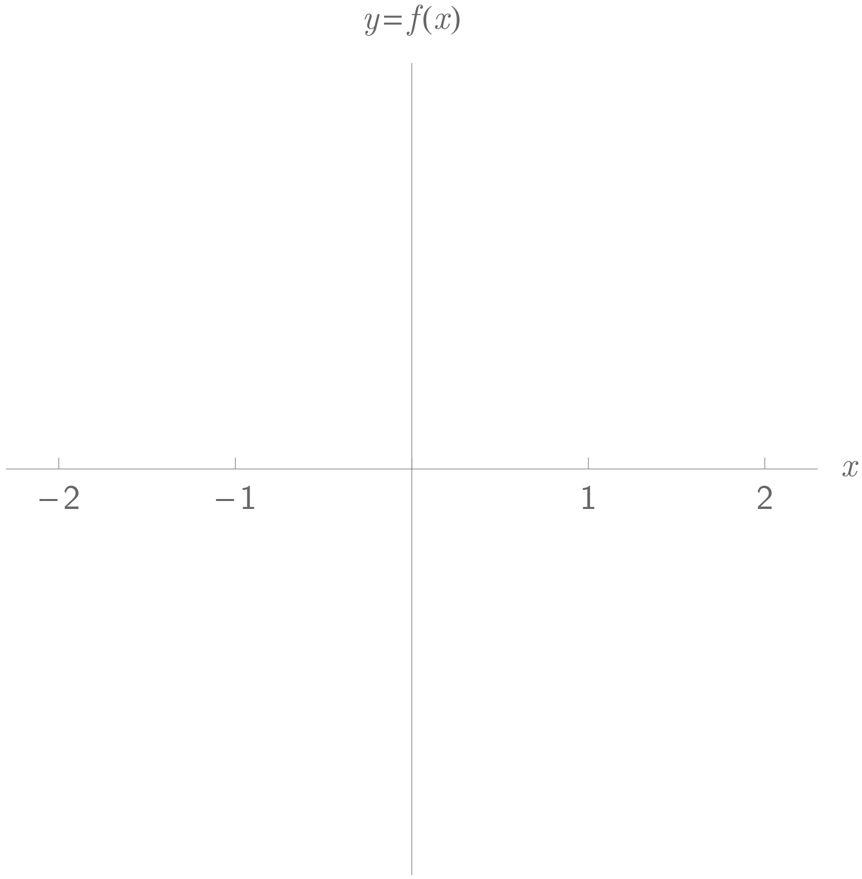
(a) Verify that g satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 9]$.

(b) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 8]$.

(c) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[2, 7]$.

Ex. 14 (§4.4—#149).

- In order to be applied to a function f, the Mean Value Theorem requires that f must be differentiable on an interval (a, b).
- Prove that differentiability is NEEDED by drawing a counterexample of a function f with domain $[-2, 2]$ on the blank coordinate system provided below.
- (That is, draw the graph of a function f which is NOT differentiable on $(-2, 2)$ and which does NOT satisfy the conclusion of the Mean Value Theorem for any number c such that $-2 < c < 2$.)



Workbook Lesson 19

§4.5, Derivatives and the Shape of a Graph

Last revised: 2020-09-29 12:44

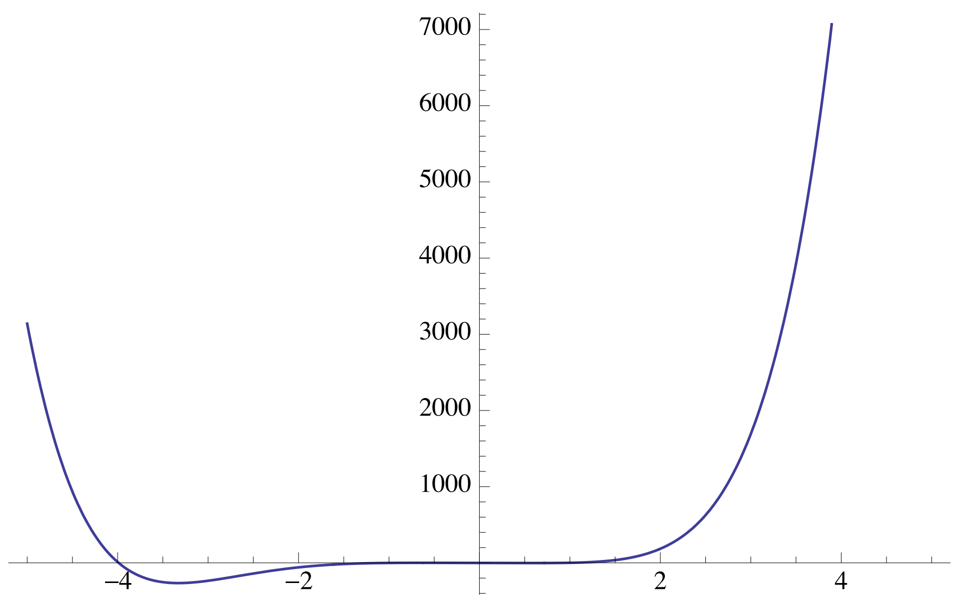
Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- Use the Increasing/Decreasing Test to determine intervals of increase/decrease.
- State the First Derivative Test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the Second Derivative Test for local extrema.

Motivation

We can graph a function using calculators, computers, and even our phones. But we still teach how to graph in calculus class. Why?

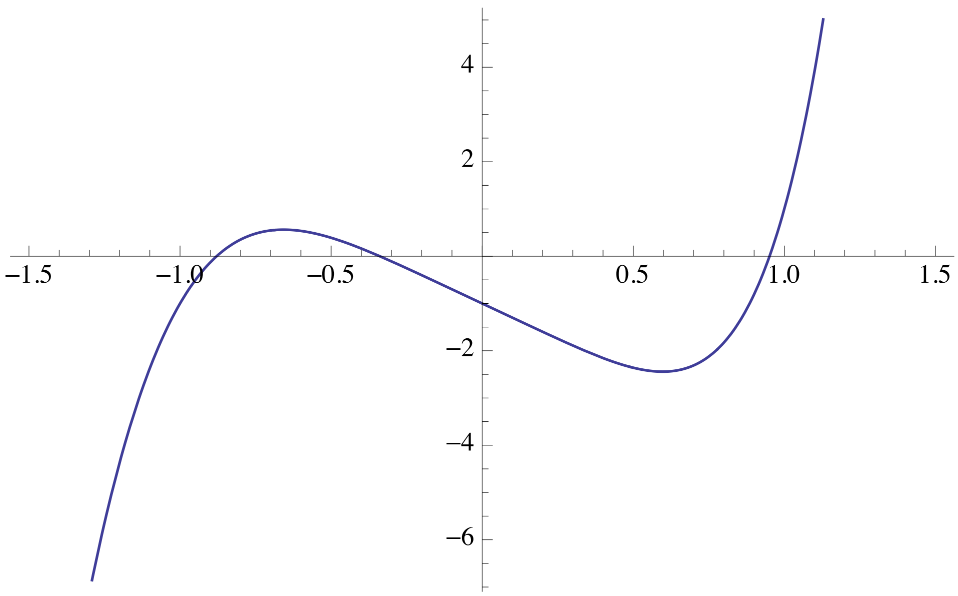
Consider the following graph of the function $y = x^6 + 4x^5 - 3x - 1$.



How many local maxima and minima are there? It looks like there's just *one*, $f(-3.329\ldots) = -265.354\ldots$.

However, *the tools of calculus tell us there must be two more.*

Now look at this graph *(top of next page)*:



This is the same function, plotted on a different range of x-values. We see two more extreme values—a local maximum $f(-0.657\ldots) \approx 0.561\ldots$ and a local minimum $f(0.597\ldots) = -2.442\ldots$.

We use calculus to ensure that we are not misled by technology. As we see in this example, a graphing calculator can easily cause us to miss important features of the graph.

Finding intervals of increase/decrease

Recall from the previous lesson:

> **Increasing/Decreasing Test.** Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
> (i) If $f'(x) > 0$ for all x in the interval (a, b), then f is increasing on $[a, b]$.
> (ii) If $f'(x) < 0$ for all x in the interval (a, b), then f is decreasing on $[a, b]$.

Let us show how to find the intervals on which a function is increasing (or decreasing).

The rough outline of the process is:

> STEP 1. Find the critical numbers and mark them on the number line.
>
> ☞ The critical numbers break up (or "partition") the number line into intervals.
>
> STEP 2. Determine the sign of $f'(x)$ in each interval.
>
> ☞ We'll go into detail about how to do this in the next exercise.
>
> STEP 3. Apply the Increasing/Decreasing Test.

Ex. 1. Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing, and the intervals on which it is decreasing.

Solution.

STEP 1. Find the critical numbers and mark them on the number line.

Critical numbers happen where $f'(x) = 0$ or $f'(x)$ is undefined.

$f'(x) = 0$:

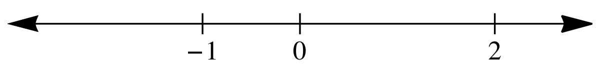
$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) \\ = 12x(x-2)(x+1)$$

We see that $f'(x) = 0$ when $x = 0$, $x = 2$, or $x = -1$.

$f'(x)$ is undefined:

This never happens, because f is a polynomial (and therefore its domain is all real numbers).

Critical numbers on the number line:



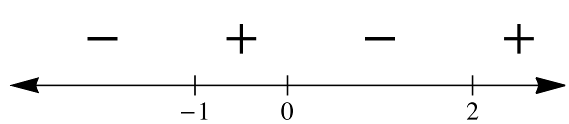
STEP 2. Determine the sign of $f'(x)$ in each interval.

We've already factored $f'(x) = 12x(x-2)(x+1)$.

This makes it easy to see where $f'(x)$ is positive or negative—we can just find where each *factor* is positive and negative, and then count the negative signs.

- An odd number of negative factors (e.g. POSITIVE $\times$ NEGATIVE) yields a negative number.
- An even number of negative factors (e.g. POSITIVE $\times$ POSITIVE) yields a positive number.

interval	sign of $f'(x)$	$12x$	$(x-2)$	$(x+1)$
$-\infty < x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$+$	$-$	$-$	$+$
$0 < x < 2$	$-$	$+$	$-$	$+$
$2 < x < \infty$	$+$	$+$	$+$	$+$



STEP 3. Apply Increasing/Decreasing Test.

f is decreasing on $(-\infty, -1)$ and $(0, 2)$, increasing on $(-1, 0)$ and $(2, \infty)$.

(We could use closed *or* open intervals here, because f' exists at each critical number.)

Ex. 2. Suppose the derivative of a function f is $f'(x) = (x-3)^2(x+1)^4(x-7)^6$. On what intervals is f increasing?

Ex. 3. Suppose the derivative of a function f is $f'(x) = (x-3)^2(x+1)^4(x-7)^5$. On what intervals is f increasing? Decreasing?

Ex. 4. Suppose the derivative of a function f is $f'(x) = \dfrac{(x-3)^2(x+1)^4}{(x-7)^5}$. On what intervals is f increasing? Decreasing? Can $[3,7]$ be one of the intervals of increase/decrease?

The First Derivative Test

Recall: for differentiable f,

$$[f(c) \text{ is a local max or min value}] \overset{\text{(Fermat)}}{\underset{\not\Leftarrow}{\Longrightarrow}} [c \text{ is a critical number of } f]$$

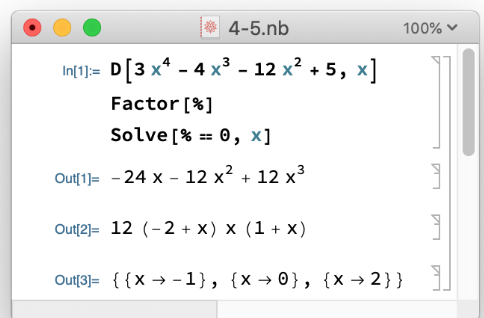
Given a critical number c, we want a test that tells us <u>whether or not</u> $f(c)$ is a local max or min.

> **<u>First Derivative Test.</u>** Suppose c is a critical number of a continuous function f.
> - If f' changes from positive to negative at c, then $f(c)$ is a local max.
> - If f' changes from negative to positive at c, then $f(c)$ is a local min.
> - If f' does not change sign at c, then $f(c)$ is not a local max or a local min.

<u>Ex. 5.</u> Find the local maximum and local minimum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

<u>Solution:</u>

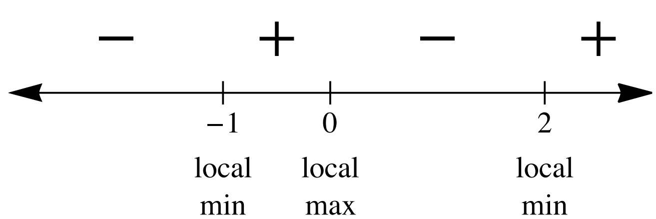
Using the technical computing system `Mathematica` (an app available on GSU computers and free for download by GSU students), we can quickly dispense with the gruntwork of finding the critical numbers of $f(x)$:



<u>$f'(x) = 0$:</u> When $x = 0$, $x = 2$, or $x = -1$

<u>$f'(x)$ is undefined:</u> Never

<u>Critical numbers on the number line:</u>



$f(-1) = 0$
$f(0) = 5$
$f(2) = -27$

Local max values:	5
Local min values:	$0, -27$

Ex. 6. Find the local maximum and local minimum values of

$$g(x) = x + 2\sin x \qquad (0 \le x \le 2\pi).$$

Solution:

The derivative

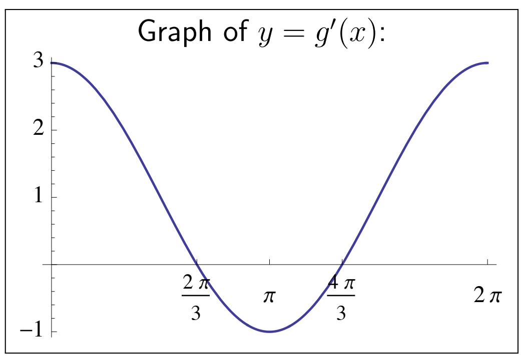
$$g'(x) = 1 + \cos x$$

exists everywhere, so the only critical numbers c are those such that $\underline{g'(c) = 0}$:

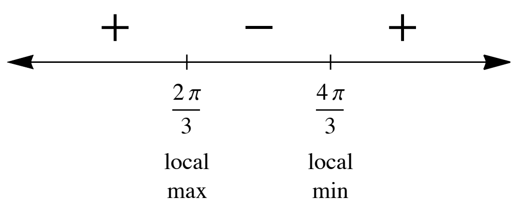
$$\begin{aligned} g'(x) &= 0 \\ 1 + 2\cos x &= 0 \\ \cos x &= -\frac{1}{2} \\ x &= \frac{2\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

Critical numbers: $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

The fastest and simplest way to determine the sign chart is by looking at the graph of $g'(x)$, which we can quickly sketch without tech based on what we know about the $\cos$ function:



Sign chart for $g'(x)$:

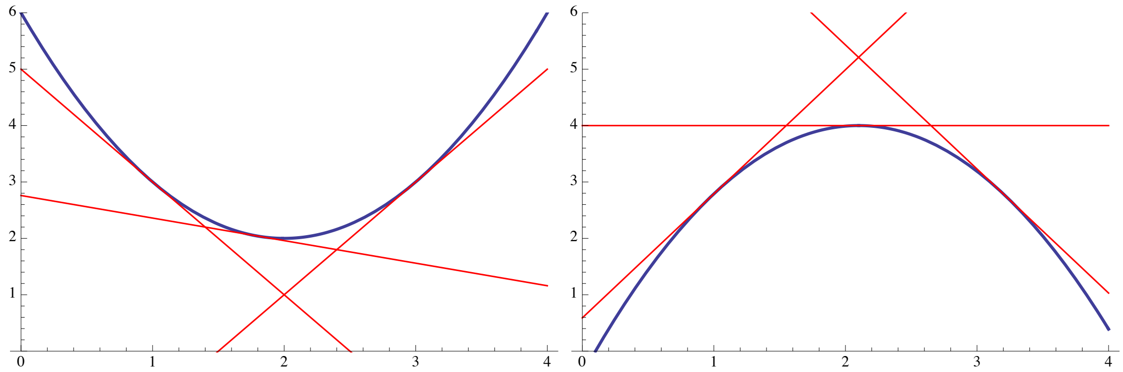


Answer: *(left as exercise)*

Local max value:
Local min value:

Concavity

Definition: If the graph of f lies above (below) all its tangents on an interval I, it is said to be **concave up (down)** on I.



Definition: A point P on a curve $y = f(x)$ is an **inflection point** if f is continuous at x and the curve changes concavity from up to down or vice versa.

Concavity Test. Suppose f is twice differentiable on an interval I. (That is, f'' exists.)
(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Ex. 7. Sketch the graph of a function f satisfying all of the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$; $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$; $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x\to-\infty} f(x) = -2$; $\lim_{x\to\infty} f(x) = 0$

Observations before graphing:

1. Horizontal asymptotes at $y = -2$ and $y = 0$.
2. f increases to its maximum at $x = 1$.
3. f has inflection points at $x = \pm 2$.

Second Derivative Test. Suppose f'' is continuous on an interval containing c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local maximum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local minimum at c.

For example, part **(a)** is true because the tangent at c is *horizontal*, and the concavity is *up*.

Ex. 8. Sketch the curve $y = x^4 - 4x^3$ by hand.

Solution:

$f'(x) = 4x^2(x - 3)$

$f''(x) = 12x(x - 2)$

Critical numbers:

f' is defined at all x in $\mathbb{R}$.

$f'(x) = 0$ at $x = 0$ and $x = 3$.

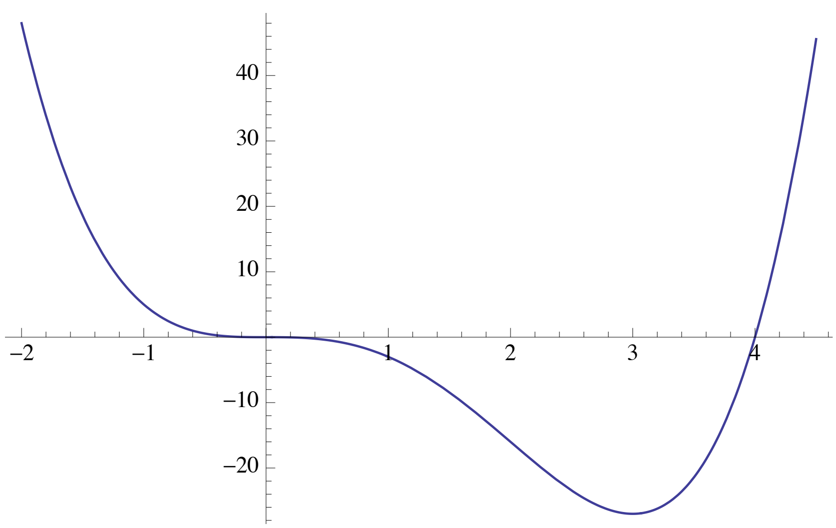
2nd derivative test:

$f''(3) = 36 > 0$ so $f(3) = -27$ is a local max

$f''(0) = 0$ so ...?

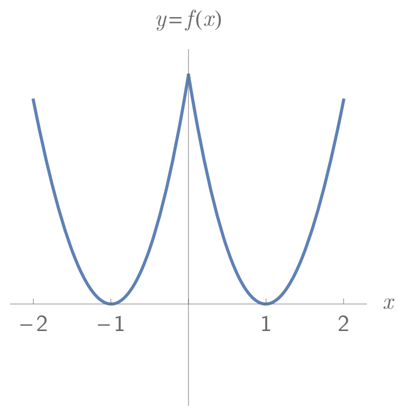
(2nd derivative test yields no information where $f'' = 0$.)

interval	sign of $f''(x) = 12(x - 2)$	concavity
$-\infty < x < 0$	$+$	up
$0 < x < 2$	$-$	down
$2 < x < \infty$	$+$	up



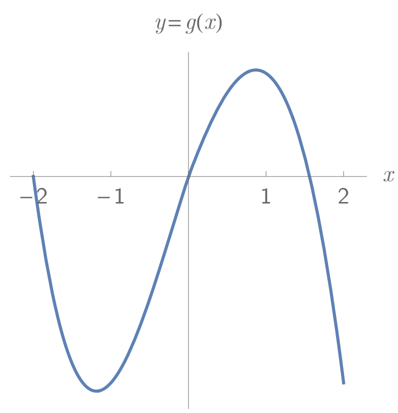
Additional exercises

Ex. 9. Use the given graph of f to find the following.



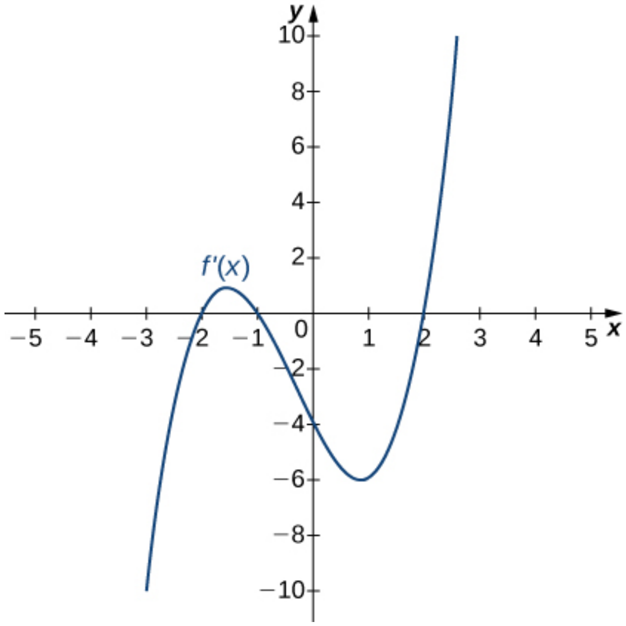
(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which the graph of f is concave upward.
(d) The open intervals on which the graph of f is concave downward.

Ex. 10. Use the given graph of g to find the following.



(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which the graph of f is concave upward.
(d) The open intervals on which the graph of f is concave downward.
(e) The x-coordinate(s) of the point(s) of inflection.

Ex. 11 (§4.5—201). The graph of the DERIVATIVE f' of a function f is given. List all intervals on which f is increasing or decreasing.



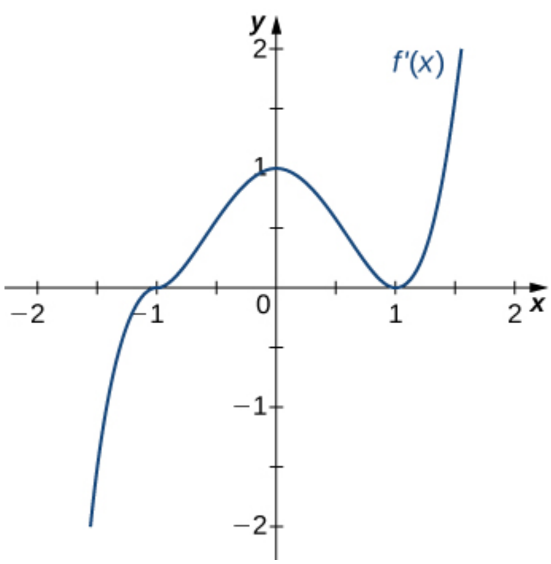
Ex. 12 (§4.5—215). The graph of the derivative f' of a continuous function f is shown.

(a) On what intervals is f is increasing? Decreasing?

(b) At what values of x does f have a local maximum? Local minimum?

(c) On what intervals is the graph of f concave upward? Concave downward?

(d) What are the x-coordinates of the inflection points?



Ex. 13 (§4.5—221). List all intervals where the function

$$\sin(x) + \sin^3(x), \qquad -\pi < x < \pi,$$

is increasing or decreasing, and list all local minimum values and all local maximum values of f.

Ex. 14 (§4.5—225, 229). For each of the following functions...

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity.

(d) Find the inflection points.

- $f(x) = x^3 - 6x^2$
- $g(x) = x^2 + x + 1$

Ex. 15.

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity.

(d) Find the inflection points.

(e) Sketch the graph of f.

- $f(x) = 36x + 3x^2 - 2x^3$
- $g(x) = \frac{1}{2}x^4 - 4x^2 + 3$
- $h(x) = x\sqrt{6-x}$
- $j(\theta) = 2\cos(\theta) + \cos^2(\theta)$, $0 \leq \theta \leq 2\pi$

Some important theorems of Differential Calculus

Last revised: 2021-03-09 12:35

§	**Theorem**	**Conditions**	**Conclusion** (true if <u>all</u> conditions are satisfied)
(2.4)	IVT	f continuous on $[a,b]$ $f(a) \neq f(b)$ z is between $f(a)$ and $f(b)$	There is some c in (a,b) such that $f(c) = z$.
(4.3)	EVT	f continuous on $[a,b]$	There is some c in $[a,b]$ such that $f(c) = \min\{f(x) \mid a \le x \le b\}$. There is some d in $[a,b]$ such that $f(d) = \max\{f(x) \mid a \le x \le b\}$.
(4.3)	Fermat	$f(c)$ local min or max value $f'(c)$ is a real number	$f'(c) = 0$
(4.3)	Closed Interval	f continuous on $[a,b]$	The min and max values of f on $[a,b]$ are members of the following set: $\{f(a), f(b)\} \cup \{f(c) \mid c \text{ is a critical number of } f\}$.
(4.4)	MVT	f continuous on $[a,b]$ f differentiable on (a,b)	There is some c in (a,b) such that $f(b) - f(a) = f'(c) \cdot (b-a)$.
(4.4)	Inc./Dec. Test	f continuous on $[a,b]$ f differentiable on (a,b)	If $f'(x) > 0$ for all x in (a,b), then f is increasing on $[a,b]$. If $f'(x) < 0$ for all x in (a,b), then f is decreasing on $[a,b]$.
(4.5)	1st Deriv. Test	c critical number of f f continuous on $[a,b]$ f' changes sign at c	If f' changes sign at c from $+$ to $-$, then $f(c)$ is a local max value. If f' changes sign at c from $-$ to $+$, then $f(c)$ is a local min value. If f' does not change sign at c, then $f(c)$ is not a local min or max value.
(4.5)	Concavity Test	f twice differentiable on I, where I is an interval	If $f''(x) < 0$ for all x in I, then the graph of f is concave up on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.
(4.5)	2nd Deriv. Test	f'' continuous on an interval that contains c	If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local max value. If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local min value.

Workbook Lesson 20

§4.6, Limits at Infinity, Asymptotes, and Curve Sketching

Last revised: 2021-03-09 07:44

Objectives

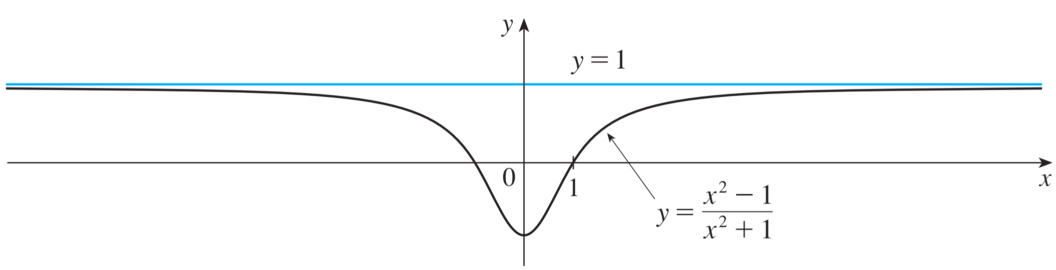
- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Analyze a function and its derivatives to draw its graph.

Limits at infinity

In the previous examples, we saw the output of a function growing arbitrarily "large" (that is, toward positive or negative infinity). Now we look at limits where the *input* approaches $\pm\infty$.

Ex. 1. What value does $f(x) = \dfrac{x^2-1}{x^2+1}$ approach as we take $x \to \infty$? As we take $x \to -\infty$?

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998



Definition (Limits at infinity): Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that for any $E > 0$, there is a number M such that $|f(x) - L| < E$ whenever $x > M$.

The statement

$$\lim_{x\to-\infty} f(x) = L$$

is defined similarly.

Definition: The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Again, the symbols $\pm\infty$ do not represent numbers.

Ex. 2. Show that $\lim_{x\to\infty} \frac{1}{x} = 0$, using the definition of a limit at infinity.

Solution: Let $e > 0$ be given. We need to find M such that

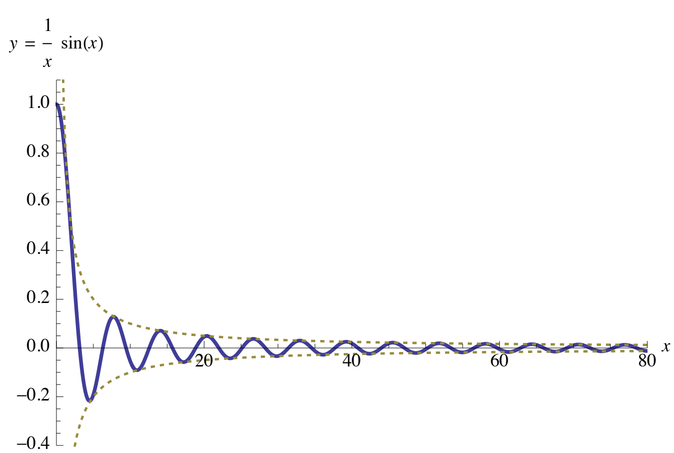
$$\left|\frac{1}{x} - 0\right| < E$$

for any $x > M$.

We may assume $x > M > 0$. *(Why?)* Then $\left|\frac{1}{x} - 0\right| = \frac{1}{x} < E \Longleftrightarrow \frac{1}{E} < x$.

Choose $M = 1/E$.

Ex. 3. By looking at the graph of $f(x) = \frac{1}{x}\sin(x)$, we make the guess that $\lim_{x\to\infty} f(x) = 0$. Can you justify this guess without referring to the graph? *(Justification given on next page.)*



Justification:

Since $\sin(x)$ is always between -1 and 1, that is,

$$-1 \leq \sin(x) \leq 1,$$

we know

$$-\frac{1}{x} \leq \frac{1}{x}\sin(x) \leq \frac{1}{x} \qquad (x > 0).$$

But we know that $\frac{1}{x} \to 0$ as $x \to \infty$. Similarly, $-\frac{1}{x} \to 0$ as $x \to \infty$.

As $x \to \infty$, the value of $\frac{1}{x}\sin(x)$ must approach 0 because it is "squeezed" between $-\frac{1}{x}$ and $\frac{1}{x}$, each of which approach 0.

(If we had to, we could make this argument into a careful formal proof.)

Ex. 4. Find $\lim_{x\to\infty} \sin(x)$, if it exists.

Solution:

As $x \to \infty$, the value of $\sin(x)$ oscillates between 1 and -1 infinitely often. Therefore, the limit does not exist: no number M is as required in the definition of a limit at infinity.

Except for the laws $\lim_{x\to a} x^n = a^n$ and $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, the Limit Laws are valid for limits as $x \to \pm\infty$.

Ex. 5. Let n be a positive integer. Taking it as known that $\frac{1}{x} \to 0$ as $x \to \pm\infty$, use the Limit Laws to justify the following facts.

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Ex. 6. Find $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$. (Use the Limit Laws.)

To find the limit of any rational function as $x \to \pm\infty$, start by dividing numerator and denominator by the largest power of x appearing in the denominator.

Solution:

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - 2\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}.$$

Ex. 7. Find $\lim_{x\to\infty} \dfrac{x^5 - 1}{x^3 + 1}$. (Use the Limit Laws.)

$$\lim_{x\to\infty} \frac{x^5 - 1}{x^3 + 1} = \lim_{x\to\infty} \frac{\frac{x^5-1}{x^3}}{\frac{x^3+1}{x^3}} = \lim_{x\to\infty} \frac{x^2 - \frac{1}{x^3}}{1 + \frac{1}{x^3}} \quad \begin{matrix} \} \xrightarrow{x\to\infty} \infty \\ \} \xrightarrow{x\to\infty} 1 \end{matrix}$$

$$= \infty.$$

Ex. 8. Find $\lim_{t\to\infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$.

Not a rational function, but the same principle works:

$$\lim_{t\to\infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5} = \lim_{t\to\infty} \frac{t - t^{3/2}}{2t^{3/2} + 3t - 5} = \lim_{t\to\infty} \frac{\dfrac{t^1 - t^{3/2}}{t^{3/2}}}{\dfrac{2t^{3/2} + 3t^1 - 5t^0}{t^{3/2}}}$$

$$= \lim_{t\to\infty} \frac{t^{-1/2} - 1}{2 + 3t^{-1/2} - 5t^{-3/2}}$$

$$= \lim_{t\to\infty} \frac{\frac{1}{\sqrt{t}} - 1}{2 + \frac{3}{\sqrt{t}} - \frac{5}{t\sqrt{t}}} = -\frac{1}{2}.$$

Ex. 9. Find $\lim_{x\to\infty} \sqrt{x^2+1} - x$.

Solution: The Subtraction Law doesn't apply, because the limits $\lim_{x\to\infty} \sqrt{x^2+1}$ and $\lim_{x\to\infty} x$ don't exist. (We would get the meaningless expression $\infty - \infty$.)

Let's try rationalizing (the numerator)...

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+1} - x &= \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}\end{aligned}$$

It is incorrect to say $\lim_{x\to\infty} \sqrt{x^2+1} + x = \infty + \infty = \infty$. We can avoid this error that by noticing that

$$\sqrt{x^2+1} + x \ > \ x \ \to \infty \quad \text{as} \quad x \to \infty. \qquad (*)$$

(Recall that $\sqrt{\ }$ means the *nonnegative* square root, so we always have $\sqrt{x^2+1} \geq 0$.)

It follows that $\lim_{x\to\infty} \sqrt{x^2+1} + x = \infty$.

Ex. 10. Find $\lim_{x\to\infty} x^2 - x$, if it exists. Justify your answer.

Solution:

$$\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1) = \infty$$

because for all $x > 1$ we have $x \cdot (x-1) > x - 1$, and $(x-1) \to \infty$ as $x \to \infty$.

Curve sketching

Consider the following functions:

- $f(x) = \dfrac{2x}{x^2 - 1}$
- $g(x) = \dfrac{\cos(x)}{2 + \sin x}$
- $h(x) = \sqrt[3]{x^3 + 1}$
- $j(x) = x^{5/3} - 5x^{2/3}$

Exercise. Sketch the graphs of the above functions.

It's understandable if this exercise makes you a little bit nervous, at first glance. These don't look anything like the kinds of functions you were asked to graph in earlier classes.

But at this stage of your mathematical education, you already know everything you need to to graph these functions. You know how to *analyze* the behavior of a function using calculus.

- You can describe the shape of the graph.
- You can say where the graph rises and falls, and where it levels off.
- You can find vertical asymptotes, which show where the function "blows up" $(F(x) \to \pm\infty)$.
- And you can find the **end behavior** of $F(x)$ is (that is, the behavior of $F(x)$ as $x \to \infty$).

Before we tackle the above **Exercise**, let's review some vocabulary and facts about functions' behavior.

Review of preliminaries for curve sketching:

Let f be a function with domain D.

- f is **even** if $f(x) = f(-x)$ for all $x \in D$.
- f is **odd** if $-f(x) = f(-x)$ for all $x \in D$.
  - Are there any functions that are both odd and even?
  - Are there any functions that are odd and periodic?
  - Are there any functions that are even and periodic?
- If $p > 0$ is a constant and $f(x + p) = f(x)$ for all $x \in D$, we say f is **periodic**.

  ☞ For example, $\cos(x + 4\pi) = \cos x$ for all x in $\mathbb{R}$, so $\cos$ is periodic.
- The smallest p as above is the **period** of f. We say f is p-**periodic** if p is the period of f.

  Period of $\cos, \sin$: 2π

  Period of $\tan$: 2π

Ex. 11. Prove that $\cot$ has period π.

Solution:

For all x in the domain of $\cot$,

$$\cot(x) = \frac{1}{\tan(x)}.$$

Since

$$\cot(x+\pi) = \frac{1}{\tan(x+\pi)} = \frac{1}{\tan(x+\pi)} = \cot(\pi),$$

the function $\cot$ is π-periodic.

- To sketch the graph of a p-periodic function, just sketch the graph an interval of length p, then repeat it.
- If f is periodic with period p, then the period of the function

$$f(Ax)$$

for a constant $A \neq 0$ is

$$\frac{p}{|A|}.$$

Ex. 12. Period of $\cos(\frac{x}{5})$: $\dfrac{2\pi}{|1/5|} = 10\pi$.

Ex. 13. Period of $\tan(-2x)+1$: $\dfrac{\pi}{|-2|} = \frac{\pi}{2}$.

Ex. 14. What is the period of $\cot(1-x)$?

- The line $y = L$ is a **horizontal asymptote (HA)** if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Ex. 15. What are the horizontal asymptotes, if any, of the function $y = 2^x$?

- The line $x = a$ is a **vertical asymptote (VA)** if $\lim_{x\to a^\pm} f(x) = \pm\infty$.

Ex. 16. What are the vertical asymptotes, if any, of $f(x) = \dfrac{x-4}{x-3}$?

There is a vertical asymptote at $x = 3$, because $\lim_{x\to a^+} f(x) = \pm\infty$.

Ex. 17. What are the vertical asymptotes, if any, of $g(x) = \dfrac{x^2-6x+9}{x-3}$?

There are none, because $\lim_{x\to a} g(x) = \lim_{x\to a} x-3$ is a number for *any* $a \in \mathbb{R}$—even for $a = 3$.

- To find the vertical asymptotes of a rational function, set the denominator equal to 0 *after canceling any common factors*.

Ex. 18. Show that $x = \dfrac{\pi}{2}$ is an asymptote of $y = \tan x$ by using the definition.

The 8-step process for sketching a curve

A. Find **domain**

B. Find x- and y-**intercepts**

C. Determine **symmetry** (even/odd, periodic)

D. Find any vertical or horizontal **asymptotes**

E. Find intervals where function is **increasing**, intervals where it is **decreasing**

F. Identify any **local maxima** and **local minima**

G. Determine intervals where function is **concave up**, intervals where it is **concave down**, and identify any **inflection points**

H. **Sketch** the curve

Ex. 19. Sketch the graph of $f(x) = \dfrac{2x^2}{x^2 - 1}$ by hand.

A. Domain

$\{x \,|\, x \neq \pm 1\}$

B. Intercepts

$(0, 0)$

C. Symmetry

Even:

We test whether or not $f(-x) = f(x)$:

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} \stackrel{\checkmark}{=} f(x)$$

$\boxed{\checkmark\ f \text{ is even.}}$

Odd:

f is even and $f(x) \not\equiv 0$, so $\boxed{f \text{ is not odd.}}$

Periodic:

f is not periodic.

D. Asymptotes

HA:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

$\boxed{y = 2}$

VA:

Noting that

- f is a rational function and
- f has no common factors in numerator and denominator,

we set the denominator equal to 0 to find two vertical asymptotes $\boxed{x = \pm 1}$.

E. Increasing/decreasing

Start by finding a formula for $f'(x)$:

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$

By the Increasing/Decreasing Test (§3.3), f is increasing on an interval I if, and only if, $f'(x) > 0$ for all $x \in I$.

$$\begin{aligned}
&f'(x) > 0 \\
&\iff \frac{-4x}{(x^2-1)^2} > 0 \\
&\iff -4x > 0 && (x^2-1)^2 \text{ is always positive} \\
&\iff x < 0 && \text{Dividing by } -4 \text{ reverses the inequality symbol}
\end{aligned}$$

(Recall that the symbol $\iff$ means "if, and only if," or that two statements are "logically equivalent," i.e. both true or both false. We often don't write this symbol, but when we solve an inequality or an equation, it is silently implied between each step.)

Thus

$$\boxed{f'(x) > 0 \iff x < 0.}$$

This means

- f is increasing on the interval $\{x \,|\, x < 0\}$.
- f is decreasing on the interval $\{x \mid x > 0\}$.

F. Local max/min

$f'(x) = 0$ if, and only if, $x = 0$.

By the 1st Derivative Test (§3.3), $\boxed{f(0) = 0}$ is a local max.

G. Concavity and inflection points

$$\begin{aligned}
f''(x) = \frac{12x^2+4}{(x^2-1)^3} &> 0 \\
&\iff (x^2-1)^3 > 0 && 12x^2+4 \text{ is always positive} \\
&\iff |x| > 1
\end{aligned}$$

Thus

$$\boxed{f''(x) > 0 \iff |x| > 1.}$$

By the Concavity Test (§3.3), this means

- f is concave up on the intervals $\{x \,|\, x > 1\}$ and $\{x \,|\, x < -1\}$.
- f is concave down on the interval $\{x \mid -1 < x < 1\}$.

Ex. 20. Sketch the graph of $f(x) = \dfrac{\cos x}{2+\sin x}$ by hand.

A. Domain

$\mathbb{R}$

B. Intercepts

$(0, \frac{1}{2})$ and $\big(\frac{(2k+1)\pi}{2}, 0\big)$ for any integer k

C. Symmetry

Even:

Want to test whether or not

$$f(-x) = f(x) = \frac{\cos x}{2+\sin x}.$$

$$f(-x) = \frac{\cos(-x)}{2+\sin(-x)} = \frac{\cos(x)}{2-\sin(x)}$$

f is not even.

Odd:

Want to test whether or not

$$-f(-x) = f(x) = \frac{\cos x}{2+\sin x}.$$

$$-f(-x) = -\frac{\cos(-x)}{2+\sin(-x)} = -\frac{\cos(x)}{2-\sin(x)}$$

f is not odd.

Periodic:

Want to test whether

$$f(x+p) = f(x)$$

for some $p > 0$ yet to be determined.

Guess: The period p of f is 2π. (We guess this because $\cos$ and $\sin$ are periodic with period 2π.)

Check:

$$f(x+2\pi) = \frac{\cos(x+2\pi)}{2+\sin(x+2\pi)} = \frac{\cos(x)}{2+\sin(x)} = f(x)$$

✓ f is periodic (with period 2π)

At this point we should pick an interval I of length $p = 2\pi$.

We'll pick

$$I = [0, 2\pi].$$

We can go back to the intercepts and note which ones have $x \in I$:

<u>x-int.</u>: $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$

<u>y-int.</u>: $\left(0, \frac{1}{2}\right)$

D. Asymptotes

<u>HA</u>: None

<u>VA</u>: None

E. Increasing/decreasing

Start by finding a formula for $f'(x)$:

$$f'(x) = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

By the Increasing/Decreasing Test (§3.3), f is increasing on an interval I if, and only if, $f'(x) > 0$ for all $x \in I$.

$$\begin{aligned} f'(x) > 0& \\ &\iff -\frac{2\sin x + 1}{(2 + \sin x)^2} > 0 \\ &\iff -2\sin x - 1 > 0 \\ &\iff \sin x < -\frac{1}{2} \\ &\iff \frac{7\pi}{6} < x < \frac{11\pi}{6} \end{aligned}$$

Thus

$$\boxed{f'(x) > 0 \iff \frac{7\pi}{6} < x < \frac{11\pi}{6}.}$$

This means

- f is increasing on the interval $\{x \mid \frac{7\pi}{6} < x < \frac{11\pi}{6}\}$.
- f is decreasing on the intervals $\{x \mid 0 < x < \frac{7\pi}{6}\}$ and $\{x \mid \frac{11\pi}{6} < x < 2\pi\}$.

F. Local max/min

Using the 1st Derivative Test, $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ is a local min, and $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$ is a local max.

G. Concavity and inflection points

Want to know when

$$f''(x) = \frac{-2\cos x(1-\sin x)}{(2+\sin x)^3}$$

is positive.

- Since $-1 \le \sin x \le 1$, we know $(1-\sin x)$ in the numerator is always nonnegative.
- Also since $-1 \le \sin x \le 1$, we know $2+\sin x$ is always positive, so $(2+\sin x)^3$ is always positive.
- Since

  $$f''(x) = -2\cos x \left.\frac{1-\sin x}{(2+\sin x)^3}\right\} \text{ fraction is always nonnegative}$$

  we conclude that

  $$f''(x) > 0 \iff -2\cos x > 0 \iff \cos x < 0 \iff \frac{\pi}{2} < x < \frac{3\pi}{2}$$

  for $x \in I = [0, 2\pi]$.

By the Concavity Test,

- f is concave up on the interval $\{x \mid -\frac{\pi}{2} < x < \frac{3\pi}{2}\}$.
- f is concave down on the intervals $\{x \,|\, 0 < x < \frac{\pi}{2}\}$ and $\{x \,|\, \frac{3\pi}{2} < x < 2\pi\}$.

<u>Ex. 21.</u> Sketch the graph of $f(x) = \sqrt[3]{x^3 + 1}$ by hand.

A. Domain

$\mathbb{R}$

B. Intercepts

$(-1, 0)$, $(0, 1)$

C. Symmetry

<u>Even</u>:

$f(-x) = \sqrt[3]{(-x)^3 + 1} \neq \sqrt[3]{x^3 + 1}$

(Take $x = 1$, then $f(-1) = 0$ but $f(1) = \sqrt[3]{2}$.)

$\boxed{f \text{ is not even.}}$

<u>Odd</u>:

$\boxed{f \text{ is not odd.}}$

<u>Periodic</u>:

$\boxed{f \text{ is not periodic}}$ because $f(x) \to \infty$ as $x \to \infty$

D. Asymptotes

<u>HA</u>:

$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

$\boxed{\text{No HA}}$

<u>VA</u>:

$\boxed{\text{No VA}}$ because f is continuous

E. Increasing/decreasing

$f'(x) = \frac{1}{3} 3x^2 (x^3 + 1)^{-2/3} = x^2 (x^3 + 1)^{-2/3}$

$$\begin{aligned} & f'(x) > 0 \\ \iff & \frac{x^2}{\sqrt[3]{(x^3+1)^2}} > 0 \\ \iff & \sqrt[3]{(x^3+1)^2} > 0 \end{aligned}$$

Always true.

$\boxed{f \text{ is increasing on } (-\infty, \infty)}$

F. Local max/min

Since $f'(x) \neq 0$ for all $x \in \mathbb{R}$, there are no local maxima or minima (hence no global maxima or minima).

G. Concavity and inflection points

$$
\begin{aligned}
f''(x) = \frac{d}{dx}\left[x^2(x^3+1)^{-2/3}\right] > 0 \\
&\iff -\frac{2}{3}x^2(x^3+1)^{-5/3}(3x^2) + 2x(x^3+1)^{-2/3} \\
&\iff \frac{-2x^4}{(x^3+1)^{5/3}} + \frac{2x}{(x^3+1)^{2/3}} > 0 \\
&\iff \frac{-2x^4 + 2x(x^3+1)}{(x^3+1)^{5/3}} \\
&\iff \frac{2x}{(x^3+1)^{5/3}} > 0 \\
&\iff x > 0
\end{aligned}
$$

Thus

$$\boxed{f''(x) > 0 \iff x > 0.}$$

By the Concavity Test (§3.3), this means

- f is concave up on the interval $\{x \mid x > 0\}$.
- f is concave down on the interval $\{x \mid x < 0\}$.

Ex. 22. Sketch the graph of $f(x) = x^{5/3} - 5x^{2/3}$ by hand.

A. Domain

$\mathbb{R}$

B. Intercepts

$(0, 0)$ and $(5, 0)$:

$$x^{5/3} - 5x^{2/3} = 0 \qquad \Longrightarrow \qquad x^{2/3}(x-5) = 0$$

C. Symmetry

No symmetry

D. Asymptotes

No asymptotes

E. Increasing/decreasing

$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2)$

$$\begin{aligned} f'(x) &> 0 \\ \iff \frac{5}{3}x^{-1/3}(x-2) &> 0 \\ \iff x < 0 &\text{ or } x > 2 \end{aligned}$$

- f is increasing on the intervals $\{x \mid x < 0\}$ and $\{x \mid x > 2\}$.
- f is decreasing on the interval $\{x \mid 0 < x < 2\}$.

F. Local max/min

Using the 1st Derivative Test, $\boxed{f(0) = 0 \text{ is a local max, and } f(2) = -3\sqrt[3]{4} \text{ is a local min.}}$

G. Concavity and inflection points

$f''(x) = \frac{d}{dx}\left[\frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}\right].$

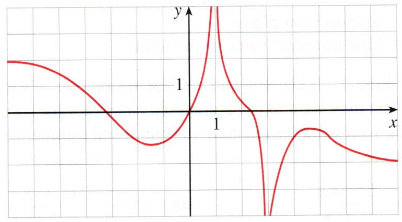
$$f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} > 0$$

if and only if $x > -1$

- f is concave up on the interval $\{x \mid x > -1\}$.
- f is concave down on the interval $\{x \mid x < -1\}$.

Additional exercises

Ex. 23. For the function f whose graph is given, state the following.



(a) $\lim_{x\to\infty} f(x)$ **(b)** $\lim_{x\to-\infty} f(x)$ **(c)** $\lim_{x\to 1} f(x)$ **(d)** $\lim_{x\to 3} f(x)$

(e) The equations of all asymptotes

Ex. 24. Find the limit or show that it does not exist.

(a) $\lim_{x\to\infty} \frac{4x+3}{5x-1}$

(b) $\lim_{t\to-\infty} \frac{3t^2+t}{t^3-4t+1}$

(c) $\lim_{x\to\infty} \frac{4-\sqrt{x}}{2+\sqrt{x}}$

(d) $\lim_{u\to-\infty} \frac{(u^2+1)(2u^2-1)}{(u^2+2)^2}$

(e) $\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1}$

(f) $\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$

(g) $\lim_{t\to\infty} \left(\sqrt{25t^2+2}-5t\right)$

(h) $\lim_{x\to\infty} \left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

(i) $\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$

Workbook Lesson 21

§4.7, Applied Optimization Problems

Last revised: 2021-03-23 11:41

Objectives

- Set up and solve optimization problems in several applied fields.

Warmup

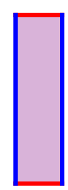
Problem:

- We have a pipe cleaner 8 inches long.
- We want to bend the pipe cleaner into the shape of a rectangle.
- How can we make the space inside the rectangle as large as possible?

We'll need to bend the pipe cleaner into four segments (the four sides of the rectangle).

Since the pipe cleaner is 8 inches long, the perimeter of the rectangle will also be 8 inches long.

perimeter = 8



We want to make the area of the rectangle as large as possible. What's a formula for the area?

$$\text{AREA} = \text{LENGTH} \times \text{WIDTH}$$
$$A = \ell w$$

In order to use the tools of single-variable calculus, we need to write the area A in terms of a single variable.

Can we write the length ℓ in terms of the width w?

$$2\ell + 2w = 8 \qquad \text{(perimeter} = 8\text{)}$$
$$2\ell = 8 - 2w$$
$$\ell = 4 - w$$

Now we can write the area A as a function of a single variable, w:

$$A(w) = (4 - w)w$$

Question: What value of w maximizes $A(w)$?

To answer the question and solve the problem, see applet on iCollege: "`Pipe cleaner`"

Optimization problems

We will now look at a variety of word problems in which a certain quantity Q (for instance, cost, profit, distance, angle, area, volume) is to be maximized or minimized.

We express the quantity as a function, $Q = Q(x)$, and find its extreme values by solving the equation $Q'(x) = 0$.

Now, most real-world quantities depend on more than one variable. Even something as simple as the area of a rectangle depends on two variables, length and width: $A = A(\ell, w)$. So, in order to pose a given question as an extreme value problem, we will often have to do some work to write Q as a function of one variable.

Exercises

Ex. 1. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r. (*Hint*: Let the base of the rectangle be a line segment in the x-axis with the origin as its midpoint.)

Ex. 2. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

Ex. 3. What is the minimal vertical distance between the parabolas $y_1 = x^2 + 1$ and $y_2 = x - x^2$?

Ex. 4. A box with a square base and open top must have a volume of $32,000$ cm^3. Find the dimensions ($\ell \times w \times h$) of the box that minimize the amount of material used.

Ex. 5. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

Ex. 6. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a maximum? (A calculator will be needed.)

☞ *The setups for* **Ex. 1** *and* **Ex. 3** *are outlined in the applets* “`Inscribed rectangle`” *and* “`Vertical distance`” *on iCollege.*

Optimization problems—General Strategy

In an **optimization problem**, a quantity Q (e.g., cost, profit, distance, angle, area, volume, ...) is to be maximized or minimized.

We express the quantity as a function, $Q = Q(x)$, and find its extreme values by solving the equation $Q'(x) = 0$.

☞ The following process is a suggestion, not a requirement. You are free to organize your work on these problems however you like, as long as your work makes sense.

#4. A box with a square base and open top must have a volume of $32,000$ cm^3. Find the dimensions of the box that minimize the amount of material used.

① Write a legend. Draw a picture, if applicable. It may be helpful to avoid using equations in this step.	*DRAW PICTURE FIRST* Key: Q: amount of material used b: side length of square base h: height of box V: volume of box
② Identify what the question asks for.	Want: Dimensions of box for min value of Q
③ Write any constraints you are given. (A **constraint** is a relation between the variables you listed in Step 1 other than Q.)	Constraints: $V = b^2h = 32,000$
④ Use the given constraints to write a formula for Q AS A FUNCTION OF 1 VARIABLE.	Q depends on b and h: $Q = b^2 + 4bh$ But h is a function of b: $h = 32,000b^{-2}$ So $Q = b^2 + 4b(32,000b^{-2}) = b^2 + 128,000b^{-1}$

⑤ Solve the extreme value problem for Q.

$$Q'(b) = 2b - 128,000b^{-2} = 0$$
$$2b^3 - 128,000 = 0$$
$$b^3 = \sqrt[3]{64,000} = 40 \ldots \text{so } b = 40 \text{ is an extreme value.}$$

Is $b = 40$ a local min or a local max? *Draw sign chart.* Test points: $Q'(1) < 0$ and $Q'(1,000,000) > 0$, so by the First Derivative Test, $b = 40$ is a local min.

⑥ Give the information asked for in Step 2.	The dimensions of the box are $\boxed{40 \times 40 \times 20}$, because if $b = 40$, then $h = \frac{32,000}{40^2} = \frac{32,000}{1,600} = 20$.

Optimization Problems—Legends

Ex. 1.

A: Area of rectangle
r: Radius of semicircle *(constant)*
$2x$: Width of rectangle
y: Height of rectangle

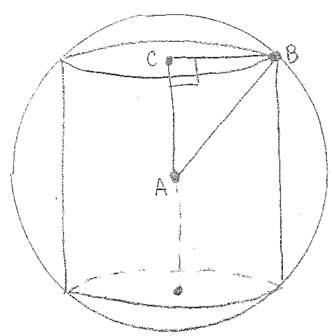
Ex. 2.

x: 1st number
y: 2nd number
Q: sum of their squares

Ex. 3.

y_1, y_2: heights of the two parabolas
$Q = Q(x)$: vertical distance between the two parabolas at a given x-coordinate

Ex. 5.



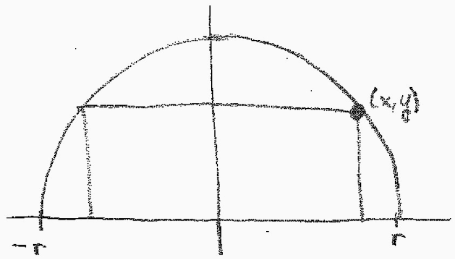
$r = AB$: radius of sphere *(constant)*
$b = BC$: radius of base of cylinder
$x = AC$: half the height of the cylinder *(height of triangle shown)*
$Q = Q(x)$: volume of cylinder

#6.

a : length of wire bent into the square
$s = \frac{1}{4}a$: side length of square
$t = \frac{1}{3}(10 - a)$: side length of triangle
h : height of triangle
Q : total area

Ex. 1.

Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



① Key:

A: Area of rectangle

r: Radius of semicircle (Constant)

2x: Width of rectangle

y: Height of rectangle

② Want: max value of A

③ Constraints:

$x^2 + y^2 = r^2$

④ A depends on x and y:

$$A = 2xy$$

y is a function of x:

$$y = \sqrt{r^2 - x^2} \quad (y \geq 0)$$

So A is a function of x:

$$A = 2x\sqrt{r^2 - x^2}$$

⑤ $A(x) = 2x\,(r^2 - x^2)^{\frac{1}{2}}$

$$A'(x) = 2(r^2-x^2)^{1/2} + 2x\left(\tfrac{1}{2}\right)(r^2-x^2)^{-1/2}(-2x)$$

$$= 2(r^2-x^2)^{1/2} - 2x^2(r^2-x^2)^{-1/2}$$

Solve

$$A'(x) = 0$$

$$2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}} = 0$$

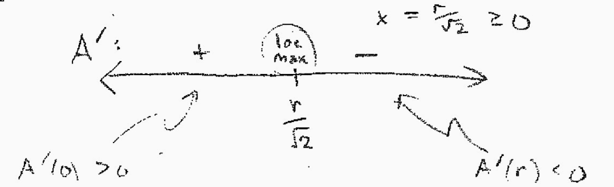
$$\frac{2(r^2-x^2) - 2x^2}{\sqrt{r^2-x^2}} = 0$$

$$2(r^2 - x^2) - 2x^2 = 0$$

$$2r^2 - 4x^2 = 0$$

$$x = \pm\sqrt{\frac{r^2}{2}} = \pm\frac{r}{\sqrt{2}}$$

$$x = \frac{r}{\sqrt{2}} \geq 0$$



⑥ $A\left(\frac{r}{\sqrt{2}}\right) = 2\left(\frac{r}{\sqrt{2}}\right)\sqrt{r^2 - \frac{r^2}{2}}$

$$= \sqrt{2}\,r \cdot \frac{1}{\sqrt{2}}\,r$$

$$= \boxed{r^2}.$$

Ex. 2.

The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

① x : 1st number

y : 2nd number

Q : sum of their squares

② Want: Minimal value of Q

③ Constraints: $x+y=16$

④ Q depends on x and y:

$Q = x^2+y^2$

y is a function of x:

$y = 16-x$

So Q is a function of x:

$Q = x^2+(16-x)^2$

⑤ Solve the extreme value problem for Q:

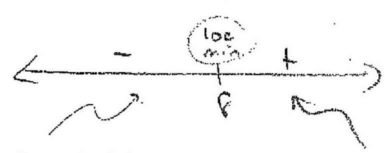
$Q'(x) = 2x + 2(16-x)(-1)$

Solve: $Q'(x) = 0$

$2x+2(x-16) = 0$

$x = 8$

Is $Q(8)$ a max or a min value?



Test values: $Q'(0) < 0$ $Q'(10) > 0$

By 1st Derivative Test, $Q(8)$ is a minimum value.

⑥ Want $Q(8)$.

$Q(x) = x^2+(16-x)^2$

$Q(8) = 8^2+8^2 = \boxed{128}$.

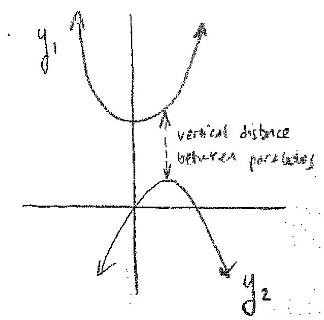
Ex. 3.

What is the minimal vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

① Sketch the graphs first.

$y_1 = x^2 + 1$ is concave up (leading coeff > 0)

$y_2 = x - x^2$ is concave down (leading coeff. < 0)



Key: y_1, y_2 : height of parabolas

Q : vertical distance between parabolas

② Want minimal value of Q.

③ Constraints: $y_1 = x^2 + 1$

$y_2 = x - x^2$

④ Equation for Q:

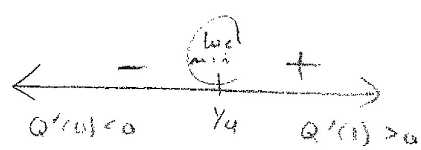
$$Q = \text{distance between heights of parabolas}$$
$$= |y_1 - y_2|$$
$$= |(x^2+1) - (x - x^2)|$$
$$= |2x^2 - x + 1|.$$

⑤ Case 1: $2x^2 - x + 1 \geq 0$.

$Q(x) = 2x^2 - x + 1$

$Q'(x) = 4x - 1 = 0$

$x = \frac{1}{4}$



Case 2: $2x^2 - x + 1 < 0$.

$Q(x) = -2x^2 + x - 1$

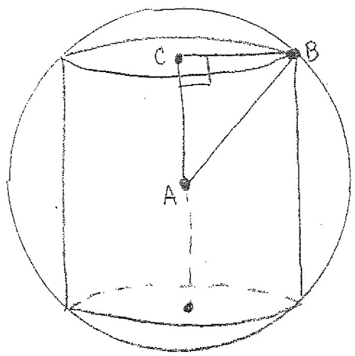
$Q'(x) = -4x + 1 = 0$

$x = \frac{1}{4}$ So no further work is needed on step ⑤.

⑥ The minimal value is $Q(\frac{1}{4}) = 2 \cdot \frac{1}{16} - \frac{1}{4} + 1 = \boxed{\frac{7}{8}}$.

Ex. 5.

A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

① 

Key: $r = AB$ = radius of sphere (constant)

$b = BC$ = radius of base of cylinder

$x = AC$ = half the height of cylinder (height of triangle shown)

② Want: Max volume of V.

③ Constraints: $b^2 + x^2 = r^2$ (Pythagorean Formula)

④ Equation for V: V depends on b and h: $V = \pi b^2 \cdot 2x$ (πb^2 = area of base)

b is a function of x: $b = \sqrt{r^2 - x^2}$ $(b \geq 0)$

So V is a function of x: $V = 2\pi x (r^2 - x^2)$

⑤ Maximize V:

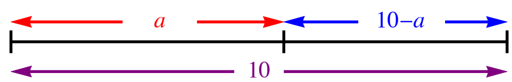
$$V(x) = 2\pi (r^2 x - x^3)$$
$$V'(x) = 2\pi (r^2 - 3x^2)$$
$$0 = r^2 - 3x^2$$
$$x = r/\sqrt{3} \quad (x \geq 0)$$

Since $V'(x) = -6\pi x^2 + 2\pi r^2$ is a quadratic with leading coeff < 0, its graph is concave down ∩, so V(x) is a maximum value.

⑥ $V(x) = 2\pi x (r^2 - x^2)$

$$V(r/\sqrt{3}) = \frac{2\pi r}{\sqrt{3}}\left(r^2 - \frac{r^2}{3}\right) = \frac{2\pi r}{\sqrt{3}} \cdot \frac{2r^2}{3} = \boxed{\frac{4\pi r^3}{3\sqrt{3}}}.$$

Ex. 6. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a maximum?



$$\begin{aligned} a &: \text{length of wire bent into the square} \\ s = \tfrac{1}{4}a &: \text{side length of square} \\ t = \tfrac{1}{3}(10-a) &: \text{side length of triangle} \\ h &: \text{height of triangle} \\ Q &: \text{total area} \end{aligned}$$

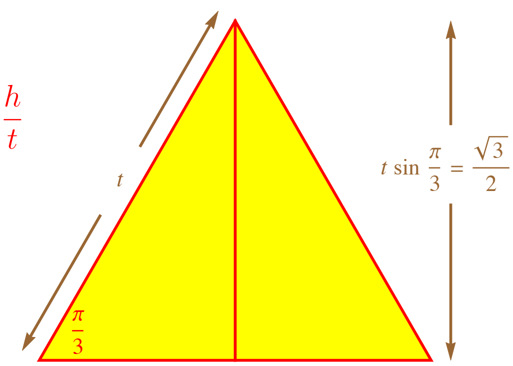
Want: How to cut wire into pieces of length a and $10 - a$ so that Q is maximized.

Constraints:

$$a \le 10$$

$$h = \tfrac{\sqrt{3}}{2}t = \tfrac{1}{2\sqrt{3}}(10-a) \qquad \Longleftarrow \qquad \sin\left(\tfrac{\pi}{3}\right) = \frac{\text{OPP.}}{\text{HYP.}} = \frac{h}{t}$$

(Since an *equilateral* triangle is required, the angle at each vertex of the triangle is $60^\circ = \frac{\pi}{3}$.)



Equation for Q:

$$\begin{aligned} Q &= \text{area of square} + \text{area of triangle} \\ &= s^2 + \frac{1}{2}th \\ &= \frac{1}{16}a^2 + \frac{1}{12\sqrt{3}}(10-a)^2 \end{aligned}$$

Solve the extreme value problem for Q:

$$\begin{aligned} Q'(a) = \frac{1}{8}a + \tfrac{1}{6\sqrt{3}}(10-a)\cdot(-1) &= 0 \\ \tfrac{1}{8}a + \tfrac{1}{6\sqrt{3}}a &= \tfrac{5}{3\sqrt{3}} \\ \tfrac{1}{8}a + \tfrac{1}{6\sqrt{3}}\left(\tfrac{\sqrt{3}}{\sqrt{3}}\right)a &= \tfrac{5}{3\sqrt{3}} \\ \tfrac{1}{8}\left(\tfrac{9}{9}\right)a + \tfrac{\sqrt{3}}{18}\left(\tfrac{4}{4}\right)a &= \tfrac{5}{3\sqrt{3}} \\ \tfrac{9+4\sqrt{3}}{72}a &= \tfrac{5}{3\sqrt{3}} \\ a &= \tfrac{120}{9\sqrt{3}+12} \approx 4.34965. \end{aligned}$$

Apply Closed Interval Theorem for $s \in [0, 10]$:

$Q(0) \approx 4.81125$
$Q\left(\frac{120}{9\sqrt{3}+12}\right) \approx 2.71853$
$Q(10) = 6.25$

Q is maximized when $a = 10$.

Answer: Don't cut the wire! The entire 10 m of the wire should be bent into a square.

Additional exercises

Ex. 7 (§4.7—#318). Find two positive integers a and b such that $a+b=10$ and such that a^2+b^2 is maximized.

Ex. 8 (§4.7—#320). You have 400 ft. of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Ex. 9 (§4.7—#315). To carry a box on an airplane, the length + width + height of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is

$$V = \left(31 - \tfrac{1}{2}h\right)^2 h.$$

What height allows you to have the largest volume?

Ex. 10 (§4.7—#347). Find the point on the line $y = 5 - 2x$ that is closest to the origin.

Ex. 11 (§4.7—#349). Find the point on the parabola

$$y = x^2$$

that is closest to the point $(2,0)$.

Ex. 12 (§4.7—#324, 325). A patient's pulse measures 70 bpm, then 80 bpm, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression

$$(x-70)^2 + (x-80)^2 + (x-120)^2.$$

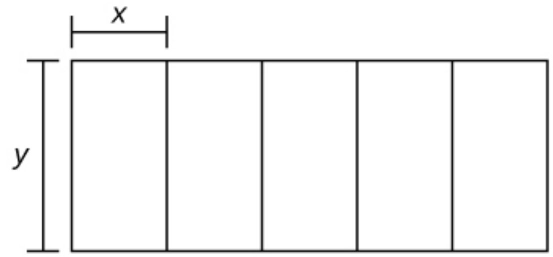
(a) What value minimizes the above expression?

(b) In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes the following expression?

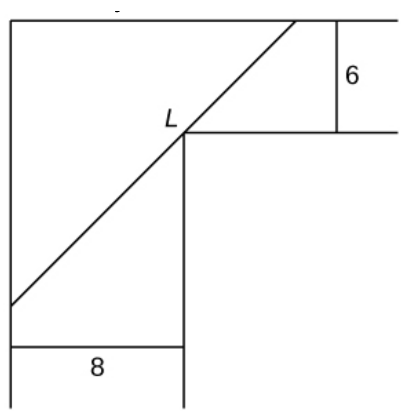
$$(x-70)^2 + (x-80)^2 + \tfrac{1}{2}(x-120)^2$$

Ex. 13 (§4.7—#355). You are the manager of an apartment complex with 50 units. When you set rent at $800/month, all apartments are rented. As you increase rent by $25/month, one fewer apartment is rented. Maintenance costs run $50/month for each occupied unit. What is the rent that maximizes the total amount of profit?

Ex. 14 (§4.7—#354). You are building five identical pens adjacent to each other with a total area of 1000m^2, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



Ex. 15 (§4.7—#323). You are moving into a new apartment and notice that there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Workbook Lesson 22

§4.8, L'Hôpital's Rule

Last revised: 2020-09-29 12:43

Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

Indeterminate quotients

Recall: We used an ad hoc geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

"Plugging in" $x = 0$ to evaluate this limit yields $\frac{0}{0}$. In general, a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \qquad \text{where } f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a$$

is called an **indeterminate form of type** $0/0$.

Similarly, we define an **indeterminate form of type** ∞/∞ to be a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \qquad \text{where both } f(x) \to \pm\infty \text{ and } g(x) \to \pm\infty \text{ as } x \to a$$

Shortly, we will see several other types of indeterminate form, including $\infty - \infty$, $0 \cdot \infty$, ∞/∞, ∞^0, and 1^∞. In this section, we'll look at a systematic method, called *L'Hôpital's rule*, that can be used to evaluate indeterminate forms.

L'Hôpital's Rule. Let f and g be differentiable functions, with $g'(x) \neq 0$ near a (except possibly at a). Here, we allow $a = \pm\infty$. Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

(In other words, we have an indeterminate form of type $\boxed{0/0}$ or $\boxed{\infty/\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right-hand side exists, is ∞, or is $-\infty$.

Ex. 1. Show that $\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$.

Ex. 2. Show that $\lim_{x \to \infty} \frac{e^x}{x^2} = \infty$. ("There is a struggle between the numerator and the denominator." Which one outraces the other as $x \to \infty$?)

Ex. 3. Show that $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = 0$.

Ex. 4. Show that $\lim_{x \to 0} \frac{\tan(x) - x}{x^3} = \frac{1}{3}$.

Ex. 5. Show that $\lim_{x \to \pi^-} \frac{\sin(x)}{1 - \cos x} = 0$.

Ex. 6. Find $\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{\csc\theta}$.

Solution:

$$\begin{aligned}
\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{\csc\theta} &= \lim_{\theta \to \pi/2} \frac{-\cos\theta}{-\cot\theta \csc\theta} \\
&= \lim_{\theta \to \pi/2} \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}} \\
&= \lim_{\theta \to \pi/2} \cos\theta \frac{\sin\theta}{\cos\theta} \sin\theta \\
&= \lim_{\theta \to \pi/2} \sin^2\theta = 0.
\end{aligned}$$

Ex. 7. Find $\lim_{x\to\infty} \frac{x^3}{e^{x^2}}$.

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3}{e^{x^2}} &= \lim_{x\to\infty} \frac{3x}{2e^{x^2}} \\ &= \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0.\end{aligned}$$

Notice that the limit

$$\lim_{x\to\infty} \frac{3}{4xe^{x^2}}$$

is *not* an indeterminate form: although the denominator approaches ∞, the numerator is finite.

☞ Expressions involving ∞ and division by 0 are not numbers.

For example, we defined

$$\lim_{x\to a} f(x) = \infty \qquad (\dagger)$$

to mean that $f(x)$ grows larger than any finite number as $x \to a$. But (†) is not a statement that two numbers are equal, if by the word "number" you mean a point on the real number line.[1]

In general, expressions that involve infinity symbols or division by zero should be understood as statements about the *behavior* of a function $f(x)$ as we take a limit.

Indeterminate products

What is the limit of $2x \ln x$ as $x \to 0^+$?

The factors $f(x) = 2x$ and $g(x) = \ln x$ have competing behaviors as $x \to 0^+$.

- As $x \to 0^+$, the factor $2x$ gets closer and closer to 0.
- But $\ln x \to \infty$ as $x \to 0^+$.
- It's not obvious which behavior will win out.
- If $2x$ wins, the limit is 0. If $\ln x$ wins, the limit is ∞.
- It's also possible in principle that the two competing behaviors reach a stalemate, in which the case the limit is some finite nonzero number.

[1] It is possible to "extend" the real numbers so that we can perform certain arithmetic operations involving ∞, but we will not do this in the calculus sequence.

When we encounter such a struggle $f(x) \cdot g(x)$, we can rewrite the product as

$$h(x) = \frac{f(x)}{1/g(x)} \qquad \text{or} \qquad h(x) = \frac{g(x)}{1/f(x)}. \tag{††}$$

Then l'Hôpital's Rule can be applied to the limit of the quotient.

Definition.

- An **indeterminate form of type** $0 \cdot \infty$ is a limit of the form

$$\lim_{x \to a} \left[f(x) \cdot g(x) \right] \qquad \text{where } f(x) \to 0 \text{ and } g(x) \to \pm\infty \text{ as } x \to a.$$

- An **indeterminate form of type** $\infty \cdot \infty$ is a limit of the form

$$\lim_{x \to a} \left[f(x) \cdot g(x) \right] \qquad \text{where both } f(x) \to \pm\infty \text{ and } g(x) \to \pm\infty \text{ as } x \to a.$$

Ex. 8. Find $\lim_{x \to 0^+} x \ln x$.

Solution:

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{1/x} \\
&= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \\
&= \lim_{x \to 0^+} \frac{1}{-1/x} \\
&= \lim_{x \to 0^+} (-x) = 0.
\end{aligned}$$

What if we had started by writing

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}?$$

We can still apply l'Hôpital's Rule, since the righthand limit is an indeterminate form of type $0/0$. But taking the derivative of the numerator and denominator yields a more complicated expression

$$\frac{1}{-1/x(\ln x)^2}.$$

When deciding which of the two ways in (††) to rewrite the product, we should try to choose the option that leads to a simpler limit.

Ex. 9. Find $\lim_{x \to \infty} x^3 e^{-x^2}$.

Solution:

$$\lim_{x \to \infty} x^3 e^{-x^2} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = 0 \qquad \text{(see above)}.$$

Ex. 10. Find $\lim_{x\to 0} \cot 2x \sin 6x$.

Solution:

$$\begin{aligned}\lim_{x\to 0} \cot 2x \sin 6x &= \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} \\ &= \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = 3.\end{aligned}$$

Indeterminate differences

I claim that

$$\lim_{x\to\infty} \Big[\underbrace{\sqrt{x}}_{\substack{\downarrow \\ \infty}} \underbrace{- \sqrt{x-1}}_{\substack{\downarrow \\ -\infty}}\Big] = 0.$$

Can you prove this?[2] This example illustrates the idea of two functions competing with each other in a limit. Here, we have a stalemate, as $\sqrt{x}$ and $\sqrt{x-1}$ turn out to be approximately equal when x grows large.

On the other hand,

$$\lim_{x\to 0} \Bigg[\underbrace{\frac{1}{x^4}}_{\substack{\downarrow \\ \infty}} \underbrace{- \frac{1}{x^2}}_{\substack{\downarrow \\ -\infty}}\Bigg] = \infty.$$

In this example, $\frac{1}{x^4} \to \infty$ wins, and $-\frac{1}{x^2} \to -\infty$ loses.

An **indeterminate form of type** $\infty - \infty$ is a limit of the form

$$\lim_{x\to a} \big[f(x) - g(x)\big] \qquad \text{where both } f(x) \to \infty \text{ and } g(x) \to \infty \text{ as } x \to a.$$

In order to evaluate an indeterminate difference, we should try to rewrite the difference as a fraction, because l'Hôpital's Rule applies only to indeterminate quotients.

For example, we find that

$$\lim_{x\to 0} \left[\frac{1}{x^4} - \frac{1}{x^2}\right] = \lim_{x\to 0} \frac{1-x^2}{x^4} \overset{\text{(l'Hôpital)}}{=} \lim_{x\to 0} \frac{-2x}{4x^3} = \lim_{x\to 0} \frac{-2}{4x^2} = 0.$$

---

[2] *Hint*:

$$\sqrt{x} - \sqrt{x-1} = \left(\sqrt{x} - \sqrt{x-1}\right) \frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}} = \frac{1}{\sqrt{x} + \sqrt{x-1}}.$$

Ex. 11. Find $\lim\limits_{x\to(\pi/2)^-} \big(\sec x - \tan x\big)$.

Solution:

$$\lim_{x\to(\pi/2)^-} \big(\sec x - \tan x\big) = \lim_{x\to(\pi/2)^-} \left(\frac{1-\sin x}{\cos x}\right) = \lim_{x\to(\pi/2)^-} \left(\frac{-\cos x}{-\sin x}\right) = 0.$$

Indeterminate powers

A limit of the form $\lim\limits_{x\to a} \big[f(x)\big]^{g(x)}$ is an **indeterminate form of**...

...**type** 0^0 if $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$,

...**type** ∞^0 if $f(x)\to\infty$ and $g(x)\to 0$ as $x\to a$,

...**type** 1^∞ if $f(x)\to 1$ and $g(x)\to\infty$ as $x\to a$.

To evaluate limits of these types, we can either take logarithms, writing

$$y = \big[f(x)\big]^{g(x)} \qquad \Longrightarrow \qquad \ln y = \underbrace{g(x)}_{\substack{\downarrow \\ \infty}} \cdot \underbrace{\ln f(x)}_{\substack{\downarrow \\ \infty}}.$$

or rewrite $\big[f(x)\big]^{g(x)}$ using the identity $b^p = e^{p\ln b}$ (recall that this identity follows from the cancellation law for $\exp$ and $\ln$).

$$\big[f(x)\big]^{g(x)} = e^{g(x)\cdot\ln f(x)} = \exp\big(\underbrace{g(x)}_{\substack{\downarrow \\ \infty}} \cdot \underbrace{\ln f(x)}_{\substack{\downarrow \\ \infty}}\big).$$

Ex. 12. Find $\lim_{x\to 0^+} x^{\sqrt{x}}$.

Solution:

Set $y(x) = x^{\sqrt{x}}$. Then $\ln y = \sqrt{x}\ln x$, and

$$\begin{aligned}\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} &= \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \\ &= \lim_{x\to 0^+} -2x^{-1}x^{3/2} \\ &= \lim_{x\to 0^+} -2\sqrt{x} = 0.\end{aligned}$$

Now use the fact that $y = e^{\ln y}$:

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = \boxed{1}.$$

Ex. 13. Find $\lim_{x\to 0}(1-2x)^{1/x}$.

Solution:

$$y := (1-2x)^{1/x}.$$

$$\ln y = \frac{1}{x}\ln(1-2x).$$

$$\begin{aligned}\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln(1-2x)}{x}\\ &= \lim_{x\to 0}\frac{-2/(1-2x)}{1} = \boxed{-2}.\end{aligned}$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{-2} = \boxed{1/e^2}.$$

Some theorems proved by l'Hôpital's Rule

Theorem If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)} = 0$.

(We might restate this informally by saying, "The form 0^∞ is not indeterminate.")

Theorem. If f' is continuous, then $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x-h)}{2h} = f'(x)$.

Theorem. If f'' is continuous, then $\lim\limits_{h\to 0}\dfrac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$.

Ex. 14. Prove the previous two theorems.

Additional exercises

Ex. 15. Suppose that

$$\lim_{x\to a} f(x) = 0, \qquad \lim_{x\to a} g(x) = 0, \qquad \lim_{x\to a} h(x) = 1,$$

$$\lim_{x\to a} p(x) = \infty, \qquad \lim_{x\to a} q(x) = \infty.$$

- Which of the following are indeterminate forms?
- For any limit that is not an indeterminate form, evaluate it if possible.

(a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$

(b) $\lim\limits_{x\to a} \dfrac{f(x)}{p(x)}$

(c) $\lim\limits_{x\to a} \dfrac{h(x)}{p(x)}$

(d) $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)}$

(e) $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$

(f) $\lim\limits_{x\to a} \big[f(x) - p(x)\big]$

(g) $\lim\limits_{x\to a} \big[p(x) - q(x)\big]$

(h) $\lim\limits_{x\to a} \big[p(x) + q(x)\big]$

Ex. 16 (§4.8—#357, 359, 363, 365, 369, 371, 375, 377, 383, 385, 387). Evaluate the limit, if possible. If L'Hôpital's Rule can't be applied, explain why not.

(a) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^k}$

(b) $\lim\limits_{x\to a} \dfrac{x-a}{x^2-a^2}$

(c) $\lim\limits_{x\to\infty} x^{1/x}$

(d) $\lim\limits_{x\to\infty} \dfrac{x^2}{1/x}$

(e) $\lim\limits_{x\to 0} \dfrac{(1+x)^{-2}-1}{x}$

(f) $\lim\limits_{x\to\pi} \dfrac{x-\pi}{\sin(x)}$

(g) $\lim\limits_{x\to 0} \dfrac{\sin(x)-\tan(x)}{x^3}$

(h) $\lim\limits_{x\to 0} \dfrac{e^x-x-1}{x^2}$

(i) $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$

(j) $\lim\limits_{x\to 0^+} x\ln x^4$

(k) $\lim\limits_{x\to\infty} x^2e^{-x}$

(l) $\lim\limits_{x\to\infty} \left(1-\frac{1}{x}\right)^x$

<u>Ex. 17.</u> If an initial amount A_0 of money is invested in an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

If we let $n \to \infty$, we refer to the **continuous compounding** of interest. Use L'Hôpital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}.$$

Workbook Lesson 24

§4.10, Antiderivatives

Last revised: 2021-04-22 12:35

Objectives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

Ex. 1. If $F'(x) = 3x^2$, what can F be? Give two different answers.

Definition. A function F is an **antiderivative** of f on an open interval I if $F'(x) = f(x)$ for all x in I.

We haven't given you any formulas for finding antiderivatives yet. This was intentional.

- At this stage, you are asked to simply *guess* what the antiderivative is.
- Check your answer by differentiating.
- If your answer is a little bit off, modify your guess and try again.

Ex. 2. Using trial and error, find an antiderivative of f.

(a) $f(x) = 0$

(b) $f(x) = \sin x$

(c) $f(x) = e^x$

(d) $f(x) = 1 - 5x^3$

(e) $f(x) = \dfrac{1}{\sqrt[3]{x-1}}$

Recall:

> *(Section 4.4)* If the derivative of a function f is 0 on an open interval $I = (a, b)$, then f is constant on I.

We can use the above result to prove the following fact:

Corollary. Let F be an antiderivative of f on an open interval I. Every antiderivative G of f on I is of the form

$$G(x) = F(x) + C$$

for some $C = \text{const}$.

Proof:

Let F and G be antiderivatives of f on I. Then

$$F'(x) = f(x) \qquad \text{and} \qquad G'(x) = f(x)$$

for all x in I, so

$$F'(x) - G'(x) = f(x) - f(x) = 0.$$

Since

$$F'(x) - G'(x)$$

is the derivative of the function

$$F(x) - G(x),$$

the Theorem tells us that

$$F(x) - G(x) = C \qquad \text{for some } C = \text{const}.$$

Therefore,

$$G(x) = F(x) + D$$

for some $D = \text{const}$.

$\square$

Definition. If F is an antiderivative of f on an open interval I, we call the expression

$$F(x) + C \qquad (C = \text{const})$$

the **most general antiderivative** of f on I.

Ex. 3. Find the most general antiderivative of

(a) $f(x) = 1$

(b) $f(x) = \sec^2 x$

(c) $f(x) = x^n$, where $n \geq 0$

(d) $f(x) = x^{-3}$

(e) $f(x) = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x}$

Commentary on Ex. 3:

An antiderivative must be defined ON AN INTERVAL.

A formula for an antiderivative for **(d)** is

$$F(x) = -\frac{1}{2}x^{-2} + C,$$

but what interval should we use?

The given function, $f(x) = x^{-3}$, is not defined on $(-\infty, \infty)$.

Domain of f: $(-\infty, 0) \cup (0, \infty)$

An antiderivative of f on $(-\infty, 0)$ is

$$F(x) = -\frac{1}{2}x^{-2} + C_1.$$

An antiderivative of f on $(0, \infty)$ is

$$F(x) = -\frac{1}{2}x^{-2} + C_2.$$

Differential equations

Definition. An equation that involves an unknown function and its derivatives is called a **differential equation**. An **initial value problem** is a differential equation for which the output of the function is specified for a single input value.

Ex. 4. Find y given $\begin{array}{l} y' = x^2 \\ y(2) = 4 \end{array}$

Solution:

$$\begin{aligned} y(x) &= \frac{1}{3}x^3 + C \\ 4 = y(2) &= \frac{1}{3} \cdot 2^3 + C \\ \frac{4}{3} &= C \end{aligned}$$

$$\boxed{y(x) = \frac{1}{3}x^3 + \frac{4}{3}}$$

Ex. 5. Find y given $\begin{array}{l} v(t) = s'(t) = \sin t - \cos t \\ s(0) = 0 \end{array}$

Solution:

$$\begin{aligned} s'(t) &= \sin t - \cos t \\ s(t) &= -\cos t - \sin t + C \\ 0 &= -\cos 0 - \sin 0 + C \\ 1 &= C \end{aligned}$$

$$\boxed{s(t) = -\cos t - \sin t + 1}$$

Rectilinear motion

Recall:

$$\underset{\text{position}}{s(t)} \overset{d/dt}{\rightsquigarrow} \underset{\text{velocity}}{v(t)} \overset{d/dt}{\rightsquigarrow} \underset{\text{acceleration}}{a(t)}$$

We can reverse the arrows by antidifferentiating.

- If acceleration and the initial position $s(0)$ and speed $v(0)$ are known, the position function $s(t)$ can be found by antidifferentiating twice.

During World War II, mechanical devices aboard rockets were used to carry out the process of antidifferentiating twice in order to guide the missile to its target.

While the position of the rocket as it flew could not be determined by the technology of the day, acceleration could be measured mechanically, as the following quote from a novelist (and former engineering student) describes:

> "a little pendulum was kept centered by a magnetic field. During launch, pulling gs, the pendulum would swing aft, off center. It had a coil attached to it. When the coil moved through the magnetic field, electric current flowed in the coil. As the pendulum was pushed off center by the acceleration of the launch, current would flow—the more acceleration, the more flow. So the Rocket . . . sensed acceleration first. . . . To get to distance from acceleration, the Rocket had to [antidifferentiate] twice—needed a moving coil, transformers, electrolytic cell, bridge of diodes, one tetrode. . . "
>
> —Thomas Pynchon, *Gravity's Rainbow*

Ex. 6. A particle moving in a straight line has acceleration

$$a(t) = 12t - 6.$$

Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm.

(a) Find its position function.

(b) When is the first time after the initial time $t = 0$ that the displacement $s(t)$ is 0?

(a)

$v'(t) = 12t - 6$

$v(t) = 6t^2 - 6t + C$

$-6 = v(0) = C$

$s'(t) = 6t^2 - 6t - 6$

$s(t) = 2t^3 - 3t^2 - 6t + C$

$9 = s(0) = C$

$\boxed{s(t) = 2t^3 - 3t^2 - 6t + 9}$

(b)

$$\begin{aligned} s(t) &= 2t^3 - 3t^2 - 6t + 9 \\ &= t^2(2t-3) - 3(2t-3) \qquad \textit{factor out } (2t-3) \\ &= (t^2-3)(2t-3) = 0 \end{aligned}$$

$$t = \pm\sqrt{3} = \pm 1.732050\ldots \qquad \text{or} \qquad t = \frac{3}{2} = 1.5$$

Ans.: $\boxed{1.5 \text{ seconds after time } t = 0.}$

Ex. 7. A stone is dropped off a cliff and hits the ground at a speed of 120 ft/s. What is the height of the cliff?

Assume downward acceleration due to gravity is the constant

$$g = 32 \text{ ft/s}^2.$$

Solution:

The acceleration is a constant function:

$$a(t) = -32$$

Assume time is $t = 0$ when the stone hits the ground.

$$\begin{aligned} v(t) &= -32t + C \\ -120 = v(0) &= C \\ v(t) &= -32t - 120 \end{aligned}$$

Assume displacement is $s(0) = 0$ at time $t = 0$.

$$\begin{aligned} s(t) &= -16t^2 - 120t + D \\ 0 = s(0) &= D \\ s(t) &= -16t^2 - 120t \end{aligned}$$

When was the stone at rest? That is, at what time $t = T$ did we have $v(T) = 0$?

$$\begin{aligned} 0 = v(T) &= -32T - 120 \\ T &= \frac{120}{-32} = -\frac{15}{4} \end{aligned}$$

$\frac{15}{4}$ seconds before the stone hit the ground.

What was the displacement at time $t = T = -\frac{15}{4}$?

$$s(T) = s\left(-\tfrac{15}{4}\right) = -16 \cdot \tfrac{225}{16} - 120 \cdot \left(-\tfrac{15}{4}\right) = -225 + 450 = \boxed{225 \text{ ft}}.$$

Ex. 8. Show that the displacement at time t for rectilinear motion with constant acceleration A, initial velocity v_0, and initial displacement s_0 is

$$s = \tfrac{1}{2}At^2 + v_0 t + s_0.$$

Solution:

$$a(t) = v'(t) = A$$
$$v(t) = At + C_1$$
$$v_0 = v(0) = C_1$$

$$v(t) = At + v_0$$
$$v(t) = s'(t) = At + v_0$$
$$s(t) = \tfrac{1}{2}At^2 + v_0 t + C_2$$
$$s_0 = s(0) = C_2$$

$$s(t) = \tfrac{1}{2}At^2 + v_0 t + s_0.$$

Ex. 9. A stone is thrown downward at a speed of 5m/s from a height of 450m above the ground How long does it take to reach the ground? (Use $g = 9.8$m/s^2.)

Solution:

We could use the formula we got in the previous exercise, but we'll do the work from scratch for practice.

$a(t) = g = -9.8$ *(Note sign.)*
$v(t) = -9.8t - 5$
$s(t) = -4.9t^2 - 5t + 450$

We solve $0 = s(t) = -4.9t^2 - 5t + 450$ by the Quadratic Formula

$$t = \frac{-(-5) \pm \sqrt{25 - 4(-4.9)(450)}}{2(-4.9)} = \frac{5 \pm \sqrt{8845}}{-9.8}.$$

$$\frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a} \text{ /. } a \to -4.9 \text{ /. } b \to -5 \text{ /. } c \to 450$$

$$-0.102041\,(5 \pm 94.0479)$$

We throw out the solution for negative time $t < 0$.

☞ *Without calculator:* Note $\sqrt{8845} > \sqrt{25} = 5$, so $5 - \sqrt{8845} > 0$. Since the denominator is negative, the positive solution is $\dfrac{5 - \sqrt{8845}}{-9.8}$.

$$\boxed{\frac{5 + \sqrt{8845}}{9.8} \approx 9.0865 \text{ seconds.}}$$

The indefinite integral

The act of finding an antiderivative of a function f is called **integrating** f.

For a function f and an antiderivative F, the family of functions

$$F(x) + C,$$

where C is any real number, is often called the **family of antiderivatives of** f.

It is also called the **indefinite integral of** f—in symbols,

$$\int f(x)\,dx = F(x) + C$$

Methods for antidifferentiation

☞ The definition of the derivative came with a formal process—the definition of the derivative—that could be used to find the derivative of a function using algebra and limits. There is no such process for finding the indefinite integral.

We find an antiderivative of a function f either

- by inspection, i.e. by thinking of a function whose derivative is f,
- using a table, e.g. in Appendix A of your textbook,
- with special techniques such as "u-substitution" and "integration by parts" (these will be discussed later in the calculus sequence) or
- "numerically," i.e. by a process of approximation (often done by computer).

Question. Can you evaluate the integrals $\int \sqrt{1+4x^2}\,dx$ and $\int \sqrt{\sin x}\,dx$ by inspection?

Answer: Almost certainly not!

We used an app (`Mathematica`) to find the first integral—you are certainly not expected to come up with this!

$$\int \sqrt{1+4x^2}\,dx = \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\log(\sqrt{1+4x^2} + 2x) + C$$

The second integral simply cannot be expressed in terms of elementary functions.

$$\int \sqrt{\sin x}\,dx = \ ? \qquad \textit{No elementary solution.}$$

A table that gives the antiderivatives of some common functions is provided at the end of this document. Some of these "*integration rules*" should be memorized—for example, you should memorize the formula

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1} + C.$$

But if you hate memorization, it is recommended that when you are asked to find an antiderivative of a function f, just ask yourself:

The derivative of *what* is f?

You can always guess, then check by differentiating.

- If your guess is a little off, adjust your guess and try again.
- If your guess is *way* off, and cannot be salvaged, it makes sense to consult the table of integration formulas and memorize the appropriate rule for future use.

Ex. 10. Verify that each of the provided integration rules logically follows from the corresponding differentiation rule indicated in the table.

Additional exercises

Ex. 11 (§4.10—#465, 467, 469). Show that $F(x)$ is an antiderivative of $f(x)$.

(a) $F(x) = 5x^3 + 2x^2 + 3x + 1,\ f(x) = 15x^2 + 4x + 3$

(b) $F(x) = x^2 e^x,\ f(x) = e^x(x^2 + 2x)$

(c) $F(x) = e^x,\ f(x) = e^x$

Ex. 12 (§4.10—#470–473). Find the most general antiderivative of the function.

- $f(x) = \dfrac{1}{x^2} + x$
- $f(x) = e^x - 3x^2 + \sin(x)$
- $f(x) = e^x + 3x - x^2$
- $f(x) = x - 1 + 4\sin(2x)$

Ex. 13 (§4.10—#475, 477, 479, 481, 483, 489). Find an antiderivative of the function.

- $f(x) = x + 12x^2$
- $f(x) = \left(\sqrt{x}\right)^3$
- $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$
- $f(x) = \sec^2(x) + 1$
- $f(x) = \sin^2(x)\cos(x)$
- $f(x) = \frac{1}{2}e^{-4x} + \sin(x)$

Ex. 14 (§4.10—#491, 493, 495). Evaluate the integral.

- $\int \sin(x)\,dx$
- $\int \frac{3x+2+2}{x^2}\,dx$
- $\int \left(4\sqrt{x}+\sqrt[4]{x}\right)dx$

Ex. 15 (§4.10—#499, 503). Solve the initial value problem.

(a) $f'(x) = x^{-3}$, $f(0) = 1$

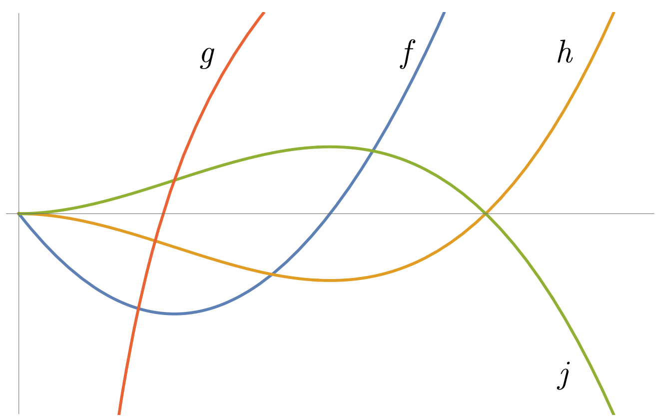
(b) $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}$, $f(1) = 0$

Ex. 16 (§4.10—#505, 507). Find two different possible functions f.

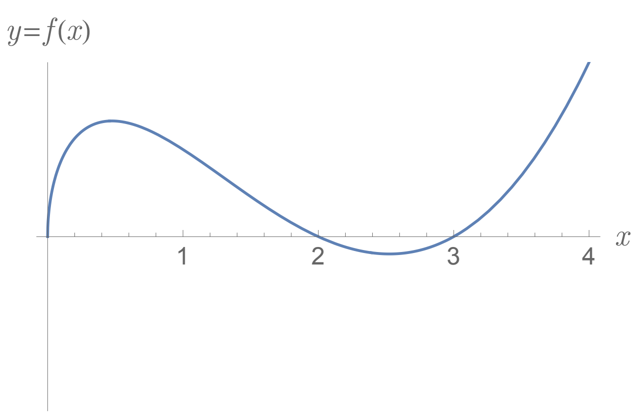
(a) $f''(x) = e^{-x}$

(b) $f'''(x) = \cos(x)$

Ex. 17. The graph of a function f is shown. Which graph is an antiderivative of f and why?



Ex. 18. The graph of a function $y = f(x)$ is shown. Make a rough sketch of its antiderivative F, given that $F(0) = 1$.



Integration rules

Integration rule	Corresponding differentiation rule
$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$ for $n \neq -1$	$\frac{d}{dx}\big[x^n\big] = nx^{n-1}$
$\int k \, dx = kx + C$ for $k = \text{const}$	Special case of previous differentiation rule $(n = 1)$
$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$	$\frac{d}{dx}\big[\ln\lvert x\rvert\big] = \frac{1}{x}$
$\int e^x \, dx = e^x + C$	$\frac{d}{dx}\big[e^x\big] = e^x$
$\int b^x \, dx = \frac{1}{\ln(b)} b^x + C$ for $0 < b \neq 1$	$\frac{d}{dx}\big[b^x\big] = b^x \ln(b)$
$\int \cos(x) \, dx = \sin(x) + C$	$\frac{d}{dx}\big[\sin(x)\big] = \cos(x)$
$\int \sin(x) \, dx = -\cos(x) + C$	$\frac{d}{dx}\big[\cos(x)\big] = -\sin(x)$
$\int \sec^2(x) \, dx = \tan(x) + C$	$\frac{d}{dx}\big[\tan(x)\big] = \sec^2(x)$
$\int \csc(x)\cot(x) \, dx = -\csc(x) + C$	$\frac{d}{dx}\big[\csc(x)\big] = -\csc(x)\cot(x)$
$\int \sec(x)\tan(x) \, dx = \sec(x) + C$	$\frac{d}{dx}\big[\sec(x)\big] = \sec(x)\tan(x)$
$\int \csc^2(x) \, dx = -\cot(x) + C$	$\frac{d}{dx}\big[\cot(x)\big] = -\csc^2(x)$
$\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C$	$\frac{d}{dx}\big[\tan^{-1}(x)\big] = \frac{1}{1+x^2}$

Workbook Lesson 25

§5.1, Approximating Areas

Last revised: 2020-09-29 12:43

Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

The method of exhaustion

In antiquity, the Greek mathematician Archimedes was fascinated with calculating the areas of irregular shapes, such as the amount of space enclosed by a curve.

He used a process that has come to be known as the *method of exhaustion* (see Lesson 1), which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area.

In this process, an area bounded by curves is filled with simple shapes like rectangles or triangles. The areas of these shapes are individually easy to calculate. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve defined by a function f and the x-axis on a closed interval $[a, b]$.

Like Archimedes, we first approximate the area under the curve using shapes of known area—namely, rectangles.

By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

In this section, we develop techniques to approximate the area between a curve $y = f(x)$ and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

We'll begin by introducing some notation to make the calculations easier to follow. We then consider the case when $f(x)$ is continuous and nonnegative for all x. Later in the chapter, we'll consider more general cases.

Sigma notation

The Greek capital letter Σ, "sigma," is used to express long sums of values in a compact form.

Say we want to add all the integers from 1 to 20. Without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a few terms and write

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better, but still tedious.

With sigma notation, we write this sum as

$$\sum_{k=1}^{20} k \qquad (\star)$$

which is both shorter and easier on the eyes.

In the above notation $(\star)$, called **sigma notation** (or **sigma notation**), the variable k is called the **index**. Each term is evaluated, then we sum all the values, beginning with the value when $k = 1$ and ending with the value when $k = 20$.

We sometimes denote the terms to be added up in the form by a_k (or b_k, etc.), where k is the index. Thus we might declare

$$a_k = \sin \frac{k\pi}{2}$$

and write

$$\sum_{k=1}^{6} a_k.$$

Ex. 1. Evaluate $\sum_{k=0}^{6} a_k$, where $a_k = \sin \frac{k\pi}{2}$.

We can use any letter we like for the index. Mathematicians most often use i, j, k, m, and n for indices.

The index is used only to keep track of the terms to be added. It does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. For example,

$$\sum_{j=1}^{20} j \quad \text{and} \quad \sum_{k=1}^{20} k$$

mean exactly the same thing, $1 + 2 + 3 + \cdots + 20$.

☞ *The values of the index are always understood to be integers.*

Ex. 2.

(a) Write the sum of terms 3^k for $k = 1, 2, 3, 4, 5$ in sigma notation and evaluate.

(b) Write the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ in sigma notation.

(c) Write the sum $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$ in sigma notation.

(d) Evaluate $\sum_{x=0}^{5} (x + 1)$.

Rules for sigma notation

Ex. 3. Which of the following "rules" is true?

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k \cdot b_k) = \left(\sum_{k=1}^{n} a_k\right) \cdot \left(\sum_{k=1}^{n} b_k\right)$$

Answer: The last formula is not true. (The remaining three are correct.)

Ex. 4. TRUE OR FALSE: If the terms a_k are all defined, then

$$\sum_{k=1}^{10} a_k = \sum_{k=0}^{9} a_{k+1} = \sum_{k=2}^{11} a_{k-1}$$

no matter what the values of the a_k are.

Some special sums

Ex. 5. What is the sum of the first n natural numbers? That is, find the value of

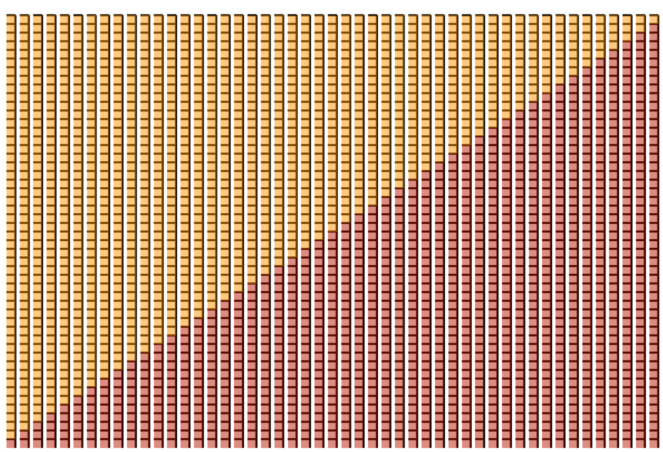
$$S = \sum_{k=1}^{n} k.$$

Solution:

This solution is very creative. It has been claimed that the mathematician Carl Gauss discovered this solution in elementary school. You are *not* expected to come up with such a novel technique—we will simply use the formula which the following argument proves.

Double the sum, and rearrange terms:

$$\begin{aligned} 2 \times S = S + S &= (1 + 2 + 3 + \cdots + (n-2) + (n-1) + n) + (n + (n-1) + \cdots + 3 + 2 + 1) \\ &= \underbrace{(1+n) + (2 + (n-1)) + (3 + (n-2)) + \cdots + ((n-1) + 2) + (n+1)}_{n \text{ bracketed terms}} \\ &= n(n+1). \end{aligned}$$



Do you see it?

Since $2S = n(n+1)$, we have

$$S = \frac{n(n+1)}{2}.$$

$$\boxed{\sum_{i=1}^{n} k = \frac{n(n+1)}{2}}$$

Two other useful formulas (which we will not prove, but may be required to complete some of the exercises to come) are:

$$\boxed{\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\sum_{i=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}}$$

Ex. 6. Write using sigma notation and evaluate:

(a) The sum of the terms $(k-3)^2$ for $k = 1, 2, 3, 4, 5, 6$.

(b) The sum of the terms $(k^3 - k^2)$ for $k = 1, 2, 3, 4, 5, 6$.

(c) The sum of the terms $4 + 3k$ for $k = 1, 2, 3, \ldots, 100$.

Approximating the area under a curve

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve.

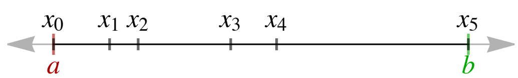
Let f be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by

- the curve $y = f(x)$ above,
- the x-axis below,
- the line $x = a$ on the left, and
- the line $x = b$ on the right.

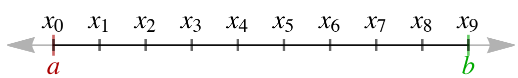
We call this area the **area under the curve** $y = f(x)$ **from** a **to** b.

<u>**Definition.**</u> Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

- The collection of n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ is a **partition** $\mathcal{P}$ of $[a, b]$.
- We write $\Delta x_k = x_k - x_{k-1}$.

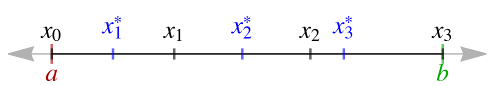


- The partition is **regular** if all the subintervals have the same width, in which we case we write Δx for the shared width.



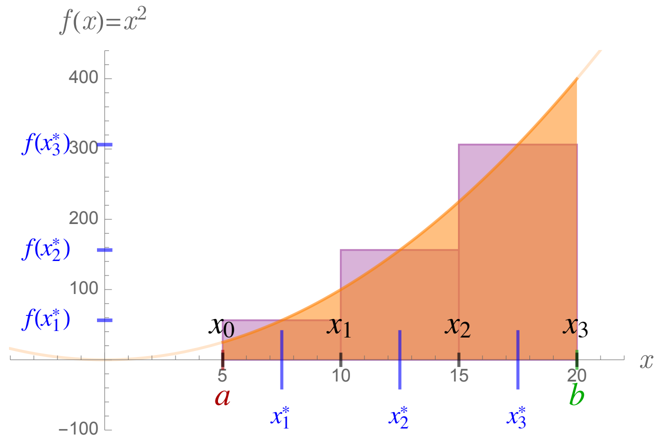
- For each subinterval $[x_{k-1}, x_k]$, $k = 1, 2, 3, \ldots, n$, choose any point $x_k^* \in [x_{k-1}, x_k]$.

  The numbers x_k^* are called **sample points** for the partition $\mathcal{P}$.

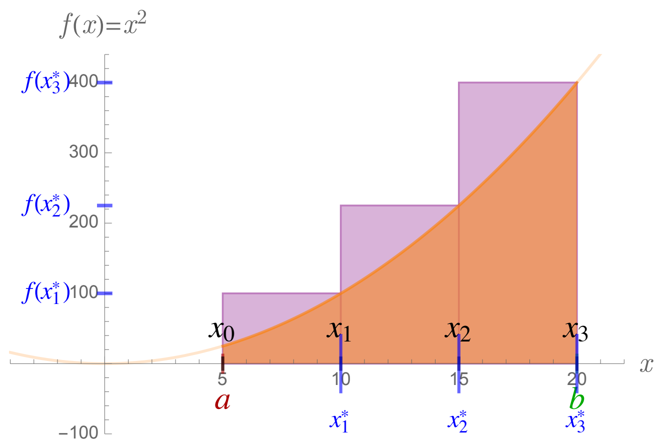


<u>**Ex. 7.**</u> What is the width of each subinterval in a *regular* partition of the interval $[a, b]$ with n subintervals?

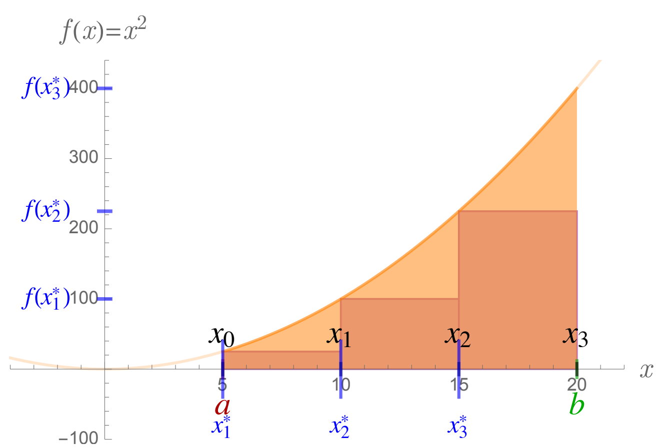
Taking the sample points x_k^* as inputs to the function $f(x)$ gives us the heights of rectangles, one rectangle for each subinterval:



If we take the sample points to be the right endpoints of the subintervals $[x_{k-1}, x_k]$, then the total area of all the rectangles is a **right-endpoint approximation** of the area under the curve from a to b:



A **left-endpoint approximation** of the area under the curve from a to b is defined analogously:



The area of each rectangle is

$$\begin{aligned}\text{BASE} \times \text{HEIGHT} &= (x_k - x_{k-1}) \cdot f(x_k^*) \\ &= f(x_k^*)\,\Delta x_k.\end{aligned}$$

The *total* area is

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

Ex. 8. Two of the three sums written below is a *left*-endpoint approximation under the curve $y = f(x)$ on $[a, b]$. The remaining sum is a *right*-endpoint approximation. Which is which? Explain your reasoning.

$$\sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k \qquad \sum_{k=1}^{n} f(x_k)\,\Delta x_k \qquad \sum_{k=0}^{n-1} f(x_k)\,\Delta x_{k+1}$$

Ex. 9.

(a) Find the right-endpoint approximation of the area under the curve $y = x^2$ from $x = 0$ to $x = 20$ with 5 rectangles.

(b) Sketch the rectangles in part **(a)** on the graph of $y = x^2$.

(c) Is this an overestimate or an underestimate of the actual area under the curve?

Ex. 10.

(a) Find the right-endpoint approximation of the area under the curve $y = x^2$ from $x = -4$ to $x = 0$ with 4 rectangles.

(b) Sketch the rectangles in part **(a)** on the graph of $y = x^2$.

(c) Is this an overestimate or an underestimate of the actual area under the curve?

Riemann sums

Let $\mathcal{P}$ be a partition of an interval $[a, b]$. In symbols, we'll write:

$$\mathcal{P}: \quad a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Suppose f is a function defined on $[a, b]$. A **Riemann sum** for the function f and the partition $\mathcal{P}$ is a sum of the form

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

where the x_k^* are sample points for the partition $\mathcal{P}$ (that is, each x_k^* is a number in the interval $[x_{k-1}, x_k]$.

Notice that a Riemann sum depends not only on the function f and the interval $[a, b]$, but also on the choice of partition $\mathcal{P}$ and the sample points x_k^* chosen for that partition.

That is, if I choose a certain partition and sample points, and you choose a different partition and different sample points, then it's unlikely that our Riemann sums will be equal, even when we're both approximating the area under the same curve $y = f(x)$ over the same interval $[a, b]$.

Our ultimate goal is not to *approximate* the area, but to find the *exact* area. To this end, we take more and more, thinner and thinner rectangles. As the number of rectangles approaches ∞, *the width of the rectangles approaches* 0.

We may express this "*limiting process*" (see Lesson 1) by writing, "the limit of the Riemann sums as the number of rectangles approaches ∞"—in symbols,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

Or we might prefer to write, "the limit of the Riemann sums as the rectangles' widths approach 0"—in symbols,

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

However we write it, we are dealing here with a type of limit that is rather peculiar, because if *I* take the limit of, say, left-endpoint approximations, and *you* take the limit of, say, right-endpoint approximations, then there is no reason for us to believe that our limits will turn out to be equal.

Fortunately, the following theorem—whose proof is beyond the scope of this class—guarantees that *whatever partitions we choose as the number of rectangles gets larger, and whatever sample points we pick for each partition, the limit of the Riemann sums is "unique."* (Here, by "unique," we mean that if two different people compute the limit in different ways, then the values they get will be equal.)

Theorem. For any function f that is continuous on the interval $[a, b]$, if the limit of the Riemann sums

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

exists for SOME particular choice of sample points and partitions, then the limit exists for EVERY choice of sample points and partitions, and in every case, the value of the limit is the same.

If this limit exists and is a real number, we say f is **integrable** on the interval $[a, b]$. (We will return to the idea of "integrability" in the next lesson.)

Types of Riemann sums

We've already seen two ways to pick sample points—by choosing the left or right endpoints of each subinterval of a partition. We now mention three other popular ways of picking sample points.

Definition. Let f be a function on $[a, b]$ and let $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a partition of $[a, b]$. Consider the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k. \qquad (\dagger)$$

- If we choose x_k^* so that $f(x_k^*)$ is the maximum value of f on $[x_{k-1}, x_k]$—in symbols,

$$f(x_k^*) = \max_{x_{k-1} \le x \le x_k} f(x)$$

—for $k = 1, 2, \ldots, n$, then the Riemann sum ($\dagger$) is called an **upper sum**.

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

- If we choose x_k^* so that $f(x_k^*)$ is the minimum value of f on $[x_{k-1}, x_k]$—in symbols,

$$f(x_k^*) = \min_{x_{k-1} \le x \le x_k} f(x)$$

—for $k = 1, 2, \ldots, n$, then the Riemann sum ($\star$) is called a **lower sum**.

☞ For an increasing function f, an upper sum is the same as a right-endpoint approximation.

Ex. 11. Explain why the previous sentence is true. Then make an analogous statement that's true for decreasing functions.

Ex. 12.

(a) Find and evaluate a lower sum with $n = 4$ rectangles that approximates the area under the curve $y = 10 - x^2$ on $[0, 1]$.

(b) Sketch the approximation.

Ex. 13.

(a) Find and evaluate an *upper* sum with $n = 4$ rectangles that approximates the area under the curve $y = 10 - x^2$ on $[0, 1]$.

(b) Sketch the approximation.

Ex. 14 (Midpoint approximation).

(a) Taking the sample points x_k^* to be the *midpoints* of the subintervals of the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

find and evaluate a Riemann sum with $n = 5$ rectangles that approximates the area under the curve $y = 3x + 1$ on $[0, 10]$.

(b) Sketch the approximation.

Ex. 15 (The limit of the approximations).

(a) Write a right-endpoint approximation that approximates the area under the curve $y = x^2$ from $a = 0$ to $b = 2$ with n rectangles and a regular partition.

Hints:

- $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$.
- The right endpoint of the k^{th} subinterval is

$$x_k = a + k \cdot \frac{2}{n} = \frac{2k}{n}.$$

- The function value at the right endpoint is therefore

$$f(x_k) = f\left(\tfrac{2k}{n}\right) = \frac{4k^2}{n^2}.$$

(b) Then take the limit as $n \to \infty$ to find the *exact* area under the curve.

Additional exercises

Ex. 16 (§5.1—#1). Are the given sums equal or unequal?

(a) $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

(b) $\sum_{k=1}^{10} k$ and $\sum_{k=6}^{15} (k-5)$

(c) $\sum_{j=1}^{10} j(j-1)$ and $\sum_{k=0}^{9} (k+1)k$

(d) $\sum_{j=1}^{10} j(j-1)$ and $\sum_{k=0}^{9} (k^2-k)$

Ex. 17 (§5.1—#2, 3). Evaluate the sums

(a) $\sum_{k=5}^{10} k$

(b) $\sum_{k=5}^{10} k^2$

Ex. 18 (§5.1—#19). Compute the left-endpoint Riemann sum L_8 with 8 rectangles for

$$f(x) = x^2 - 2x + 1$$

on $[0, 2]$. Illustrate with a graph.

Ex. 19 (§5.1—#23). Compute the left-endpoint Riemann sum and the right-endpoint Riemann sum with 6 rectangles (L_6 and R_6, respectively) for

$$f(x) = \sqrt{9 - (x-3)^2}$$

on $[0, 6]$, and compare the values of L_6 and R_6. Illustrate with a graph.

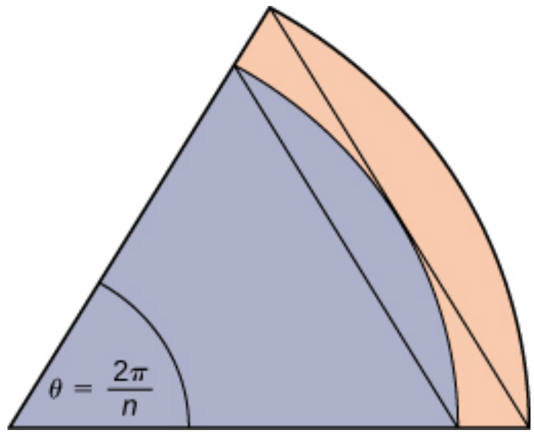
Ex. 20 (§5.1—#27). Express the left-endpoint sum R_{100} for $f(x) = \ln(x)$ on $[1, e]$, but do not evaluate the sum.

Ex. 21 (§5.1—#39). The following table gives the approximate increase in mean sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

starting year	1870	1890	1910	1930	1950	1970	1990
20-year change	0.3	1.5	0.2	2.8	0.7	1.1	1.5

Data source: Church & White, Sea-level rise from the late 19th to the early 21st century, *Surv Geophys* 32 (2011), pp. 585–602

Ex. 22 (§5.1—#59). A unit circle is made up of n sectors equivalent to the inner sector in the figure.



The base of the inner triangle is

$$b = 1$$

unit and its height is

$$h = \sin\left(\frac{\pi}{n}\right).$$

The base of the outer triangle is

$$B = \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$$

and its height is

$$H = B\sin\left(\frac{2\pi}{n}\right).$$

Use this information to argue that the area of a unit circle is equal to π.

Workbook Lesson 26

§5.2, The Definite Integral

Last revised: 2021-04-14 07:06

Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

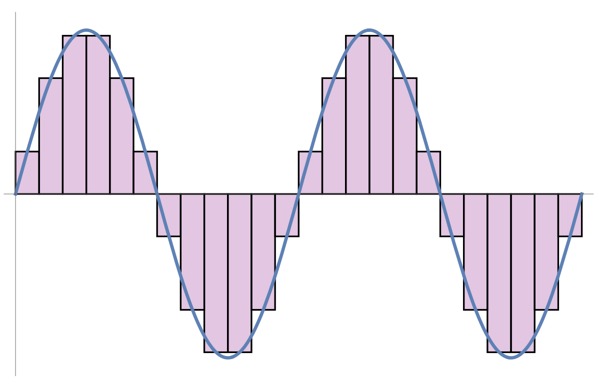
Definition of the definite integral

Recall: Last time, we saw that the *area between a curve* $y = f(x)$ *and the* x*-axis*—which is usually called the "area under the curve"—can be found as the the limit of Riemann sums:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, the phrase "area under the curve" only makes sense if the values of $f(x)$ are nonnegative—that is, if the graph of f never dips below the x-axis.

If f has both positive values and negative values, the Riemann sum can include the area of rectangles that lie below the x-axis.



We now lift the restriction that $f(x) \geq 0$ for all x in $[a, b]$, and define the *definite integral* to be the limit of the Riemann sums.

Definition. For a function f defined on an interval $[a, b]$, we define the **definite integral of** f **from** a **to** b to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

☞ The integration symbol $\int$ is an elongated S, meant to suggest summation (which is happening behind the scenes in the Riemann sums whose limit is the definite integral).

☞ Think of the symbols $\int_a^b$ and dx in the definite integral to be like parentheses—it is an error to write $\int$ and omit the dx at the end. The expression $f(x)$ between $\int_a^b$ and dx is called the **integrand**.

☞ If we use a variable other than x as the input to the function f, the expression dx should be changed accordingly. For example, the symbols

$$\int_a^b f(t)\,dt \qquad \text{and} \qquad \int_a^b f(x)\,dx$$

mean the same thing.

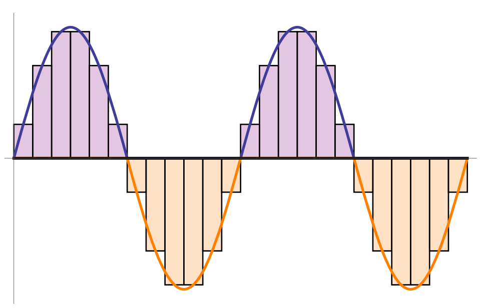
☞ The variable $\square$ in the definite integral $\int_a^b f(\square)\,d\square$ is called the **variable of integration**. The numbers a and b are called the **limits of integration**.

Net signed area and total area

Initially, we interpreted "area under the curve" only for continuous functions f that were everywhere nonnegative.

Now that we are allowing the curve $y = f(x)$ to dip below the x-axis, how can we interpret the definite integral in terms of area?

Intuitively, the answer is simple: simply subtract the area under the x-axis *(shaded below in purple)* from the area above the x-axis *(shaded below in orange)*.



__**Definition.**__ The **net signed area "under" the curve $y = f(x)$ over** $[a, b]$ is defined to be

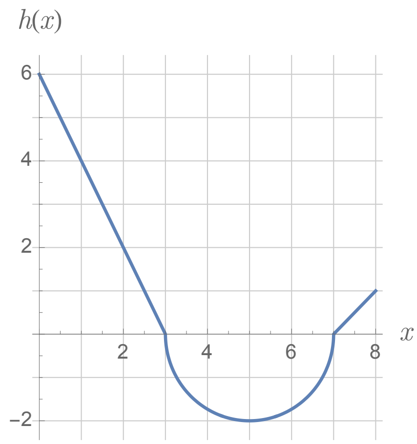
$$A_1 - A_2$$

where A_1 is the area of the portion of the region between the curve $y = f(x)$ $(a \le x \le b)$ and the x-axis that lies above the x-axis, and A_2 is the area of the portion that lies below the x-axis.

__**Theorem.**__ The definite integral of a continuous function f from a to b is equal to the net signed area over $[a, b]$:

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

Ex. 1. The graph of h consists of two straight lines and a semicircle. Evaulate the integrals by interpreting them in terms of areas.



(a) $\displaystyle\int_0^3 h(x)\,dx$ **(b)** $\displaystyle\int_3^6 h(x)\,dx$ **(c)** $\displaystyle\int_0^8 h(x)\,dx$

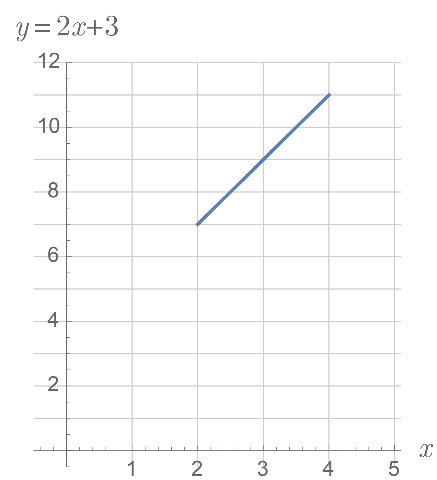
Ex. 2. Recall that the formula for the area of a trapezoid is

$$A = \frac{a+b}{2}h,$$

where h is the distance between the parallel sides of lengths a and b.

Use the formula to find

$$\int_2^4 2x + 3\,dx.$$



Ex. 3. Evaluate $\int_0^{10} |x-5|\,dx$.

Ex. 4.

(a) Write a formula for a Riemann sum using n rectangles and a right-endpoint approximation of $f(x) = 2x - 1$ on the interval $[0, 3]$.

(b) Use the definition of the integral to evaluate $\int_0^3 2x - 1\,dx$.

Solution (Ex. 4):

By definition of the definite integral $\int_a^b f(\Box)\,d\Box$,

$$\int_a^b f(x)\,dx = \int_0^3 \big(2x-1\big)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\big(2x_k^*-1\big)\,\Delta x,$$

where $x_1^*,\ldots,x_n^*$ are sample points for a partition $0=x_0<\cdots<x_n=3$ of $[0,3]$.

For a right-endpoint approximation, we take:

$$\Delta x = \frac{b-a}{n}$$

$$\begin{aligned} x_0 &= a \\ x_1 &= a+\Delta x \\ x_2 &= a+2\Delta x \\ &\vdots \\ x_k &= a+k\Delta x \\ &\vdots \\ x_n &= a+n\Delta x \end{aligned}$$

$$x_1^* = \left(\begin{array}{c}\text{right endpoint of the}\\ \text{first subinterval } [x_0,x_1]\end{array}\right) = x_1$$

$$x_2^* = \left(\begin{array}{c}\text{right endpoint of the}\\ \text{next subinterval } [x_1,x_2]\end{array}\right) = x_2$$

$$\vdots$$

$$x_k^* = \left(\begin{array}{c}\text{right endpoint of the}\\ k^{\text{th}} \text{ subinterval } [x_{k-1},x_k]\end{array}\right) = x_k$$

$$\vdots$$

$$x_n^* = \left(\begin{array}{c}\text{right endpoint of the}\\ \text{last subinterval } [x_{n-1},x_n]\end{array}\right) = x_n$$

We now substitute and evaluate the limit, using a special sum formula along the way:

$$\int_0^3 f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\big(2x_k^*-1\big)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\big(2x_k-1\big)\cdot\frac{3}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\big(2k\Delta x-1\big)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{6k}{n}-1\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{18k}{n^2}-\frac{3}{n}\right)$$

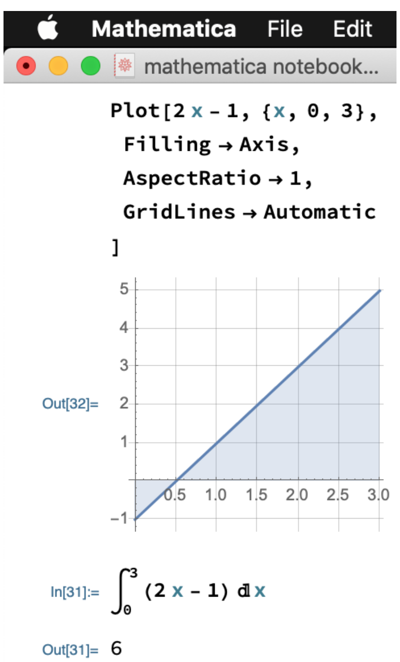
$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{18k}{n^2}-\sum_{k=1}^{n}\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{18}{n^2}\left(\sum_{k=1}^{n}k\right)-3$$

$$= \lim_{n\to\infty}\frac{18}{n^2}\,\frac{n(n+1)}{2}-3$$

$$= \lim_{n\to\infty}\frac{18(n+1)}{2n}-3$$

$$= 9-3 = \boxed{6}$$



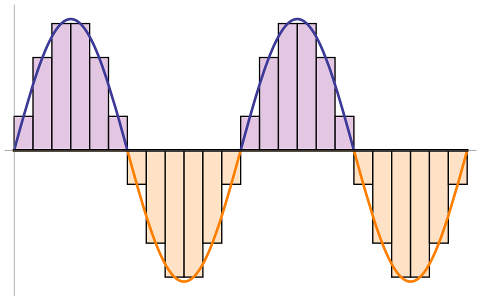
Ex. 5.

Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1 - x_i^2}{4 + x_i^2} \Delta x$ as a definite integral on the interval $[2, 6]$.

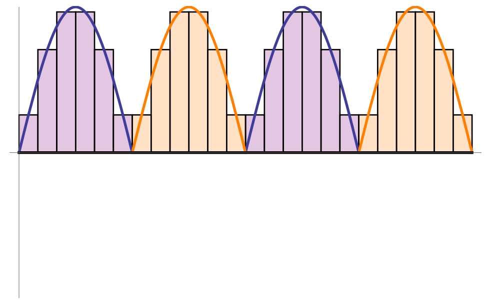
Total area

Definition. Let A_1 and A_2 be as above. The **total area "under" the curve** $y = f(x)$ over $[a, b]$ is

$$\int_a^b |f(x)|\, dx = A_1 + A_2.$$



Net signed area



Total area

Ex. 5. Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$. (*Hint*: Sketch the graph!)

Ex. 6. Find the signed area and the total area for $g(x) = 1 - |x|$ over the interval $[0, 4]$. (*Hint*: Sketch the graph!)

Conditions for integrability

If the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

exists, we say f is **integrable** on $[a, b]$.

When can we be guaranteed that this limit exists? The following theorem gives a useful, but somewhat restrictive condition:

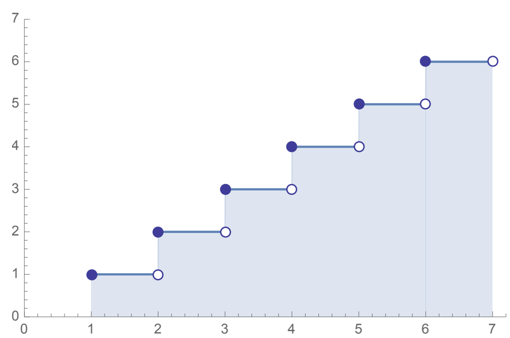
Theorem. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Counterexample. The area under the curve $y = x$ for $x \geq 0$ is infinite. Why is this not a violation of the Theorem?

We can relax the continuity requirement as follows:

Theorem. If f has a finite number of jump discontinuities, but is continuous at x for every other $x \in [a, b]$, then f is integrable on $[a, b]$.

This makes intuitive sense: the area under a graph with finitely many jump discontinuities can be split up into pieces that don't include a discontinuity, then the area of the pieces can be added.



Properties of the definite integral

The definition of a Riemann sum makes sense even if $a > b$. To see this, swap a and b. Then $\Delta x = \dfrac{b-a}{n}$ becomes $\dfrac{a-b}{n} = -\Delta x$ instead. Then, when we take the limit on both sides of the equation

$$\sum_{i=1}^{n} f(x_i^*)(-\Delta x) = -\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right)$$

we get

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{1}$$

Now, what happens if $a = b$?

$$\int_a^a f(x)\,dx = 0. \qquad (2)$$

Some additional properties of the definite integral (assuming all integrals exist):

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \text{ for } c = \text{const} \qquad (3)$$

$$\int_a^b \big[f(x) + g(x)\big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad (4)$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \qquad (5)$$

Ex. 6. Prove that

$$\int_a^b c\,dx = c(b-a) \text{ for } c = \text{const}.$$

(Use rule (3)*, and interpret the definite integral as area under a curve.)*

Ex. 7. Prove that

$$\int_a^b \big[f(x) - g(x)\big]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Hint: Use rule (3), rule (4), and the definition of subtraction.

<u>Ex. 8.</u> Find $\int_0^1 (4+3x^2)\,dx$.

Hint: $\int_0^1 x^2\,dx = \frac{1}{3}$.

<u>Ex. 9.</u> Given that $\int_0^{17} f(x)\,dx = 3$ and $\int_0^{12} f(x)\,dx = 2$, find $\int_{12}^{17} f(x)\,dx$.

Hint: Use rules (1) and (5).

Comparison theorems for integrals

The following facts can be used to estimate lower or upper bounds for the value of a definite integral.

$$\text{If } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x)\,dx \geq 0. \tag{6}$$

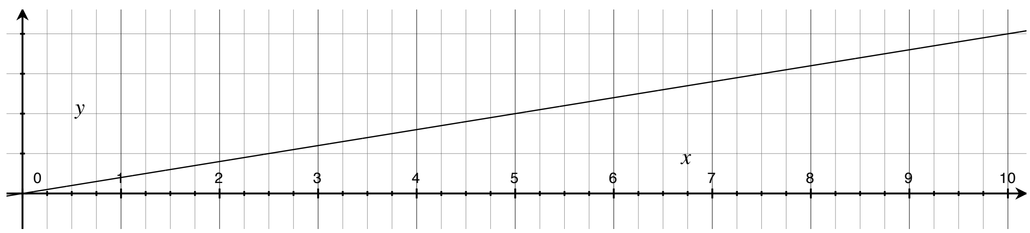
$$\text{If } f(x) \geq g(x) \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx. \tag{7}$$

$$\text{If } m \leq f(x) \leq M \text{ for } a \leq x \leq b, \text{ then } m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a). \tag{8}$$

Ex. 10. Find lower and upper bounds for $\int_1^4 \sqrt{x}\,dx$. *(Use the fact that $f(x) = \sqrt{x}$ is increasing to find the minimum and maximum values of f on $[1, 4]$.)*

Average value of a function

Suppose the temperature inside an oven changes at a constant speed. Then the graph of the temperature function would be a straight line:



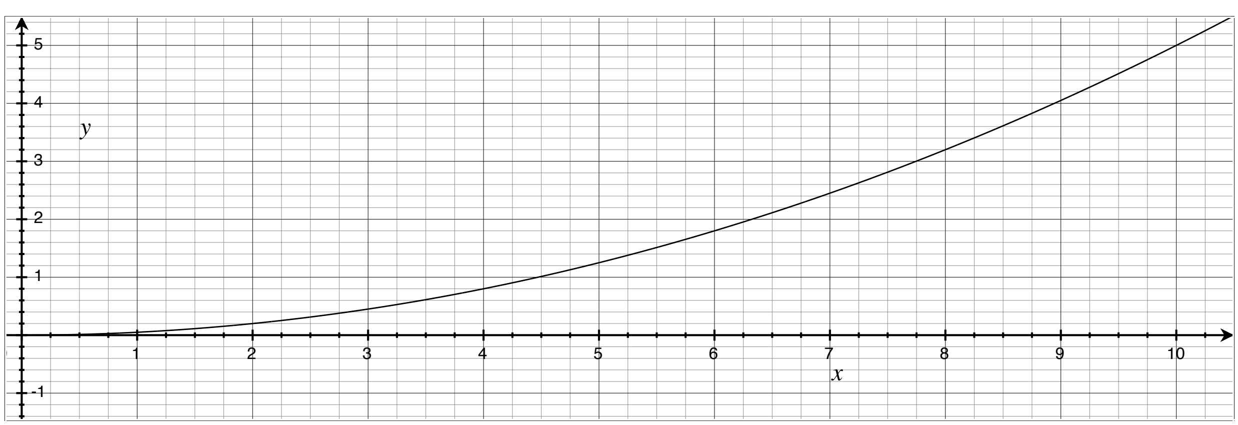
What is the average temperature inside the oven over the ten-minute period we've graphed?

If we measure the temperature n times, at evenly spaced intervals, then the average temperature is

$$\frac{y_1 + y_2 + \cdots + y_n}{n}, \qquad (\dagger)$$

no matter what n we use.

But what if the temperature changes "quadratically"? That is, what if the graph of the temperature function follows a parabola?



Now formula (†) can't always give us the exact average temperature. After all, if AL takes the average of $n = 2$ measurements of the temperature, at time $x = 5$ and $x = 10$, he gets

$$\frac{1.2 + 5}{2} = 3.1,$$

but if ANA takes the average of $n = 5$ measurements, estimating the height of the graph at $x = 2, 4, 6, 8, 10$, she gets

$$\frac{0.2 + 0.8 + 1.8 + 3.2 + 5}{5} = \frac{11}{5} = 5.5$$

—and these can't both be right.

AL and ANA have *approximated* the average value. It's reasonable to think that if we take more and more measurements, our approximation will get more and more accurate.

Our formula for the average of finitely many numbers is

$$\frac{y_1 + \cdots + y_n}{n}.$$

If these represent the values of a function f over the interval $[a, b]$, this becomes

$$\frac{y_1 + \cdots + y_n}{n} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

for some values $x_1^*, \ldots, x_n^*$ chosen from subintervals of equal width $\Delta x = \dfrac{b-a}{n}$.

Solving for n, we have

$$n = \frac{b-a}{\Delta x},$$

so

$$\begin{aligned}\frac{y_1 + \cdots + y_n}{n} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{n} &= \frac{f(x_1^*) + \cdots + f(x_n^*)}{(b-a)/\Delta x} \\ &= \frac{1}{b-a}\left[f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x\right] \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x.\end{aligned}$$

If we take more and more measurements, we are letting $n \to \infty$...

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

which, using the definition of a definite integral, is

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We therefore define the **average value** of a function f on the interval $[a, b]$ as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex. 11. Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Ex. 12. Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Ex. 13. Find the average value of $g(t) = -\csc(t)\cot(t)$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$.

Solution:

$$\int_{\pi/6}^{\pi/4} g(t)\,dt = [\csc(t)]_{\pi/6}^{\pi/4} = \frac{1}{\sin(\pi/4)} - \frac{1}{\sin(\pi/6)} = \frac{\sqrt{2}-1}{2}.$$

Additional exercises

Ex. 14 (§5.2—#60, 61, 62, 63). Express the limits as definite integrals over the indicated interval.

(a) $\lim_{n\to\infty} \sum_{k=1}^{n} x_k^* \, \Delta x$ over $[1, 3]$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \left[5(x_k^*)^3 - 4x_k^*\right] \Delta x$ over $[2, 7]$

(c) $\lim_{n\to\infty} \sum_{k=1}^{n} \sin^2(2\pi x_k^*) \, \Delta x$ over $[0, 1]$

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{e^{x_k}}{1 + x_k} \Delta x$ over $[0, 1]$

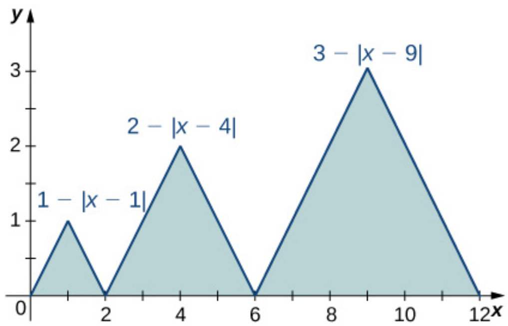
Ex. 15 (§5.2—#64, 67). Given the left Riemann sum L_n or the right Riemann sum R_n as indicated, express as a definite integral the limit of the sum as $n \to \infty$, identifying the correct intervals.

(a) $L_n = \dfrac{1}{n} \sum_{k=1}^{n} \dfrac{k-1}{n}$

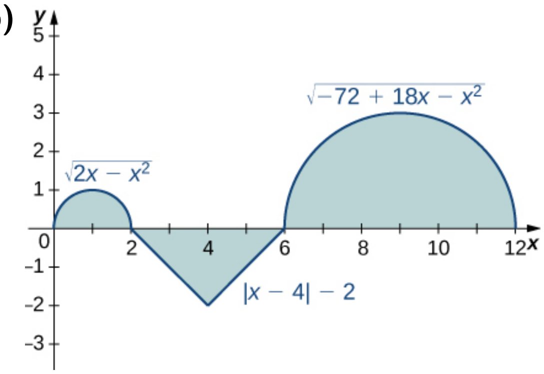
(b) $R_n = \dfrac{3}{n} \sum_{k=1}^{n} \left(3 + 3\dfrac{k}{n}\right)$

Ex. 16 (§5.2—#71, 72). Evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.

(a)



(b)



Ex. 17 (§5.2—#76, 79, 81, 83). Evaluate the integral using area formulas.

(a) $\int_0^3 (3-x)\,dx$

(b) $\int_0^6 \left(3-|x-3|\right)dx$

(c) $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

(d) $\int_{-2}^3 \left(3-|x|\right)dx$

Ex. 18 (§5.2—#98, 101). Given that

$$\int_0^1 x\,dx = \frac{1}{2}, \qquad \int_0^1 x^2\,dx = \frac{1}{3}, \qquad \text{and} \int_0^1 x^3\,dx = \frac{1}{4},$$

compute the integrals

$$\int_0^1 (1 + x + x^2 + x^3)\,dx$$

and

$$\int_0^1 (1 - 2x)^3\,dx.$$

Ex. 19 (§5.2—#104). Use the Comparison Theorems to show that

$$\int_0^3 (x^2 - 6x + 9)\,dx \geq 0.$$

Ex. 20 (§5.2—#107). Use the Comparison Theorems to show that

$$\int_1^2 \sqrt{1+x}\,dx \leq \int_1^2 \sqrt{1+x^2}\,dx.$$

Ex. 21. Write as a single integral of the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

Ex. 22 (§5.2—#110). Find the average value f_{ave} of

$$f(x) = x^2$$

between $a = -1$ and $b = 1$, and find a number c such that $f(c) = f_{\text{ave}}$.

Ex. 23 (§5.2—#112). Find the average value f_{ave} of

$$f(x) = \sqrt{4 - x^2}$$

between $a = 0$ and $b = 2$, and find a number c such that $f(c) = f_{\text{ave}}$.

Workbook Lesson 27

§5.3, The Fundamental Theorem of Calculus

Last revised: 2020-10-01 10:37

Objectives

- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.
- Explain the meaning of the Mean Value Theorem for Integrals.

Evaluation Theorem

Not all integrable functions are continuous. But for continuous functions, the value of a definite integral can be very easy to determine.

Evaluation Theorem. Let f be a function that is continuous on $[a, b]$. If F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Considering how complicated the definition of the definite integral was, this formula seems to be a minor miracle. The definition of the definite integral involved infinitely many values of $f(x)$. But this theorem says we can evaluate it knowing only two (!) values of an antiderivative $F(x)$.

On the other hand, if we look at a certain physical application, this theorem becomes quite believable. We know that the position function is an antiderivative of velocity:

$$s(t) \overset{d/dt}{\rightsquigarrow} v(t).$$

The area under the velocity curve is equal to the change in distance:

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

This is exactly what the Evaluation Theorem says.

Ex. 1. Evaluate:

(a) $\displaystyle\int_{-2}^{1} x^2\,dx$ **(b)** $\displaystyle\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta\,d\theta$ **(c)** $\displaystyle\int_{1}^{4} \sqrt{t}(1+t)\,dt$

☞ The notations $\Big[\dots\Big]_a^b$ means "evaluate what's inside the brackets when b and a are substituted for the variable, and then subtract." The notation $\dots\Big|_a^b$ is defined similarly. For example,

$$\Big[F(x)\Big]_a^b = F(x)\Big|_a^b = F(b) - F(a).$$

Solution to Ex. 1:

(a) An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$.

$$\int_{-2}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-2}^{1} = \frac{1^3}{3} - \frac{(-2)^3}{3} = \frac{1}{3} + \frac{8}{3} = 3.$$

(b) An antiderivative of $f(\theta) = \sec\theta\tan\theta$ is $F(\theta) = \sec\theta$.

$$\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta\,d\theta = \Big[\sec\theta\Big]_{\pi/4}^{\pi/3} = \frac{1}{\cos\pi/3} - \frac{1}{\cos\pi/4} = \frac{1}{1/2} - \frac{1}{\sqrt{2}/2} = 2 - \sqrt{2}.$$

(c) An antiderivative of $f(t) = \sqrt{t}(1+t) = t^{1/2} + t^{3/2}$ is $F(t) = \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}$.

$$\begin{aligned}
\int_1^4 \sqrt{t}(1+t)\,dt &= \left[\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}\right]_1^4 \\
&= \left[\frac{2}{3}4^{3/2} + \frac{2}{5}4^{5/2}\right] - \left[\frac{2}{3}1^{3/2} + \frac{2}{5}1^{5/2}\right] \\
&= \left[\frac{16}{3} + \frac{64}{5}\right] - \left[\frac{2}{3} + \frac{2}{5}\right] \\
&= \frac{80}{15} + \frac{192}{15} - \frac{10}{15} + \frac{6}{15} \\
&= \frac{256}{15}.
\end{aligned}$$

The relationship between the indefinite integral and the definite integral

The relationship between the indefinite and definite integral of a continuous function $f(x)$ is

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b.$$

Notice that

$$\left[\int f(x)\,dx\right]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

no matter what constant C we choose.

Rectilinear motion problems

Ex. 2. A particle moves along a horizontal line with velocity

$$v(t) = t^2 - t - 6$$

meters per second at time t.

(a) Find and interpret the displacement $s(4) - s(1)$ of the particle during the time period $t \in [1, 4]$.

(b) Find the distance traveled $\int_1^4 |v(t)|\, dt$ during this time period.

Solution:

(a) Since $v(t) = s'(t)$, by the Evaluation Theorem we have

$$\begin{aligned} s(4) - s(1) &= \int_1^4 v(t)\, dt \\ &= \int_1^4 t^2 - t - 6\, dt \\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \\ &= \left[\frac{64}{3} - 8 - 24\right] - \left[\frac{1}{3} - \frac{1}{2} - 6\right] \\ &= -\frac{9}{2}. \end{aligned}$$

(b) How should we evaluate $\int_1^4 |v(t)|\, dt$?

We know that $|v(t)| = v(t)$ when $v(t) > 0$, and $|v(t)| = -v(t)$ when $v(t) < 0$.

So where is $v(t) > 0$ and where is $v(t) < 0$?

Recall that the parabola

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

opens upward and meets the horizontal axis at $t = -2$ and $t = 3$.

Therefore,

$$\begin{aligned} \int_1^4 |v(t)|\, dt = \int_1^3 |v(t)|\, dt + \int_3^4 |v(t)|\, dt &= \int_1^3 \big(-v(t)\big)\, dt + \int_3^4 v(t)\, dt \\ &= -\int_1^3 \big(t^2 - t - 6\big)\, dt + \int_3^4 \big(t^2 - t - 6\big)\, dt \\ &= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_2^4 \\ &= \frac{22}{3} + \frac{17}{6} \\ &= \frac{61}{6} \end{aligned}$$

Functions defined by an integral

Consider the following function:

$$f(x) = \begin{pmatrix} \text{area under the curve} \\ y = t^2 \\ \text{from } t = 0 \text{ to } t = x \end{pmatrix}.$$

A formula for this function is...

$$f(x) = \int_0^x t^2 \, dt.$$

Functions defined by integrals appear often in physics, statistics, chemistry, electrical and civil engineering...

Today we will see how to analyze a function like this by using differential calculus. For example, we'll find the local maximum values of the **sine integral function**

$$\mathrm{Si}(x) = \int_0^x \tfrac{\sin t}{t} \, dt.$$

☞ We will also see that the natural logarithmic function $\ln$ can be defined by an integral.

<u>**Ex. 3.**</u> Define a function

$$g(x) = \int_a^x f(t) \, dt,$$

where $a = 1$ and $f(t) = 4t^3$. What is the derivative of $g(x)$?

<u>**Solution:**</u>

First we substitute for $f(t)$ and a. We get

$$g(x) = \int_1^x 4t^3 \, dt. \qquad (*)$$

Now use the Evaluation Theorem to get something we can differentiate.

$$g(x) = \int_1^x 4t^3 \, dt = \left[t^4\right]_1^x = x^4 - 1.$$

We know the derivative of $x^4 - 1$. Our answer is

$$g'(x) = \frac{d}{dx}\left[x^4 - 1\right] = 4x^3.$$

The Fundamental Theorem of Calculus, Part 1

Notice in the previous exercise that the expression we got for $g'(x)$ was exactly the integrand in the original equation (*) defining g.

That is,

$$g'(x) = 4x^3 = f(x).$$

Why is this?

Let's try to understand why this is true by looking at the situation geometrically.

- The definition of the derivative says

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

- We know

$$g(x) = \left(\begin{array}{c} \text{area under the curve} \\ y = f(t) \\ \text{from } t = 0 \text{ to } t = x \end{array} \right)$$

and

$$g(x+h) = \left(\begin{array}{c} \text{area under the curve} \\ y = f(t) \\ \text{from } t = 0 \text{ to } t = x + h \end{array} \right).$$

Therefore, the expression

$$g(x+h) - g(x)$$

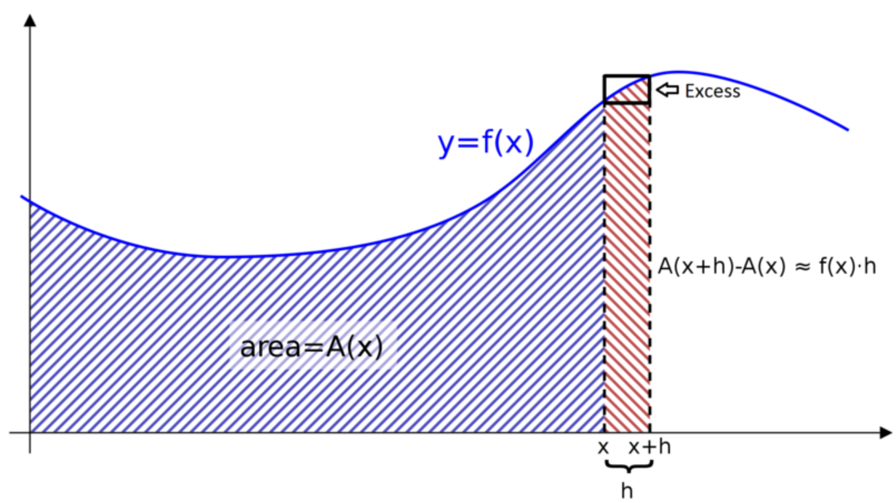
is the subtraction of two areas. This difference is the area under the curve from x to $x + h$.

- How else can we express the area under the curve $y = f(t)$ between x and $x + h$? Let's try to approximate it with a rectangle. The rectangle has to have base h. Let's give it the height $f(x)$. Now we have:

$$g(x+h) - g(x) \approx f(x) \cdot h.$$

- Therefore,

$$\frac{g(x+h) - g(x)}{h} \approx f(x). \qquad (**)$$



For *continuous* functions, taking the limit as $h \to 0$ in this last approximation (**) turns it into an equality:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

Notice that this equation establishes a connection between INTEGRATION (since g was defined by an integral), and DIFFERENTIATION, the two branches of the science of calculus.

$$g'(x) = \left[\frac{d}{dx}\int_a^x f(t)\,dt\right] = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

The above argument will serve as our proof for the first part of the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus, Part 1.

Let f be a continuous function defined on $[a,b]$. Define a function g as

$$g(x) = \int_a^x f(t)\,dt \qquad (a \le x \le b),$$

Then g is differentiable on the interval (a,b), and

$$g'(x) = f(x) \qquad (a < x < b).$$

Corollary.

Let f be a continuous function defined on $[a,b]$. If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Ex. 4. Define a function

$$\ell(x) = \int_1^x \frac{1}{t}\,dt.$$

(Many authors define the natural logarithm function $\ln(x)$ by this integral.)

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $\ell(x)$.

Ex. 5. Find the derivative of the **Fresnel function**

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\, dt.$$

Solution:

$$\frac{d}{dx}\big[S(x)\big] = \sin\left(\frac{\pi t^2}{2}\right).$$

Ex. 6. Let

$$h(t) = 12t - 8\sin(2t) + \sin(4t).$$

Given the fact that

$$h'(t) = 32\sin^4(x), \qquad (***)$$

find

$$\int_0^{\pi/2} \sin^4(t)\, dt.$$

Solution:

Equation (***) tells us that the function

$$h(t) = 12t - 8\sin(2t) + \sin(4t)$$

is an antiderivative of

$$h'(t) = 32\sin^4(x),$$

so

$$\begin{aligned}\int_0^{\pi/2} \sin^4(t)\, dt = \int_0^{\pi/2} \frac{1}{32} h'(t)\, dt = \frac{1}{32}\int_0^{\pi/2} h'(t)\, dt \\ &= \frac{1}{32}\left[12t - 8\sin(2t) + \sin(4t)\right]_0^{\pi/2} \\ &= \left[6\pi - 8\sin(\pi) + \sin(2\pi)\right] - 0 \\ &= 6\pi - 8.\end{aligned}$$

Ex. 7. At what values of x does the following function have local maximum values?

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt \qquad (x > 0)$$

Solution:

$$\frac{d}{dx}\left[\mathrm{Si}(x)\right] = \frac{\sin x}{x} = 0 \quad (x > 0) \qquad \Longleftrightarrow \qquad \sin x = 0 \quad (x > 0)$$

$$\Longleftrightarrow \qquad x = k\pi \ (k \text{ positive integer}).$$

So the critical numbers of $\mathrm{Si}(x)$ are

$$\pi,\ 2\pi,\ 3\pi,\ \ldots$$

Does $\mathrm{Si}(x)$ have local maxima or minima at these values of x?

Sign chart:

	$\sin(x)$	x	$\mathrm{Si}'(x) = \frac{\sin x}{x}$	
$(0, \pi)$	$+$	$+$	$+$	$\mathrm{Si}(x)$ increasing
$(\pi, 2\pi)$	$-$	$+$	$-$	$\mathrm{Si}(x)$ decreasing
$(2\pi, 3\pi)$	$+$	$+$	$+$	$\mathrm{Si}(x)$ increasing
$(3\pi, 4\pi)$	$-$	$+$	$-$	$\mathrm{Si}(x)$ decreasing
$(4\pi, 5\pi)$	$+$	$+$	$-$	$\mathrm{Si}(x)$ increasing
$(5\pi, 6\pi)$	$-$	$+$	$-$	$\mathrm{Si}(x)$ decreasing

Answer:

Si(x) has local maximum values at $x = \pi,\ 3\pi,\ 5\pi,\ \ldots.$

The Fundamental Theorem of Calculus, Part 1

We actually already know the second part of the Fundamental Theorem of Calculus—we've been calling it the Evaluation Theorem *(see beginning of this document)*.

Fundamental Theorem of Calculus, Part 2.

Let f be an integrable function defined on $[a, b]$. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The observant student will notice that this looks very much like the Corollary to Part 1 of the FTC. The only difference is that, in the Corollary, we required f to be continuous—here, we only require that f is integrable on the interval $[a, b]$.

☞ We've seen functions that are integrable on a closed interval $[a, b]$, but not continuous. Can you sketch the graph of such a function?

Mean Value Theorem for Integrals

Mean Value Theorem for Integrals. For a continuous function f defined on an interval $[a, b]$, there is a number c in $[a, b]$ such that the rectangle with base $[a, b]$ and height $f(c)$ that has the same area as the region under the graph of f from a to b.

☞ One way to understand this fact is: "You can always chop off the top of a (two-dimensional) mountain at a certain height, and use it to fill in the valleys so that the mountaintop becomes completely flat." The height will be $f(c)$.

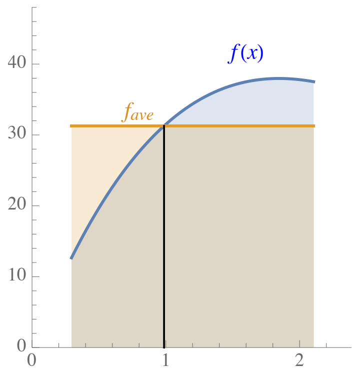
Ex. 8. As written above, the Mean Value Theorem for Integrals says that, for a continuous function f defined on the interval $[a, b]$,

$$f(c) \cdot (b - a) = \int_a^b f(x)\,dx$$

for some number c in $[a, b]$. Show that this means exactly that:

> A continuous function defined on $[a, b]$ takes on
> its average value for some number c in $[a, b]$.

(*See applet on iCollege:* "`Mean Value Theorem for Integrals`"—image on next page)



"Shave the mountain"

Ex. 9. Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c in $[0, 4]$ such that $f(c)$ equals the average value of f over $[a, b]$.

Ex. 10. Given the fact that $\int_0^3 2x^2 - 1\, dx = 15$, find c in the interval $[0, 3]$ such that $2c^2 - 1$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Additional exercises

Ex. 11. Find a function f and a number a such that

$$6+\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}.$$

(*Hint*: Use the Fundamental Theorem of Calculus, Part 1.)

Solution:

$$\begin{aligned}\frac{d}{dx}\left[6+\int_a^x \frac{f(t)}{t^2}\,dt\right] &= \frac{d}{dx}\left[2\sqrt{x}\right]\\ \frac{f(x)}{x^2} &= 2\left(\frac{1}{2}\right)x^{-1/2} = x^{-1/2}\\ f(x) &= x^{3/2}\end{aligned}$$

Now we find a by substituting $x = a$.

$$\begin{aligned}6+\int_a^a \frac{f(t)}{t^2}\,dt &= 2\sqrt{a}\\ 3+0 &= \sqrt{a}\end{aligned}$$

$$\boxed{f(x) = x^{3/2} \text{ and } a = 3}$$

Ex. 12. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous. Find

$$\int_1^2 h''(u)\,du,$$

and justify each step.

Solution:

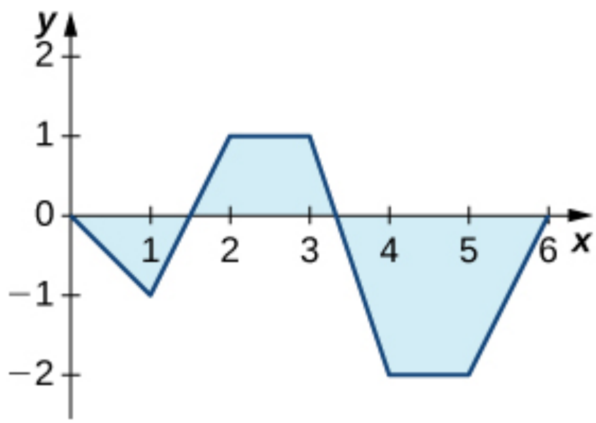
$$\begin{aligned}\int_1^2 h''(u)\,du &= h'(2) - h'(1) \qquad \text{(Corollary to FTC applies since } h'' \text{ is continuous)}\\ &= 5 - 2 = 3.\end{aligned}$$

"The other information is unnecessary."

Ex. 13 (§5.3—#161). Below is the graph of

$$g(x) = \int_0^x f(t)\,dt,$$

where f is a piecewise constant function.



(a) Over which interval(s) is f positive? Over which interval(s) is g negative?

(b) What are the maximum and minimum values of f?

(c) What's the average value of f?

Ex. 14 (§5.3—#149, 151). Find the derivative.

(a) $g(x) = \displaystyle\int_1^x e^{\cos(t)}\,dt$

(b) $g(x) = \displaystyle\int_4^x \frac{ds}{\sqrt{16 - s^2}}\,ds$

Ex. 15 (§5.3—#171, 173, 175, 177, 179, 181, 182, 183, 189). Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(a) $\displaystyle\int_{-2}^{3}(x^2+3x-5)\,dx$

(b) $\displaystyle\int_{2}^{3}(t^2-9)(4-t^2)\,dt$

(c) $\displaystyle\int_{0}^{1}x^{99}\,dx$

(d) $\displaystyle\int_{1/4}^{4}\left(x^2-\frac{1}{x^2}\right)\,dx$

(e) $\displaystyle\int_{1}^{4}\frac{1}{2\sqrt{x}}\,dx$

(f) $\displaystyle\int_{1}^{16}\frac{dt}{t^{1/4}}$

(g) $\displaystyle\int_{0}^{2\pi}\cos(\theta)\,d\theta$

(h) $\displaystyle\int_{0}^{\pi/2}\sin(\theta)\,d\theta$

(i) $\displaystyle\int_{-2}^{-1}\left(\frac{1}{t^2}-\frac{1}{t^3}\right)\,dt$

Ex. 16 (§5.3—#191, 193). Use the evaluation theorem to express the integral as a function of x.

(a) $\displaystyle\int_{1}^{x}e^t\,dt$

(b) $\displaystyle\int_{-x}^{x}\sin(t)\,dt$

Workbook Lesson 28

§5.4, Integration Formulas and the Net Change Theorem

Last revised: 2020-09-29 12:42

Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

Applying the basic integration formulas

Rules for integration were given in Lesson 24 (Section 4.10). Let's warm up with a couple of exercises that apply one of those formulas.

Ex. 1. Use the Power Rule for integration,

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \text{ for } n \neq -1,$$

to find

$$\int_1^4 \sqrt{t}(1+t)\, dt.$$

Ex. 2. Find the definite integral of $f(x) = x^2 - 3x$ over the interval $[1, 3]$.

The Net Change Theorem

The Net Change Theorem considers the integral of a *rate of change* $F'(x)$.

It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity.

Net Change Theorem. The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)\,dx.$$

This is not news. If the function F is continuously differentiable (that is, if the derivative of F is continuous), then

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

by the Evaluation Theorem (also known as the Fundamental Theorem of Calculus, Part 2). Simply adding $F(a)$ to both sides yields the Net Change Theorem.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral.

For example, suppose we are given the velocity function of a particle in motion. The velocity function accounts for both forward distance ($v(t) > 0$) and backward distance ($v(t) < 0$). To find the change in position—that is, the **net displacement**—we integrate $v(t)$.

Notice, however, that if we want the total *distance traveled*, we must count both forward distance and backward distance as positive quantities. When the total distance traveled is asked for, we must integrate $|v(t)|$, which is always nonnegative.

Ex. 3. Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Ex. 4. Find the total difference traveled by the particle in the previous exercise.

Ex. 5. Find the net displacement and total distance traveled in meters given the velocity function $f(t) = 12e^t - 2$ over the interval $[0, 2]$.

We can apply the Net Change Theorem to rates of change other than the velocity of a moving particle. For example, in the next exercise, we apply the Net Change Theorem to the rate of fuel consumption.

Ex. 7. If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at $5 - t^3$ gal./hr for the first hour, how much gasoline is used in the first hour?

Integrating even and odd functions

Recall:

A function f is **even** if

$$f(-x) = f(x)$$

for all x in the domain of f, **odd** if

$$f(-x) = -f(x)$$

for all x in the domain of f.

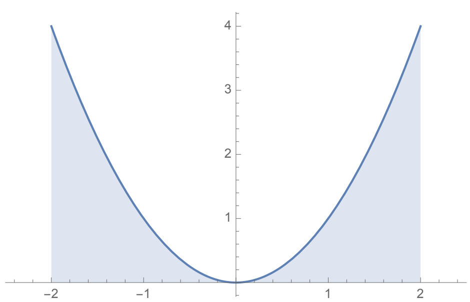
When integrating an even or odd function, symmetry ensures that the following rule holds true.

Theorem. For a continuous even function $f(x)$,

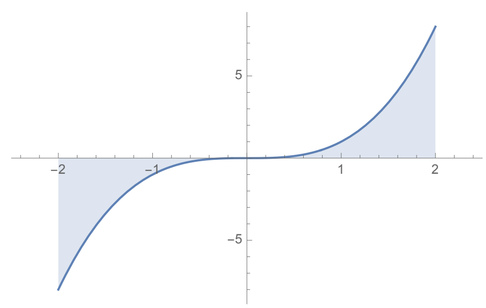
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For a continuous odd function $g(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$



For an even function (e.g., x^2), the signed area from 0 to a is equal to the signed area from $-a$ to 0.



For an odd function (e.g., x^3), the signed area from $-a$ to 0 cancels with the signed area from 0 to a.

Ex. 8. Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

Ex. 9. Integrate the even function $3x^8 - 2$ from -2 to 2 and verify that the integration formula for even functions holds.

Additional exercises

Ex. 10 (§5.4—#207, 209, 211). Evaluate.

(a) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **(b)** $\int \frac{1}{2x}\, dx$ **(c)** $\int (\sin x - \cos x)\, dx$

Ex. 11 (§5.4—#213). Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

Ex. 12 (§5.4—#219). Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s units to $2s$ units and evaluate the integral.

Ex. 13 (§5.4—#221). Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R units to $2R$ units and evaluate the integral.

Ex. 14 (§5.4—#225). Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \leq t \leq 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Ex. 15 (§5.4—#239). For a given motor vehicle, the maximum achievable deceleration from braking is approximately 7 m/sec^2 on dry concrete. On wet asphalt, it is approximately 2.5 m/sec^2. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

Workbook Lesson 29

§5.5, Substitution

Last revised: 2021-04-22 12:38

Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

Review: Change of variable

You may have encountered the idea of a *change of variable* in earlier math classes.

The idea is this: introducing a new variable sometimes makes an algebraic expression easier to work with.

For example, solving the equation

$$(x-3)^4 - (x-3)^2 = 0 \qquad (\star)$$

becomes easier if we substitute as follows:

$$\text{Let } u = (x-3)^2.$$

Then equation $(\star)$, which is equivalent to

$$\left((x-3)^2\right)^2 - (x-3)^2 = 0,$$

becomes

$$u^2 - u = 0,$$

which can be solved quickly:

$$u(u-1) = 0$$

$$u = 0 \qquad\qquad u = 1$$

$$(x-3)^2 = 0 \qquad\qquad (x-3)^2 = 1$$

$$x = 3 \qquad\qquad x - 3 = \pm 1$$

$$x = 2 \qquad x = 4$$

Notice that we gave the solutions in terms of the <u>original</u> variable x.

The statement

$$\text{Let } u = (x-3)^2.$$

is an example of a **change of variable**.

☞ Note that, when you make a change of variable in a math problem, you must *tell the reader* by writing a statement like "Let $u = \ldots$"

The Substitution Rule

As we've mentioned, the definition of the derivative comes with a formal process for differentiating a function, whereas antidifferentiation does not (see Lesson 24). So far, we have only been able to evaluate indefinite integrals "by inspection." That is, to evaluate

$$\int f(x)\,dx,$$

we had to stare at $f(x)$ until we thought of an antiderivative $F(x)$, that is, a function F such that

$$F'(x) = f(x)$$

on some interval. If we couldn't think of such a function F (or find one in a table of integrals), we couldn't evaluate the integral.

Special techniques for evaluating complicated integrals exist, however. We'll look at one such technique today, often called u-*substitution*.

Formally, this technique amounts to reversing the Chain Rule.

In practice, we think of this technique as a "change of variable" that simplifies the integrand. That is, we rewrite the integrand

$$y = y(x)$$

as a function of a new variable

$$y = y(u).$$

The u-substitution formula:

$$\int y(x)\,dx = \int y\big(u(x)\big)\cdot\frac{du}{dx}\,dx = \int y(u)\,du.$$

<u>Ex. 1.</u>

(a) Find a function $u(x)$ such that both u and $\dfrac{du}{dx}$ appear in the expression $2x\sqrt{1+x^2}$.

(b) Evaluate $\int 2x\sqrt{1+x^2}\,dx$. Make sure to state the change of variables you used, e.g. by writing "Let $u = \ldots$"

Solution:

Set $u = 1 + x^2$. Then $\dfrac{du}{dx} = 2x$.

$$\int \underbrace{(1+x^2)^{1/2}}_{y=u^{1/2}} \cdot \underbrace{2x}_{\frac{du}{dx}}\, dx = \int u^{1/2} \frac{du}{dx}\, dx = \int u^{1/2}\, du$$
$$= \tfrac{2}{3} u^{3/2} + C$$
$$= \tfrac{2}{3}(1+x^2)^{3/2} + C$$

Notice that we give our answer in terms of the original variable x.

Why does the u-substitution formula work?

Justification: Let Y be an antiderivative of y, so that $Y' = y$. Then

$$\int y\big(u(x)\big) \cdot \frac{du}{dx}\, dx = \int Y'\big(u(x)\big) \cdot u'(x)\, dx \overset{\text{(Chain Rule)}}{=} Y\big(u(x)\big) + C = Y\big(u(x)\big) + C$$
$$= \int Y'(u)\, du = \int y(u)\, du.$$

Theorem (Substitution Rule). If $u = u(x)$ is a differentiable function whose range is an interval I, and y is a continuous function defined on I, then

$$\int y\big(u(x)\big) \cdot \frac{du}{dx}\, dx = \int y(u)\, du.$$

Ex. 2. Evaluate $\int \sec^2 x \tan^3 x\, dx$.

Solution:

Set $u = \tan x$. Then $\dfrac{du}{dx} = \sec^2 x$.

$$\int \sec^2 x \tan^3 x\, dx = \int (\tan x)^3 (\sec x)^2\, dx = \int u^3 \frac{du}{dx}\, dx$$
$$= \frac{1}{4} u^4 + C$$
$$= \frac{1}{4} \tan^4 x + C.$$

A popular trick for applying the Substitution Rule is as follows. It uses the notation for differentials.

Trick. If $u = u(x)$, then $du = u'(x)\, dx$, so

$$\int y(u(x)) \cdot u'(x)\, dx = \int y(u)\, du.$$

Group work: Substitution Rule, Day 1

I. $\displaystyle\int x^2 \cos x^3 \, dx$

II. $\displaystyle\int \frac{\sin\sqrt{t}}{\sqrt{t}} \, dt$

III. $\displaystyle\int \frac{\cos(\pi/x)}{x^2} \, dx$

IV. $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

V. $\displaystyle\int \sin\theta \sec^2(\cos\theta) \, d\theta$

VI. $\displaystyle\int_0^{\sqrt{\pi}} x \cos x^2 \, dx$

VII. $\displaystyle\int_{-1}^{1} \cos\frac{\pi t}{4} \, dt$

VIII. $\displaystyle\int_{-1}^{1} \frac{\tan x \, dx}{1+x^2+x^4}$

IX. $\displaystyle\int \csc \pi t \cos \pi t \, dt$

X. Average value of $\sin 4x$ on $[-\pi, \pi]$

Substitution Rule, Day 1—Solutions

I.

$$u = x^3$$
$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}\,du = x^2\,dx$$

$$\int x^2\cos(x^3)\,dx = \int \frac{1}{3}\cos(u)\,du = \frac{1}{3}\sin(u) + C = \boxed{\frac{1}{3}\sin(x^3) + C}$$

II.

$$u = \sqrt{t} = t^{1/2}$$
$$\frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$$
$$2\,du = \frac{1}{\sqrt{t}}\,dt$$

$$\int \frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}\,dt = \int \frac{1}{\sqrt{t}}\sin\left(\sqrt{t}\right)\,dt = -2\cos(u) + C = \boxed{-2\cos\sqrt{t} + C}$$

III.

$$u = \frac{\pi}{x} = \pi x^{-1}$$
$$\frac{du}{dx} = -\pi x^{-2}$$
$$-\frac{1}{\pi}\,du = x^{-2}\,dx$$

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int x^{-2}\cos\left(\frac{\pi}{x}\right)\,dx = \int -\frac{1}{\pi}\cos(u)\,du = -\frac{1}{\pi}\int \cos(u)\,du = -\frac{1}{\pi}\sin(u) + C = \boxed{-\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C}$$

IV. $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

$$u = 1+\tan(t)$$
$$du = \sec^2(t)\,dt$$

$$\begin{aligned}\int \frac{dt}{\cos^2(t)\sqrt{1+\tan(t)}} &= \int \frac{\sec^2(t)}{\sqrt{1+\tan(t)}}\,dt\\ &= \int u^{-1/2}\,du\\ &= 2u^{1/2}+C\\ &= \boxed{2\sqrt{1+\tan(t)}+C}\end{aligned}$$

V.

$$u = \cos(\theta)$$
$$du = -\sin(\theta)\,d\theta$$
$$-du = \sin(\theta)\,d\theta$$

$$\begin{aligned}\int \sin(\theta)\,\sec^2\left(\cos(\theta)\right)\,d\theta &= -\int \sec^2(u)\,du\\ &= -\tan(u)+C\\ &= \boxed{-\tan\left(\cos(\theta)\right)+C}\end{aligned}$$

VI.

$$u = x^2$$
$$du = 2x\,dx$$
$$\frac{1}{2}\,du = x\,dx$$

$$x=\sqrt{\pi} \rightsquigarrow u=\pi$$
$$x=0 \rightsquigarrow u=0$$

$$\begin{aligned}\int_{x=0}^{x=\sqrt{\pi}} x\cos(x^2)\,dx &= \frac{1}{2}\int_{u=0}^{u=\pi} \cos(u)\,du\\ &= \frac{1}{2}\Big[\sin(u)\Big]_{u=0}^{u=\pi}\\ &= \frac{1}{2}\sin(\pi)-\sin(0)\\ &= \boxed{0}\end{aligned}$$

VII.

$$u = \frac{\pi t}{4} = \frac{\pi}{4}t$$
$$du = \frac{\pi}{4}\,dt$$
$$\frac{4}{\pi}\,du = dt$$

$$x = 1 \rightsquigarrow u = \frac{\pi}{4}$$
$$x = -1 \rightsquigarrow u = -\frac{\pi}{4}$$

$$\begin{aligned}\int_{-1}^{1} \cos\left(\frac{\pi t}{4}\right) dt &= \frac{4}{\pi}\int_{-\pi/4}^{\pi/4} \cos(u)\,du \\ &= \frac{4}{\pi}\Big[\sin(u)\Big]_{-\pi/4}^{\pi/4} \\ &= \frac{4}{\pi}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right] \\ &= \frac{4}{\pi}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= \boxed{\frac{4\sqrt{2}}{\pi}}\end{aligned}$$

VIII.

$\displaystyle\int_{-1}^{1} \frac{\tan x\,dx}{1+x^2+x^4} = \boxed{0}$ because $f(x) = \dfrac{\tan x}{1+x^2+x^4}$ is odd:

$$f(-x) \stackrel{?}{=} -f(x)$$
$$\frac{\tan(-x)}{1+(-x)^2+(-x)^4} \stackrel{?}{=} -\frac{\tan x}{1+x^2+x^4}$$
$$\frac{-\tan(x)}{1+x^2+x^4} \stackrel{\checkmark}{=} -\frac{\tan x}{1+x^2+x^4}$$

Note that the last equation follows from the prior equation because $\tan$ is odd on $[-1, 1]$.

IX.

$$u = \sin(\pi t)$$
$$du = \pi\cos(\pi t)\,dt$$
$$\frac{1}{\pi}\,du = \cos(\pi t)\,dt$$

$$\begin{aligned}\int \csc(\pi t)\cos(\pi t)\,dt &= \int \frac{1}{\sin(\pi t)} \cos(\pi t)\,dt \\ &= \int u^{-1}\,du \\ &= \boxed{\frac{1}{\pi}\ln\big(\sin(\pi t)\big) + C}\end{aligned}$$

X. Average value of $\sin 4x$ on $[-\pi, \pi]$

$$u = 4x$$
$$du = 4\,dx$$
$$\frac{1}{4}\,du = dx$$

$$x = \pi \rightsquigarrow u = 4\pi$$
$$x = -1 \rightsquigarrow u = -4\pi$$

$$\frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin(4x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(4x)\,dx$$
$$= \frac{1}{8\pi} \int_{-4\pi}^{4\pi} \sin(u)\,du$$
$$= \frac{1}{8\pi} \Big[\cos(4\pi) - \cos(-4\pi) \Big]$$
$$\overset{\text{(cos is even)}}{=} \boxed{0}$$

Substitution Rule, Day 2—Solutions

I.

$$u = \frac{\pi}{y} = \pi y^{-1}$$
$$du = -\pi y^{-2}\, dy$$
$$-\frac{1}{\pi}\, du = dy$$

$$\int \cos\left(\frac{\pi}{y}\right)\, dy = -\frac{1}{\pi}\int \cos(u)\, du \quad = \boxed{-\frac{1}{\pi}\sin\left(\tfrac{\pi}{y}\right) + C}$$

II.

$$\sin^2(\beta) + \cos^2(\beta) = 1$$
$$\sin^2(\beta) = 1 - \cos^2(\beta)$$
$$-\sin^2(\beta) = \cos^2(\beta) - 1$$

$$u = \sin(\beta)$$
$$du = \cos(\beta)\, d\beta$$

$$\begin{aligned}
\int \cos\beta\big(\cos^2(\beta) - 1\big)^{11}\, d\beta &= \int \cos\beta\big(-\sin^2(\beta)\big)^{11}\, d\beta \\
&= -\int \cos\beta\big(\sin(\beta)\big)^{22}\, d\beta \\
&= -\int u^{22}\, du \\
&= -\frac{1}{23}u^{23} + C \\
&= \boxed{-\frac{\sin^{23}\beta}{23} + C}
\end{aligned}$$

III.

$$u = 1 + 5t$$
$$du = 5\, dt$$
$$\frac{1}{5}\, du = dt$$

$$\begin{aligned}
\int \frac{-1}{(1+10t+25t^2)(1+5t)}\, dt &= \int \frac{-1}{(1+5t)^3}\, dt \\
&= -\frac{1}{5}\int u^{-3}\, du \\
&= \frac{1}{10}\frac{1}{(1+5t)^2} + C \\
&= \boxed{\frac{1}{10+100t+250t^2} + C}
\end{aligned}$$

IV.

$$u = v - 1$$
$$u + 4 = v + 3$$
$$du = dv$$

$$\begin{aligned}\int (v+3)(v-1)^5\,dv &= \int (u+4)u^5\,du \\ &= \int u^6 + 4u^5\,du \\ &= \frac{u^7}{7} + \frac{2u^6}{3} + C \\ &= u^6\left(\frac{3u+14}{21}\right) + C \\ &= \boxed{\frac{1}{21}(v-1)^6(3v+11) + C}\end{aligned}$$

V.

$$u = 4 - \sqrt{\theta}$$

Now solve for $d\theta$:

$$\sqrt{\theta} = 4 - u$$
$$\theta = (4-u)^2$$
$$d\theta = -2(4-u)\,du$$

$$\begin{aligned}\int \sqrt{4-\sqrt{\theta}}\,d\theta &= -2\int \sqrt{u}(4-u)\,du \\ &= -2\int 4u^{1/2} - u^{3/2})\,du \\ &= -2\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \\ &= -4u^{3/2}\left(\frac{20-3u}{15}\right) + C \\ &= \boxed{-\frac{4}{15}(4-\theta)^{3/2}(8+3\sqrt{\theta}) + C}\end{aligned}$$

VI.

$$u = 1 + 4x$$
$$2u - 2 = 2x$$
$$\frac{1}{4}\,du = dx$$

$$\begin{aligned}\int 8x\sqrt{1+4x}\,dx &= \frac{1}{4}\int (2u-2)u^{1/2}\,du \\ &= \frac{1}{2}\int (u-1)u^{1/2}\,du \\ &= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= u^{3/2}\left(\frac{3u-5}{15}\right) + C \\ &= \boxed{\frac{2}{15}(1+4x)^{3/2}(6x-1) + C}\end{aligned}$$

VII.

$$u = 2 - 3x$$
$$-\frac{1}{3}(u-2) = x$$
$$-\frac{1}{3}\,du = dx$$

$$\begin{aligned}
\int x\sqrt{2-3x}\,dx &= -\frac{1}{3}\int (u-2)u^{1/2}\cdot\left(-\frac{1}{3}\right)du \\
&= \frac{1}{9}\int u^{3/2} - 2u^{1/2}\,du \\
&= \frac{1}{9}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right) + C \\
&= \frac{1}{9}u^{3/2}\left(\frac{6u-20}{15}\right) + C \\
&= \boxed{-\frac{2}{135}(2-3x)^{3/2}(9x+4) + C}
\end{aligned}$$

VIII.

$$u = x^2 + 5$$
$$u - 5 = x^2$$
$$\frac{1}{2}\,du = x\,dx$$

$$\begin{aligned}
\int x^3\sqrt{x^2+5}\,dx &= \int x\cdot x^2\cdot\sqrt{x^2+5}\,dx \\
&= \frac{1}{2}\int (u-5)u^{1/2}\,du \\
&= \frac{1}{2}\int u^{3/2} - 5u^{1/2}\,du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{10}{3}u^{3/2}\right) + C \\
&= \frac{1}{2}u^{3/2}\left(\frac{6u-50}{15}\right) + C \\
&= \frac{1}{2}(x^2+5)^{3/2}\left(\frac{6x-20}{15}\right) + C \\
&= \boxed{\frac{1}{15}(x^2+5)^{3/2}\,(3x-10) + C}
\end{aligned}$$

IX.

$$u = 2x^3 + 4$$
$$\frac{1}{6}\,du = x^2\,dx$$
$$\frac{1}{2}(u-4) = x^3$$

$$\begin{aligned}
\int x^5\sqrt{2x^3+4}\,dx &= \int x^2 \cdot x^3 \cdot \sqrt{2x^3+4}\,dx \\
&= \frac{1}{12}\int (u-4)u^{1/2}\,dx \\
&= \frac{1}{12}\int u^{3/2} - 4u^{1/2}\,du \\
&= \frac{1}{12}\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right) + C \\
&= \frac{1}{6}u^{3/2}\left(\frac{3u-20}{15}\right) + C \\
&= \frac{1}{6}(2x^3+4)^{3/2}\left(\frac{6x^3-8}{15}\right) + C \\
&= \frac{1}{45}\big(2(x^3+2)\big)^{3/2}\left(\frac{3x^3-4}{15}\right) + C \\
&= \boxed{\frac{2\sqrt{2}}{45}(x^3+2)^{3/2}\,(3x^3-4) + C}
\end{aligned}$$

Back-substitution

[S] #29. Show that $\displaystyle\int \frac{\cos(\pi/y)}{y^2}\,dy = -\frac{1}{\pi}\sin\frac{\pi}{y} + C.$

$u = \pi/y$

☞ Show that $\displaystyle\int \cos\alpha(1-\cos^2\alpha)^{10}\,d\alpha = \frac{\sin^{21}\alpha}{21} + C.$

$u = \sin x$

☞ Show that $\displaystyle\int \cos\beta(\cos^2\beta - 1)^{11}\,d\beta = -\frac{\sin^{23}\alpha}{23} + C.$

$u = \sin x$

☞ Show that $\displaystyle\int \frac{du}{(1-7u)^2} = \frac{1}{7-49u} + C.$

$u = 1 + 7u$

☞ Show that $\displaystyle\int \frac{-1}{(1+10t+25t^2)(1+5t)}\,dt = \frac{1}{10+100t+250t^2} + C.$

$u = 1 + 5t$

☞ Show that $\displaystyle\int (x-3)(x+2)^7\,dx = \frac{1}{72}(x+2)^8(8x-29) + C.$

$u = 1 + 5t$

[D] #13. Show that $\int (v+3)(v-1)^5\,dv = \frac{1}{21}(v-1)^6(3v+11)+C$.

$u = v - 1$

[D] #18. Show that $\int \sqrt{4-\sqrt{\theta}}\,d\theta = -\frac{4}{15}(4-\sqrt{\theta})^{3/2}(8+3\sqrt{\theta})+C$.

$u = 4 - \sqrt{\theta}$

[M] #1. Show that $\int 8x\sqrt{1+4x}\,dx = \frac{2}{15}(1+4x)^{3/2}(6x-1)+C$.

$u = \sqrt{1+4x}$

[M] #2. Show that $\int x\sqrt{2-3x}\,dx = -\frac{2}{135}(2-3x)^{3/2}(9x+4)+C.$

$u = \sqrt{2+3x}$

[M] #13. Show that $\int x^3\sqrt{x^2+5}\,dx = \frac{1}{15}(x^2+5)^{3/2}(3x^2-10)+C.$

$u = \sqrt{x^2+5}$

[M] #12. Show that $\int x^5\sqrt{2x^3+4}\,dx = \frac{2\sqrt{2}}{45}(x^3+2)^{3/2}(3x^3-4)+C.$

$u = 2x^3+4$

Group work: Substitution Rule, Day 2

I. Show that $\displaystyle\int \frac{\cos(\pi/y)}{y^2}\,dy = -\frac{1}{\pi}\sin\frac{\pi}{y} + C.$

II. Show that $\displaystyle\int \cos\beta(\cos^2\beta - 1)^{11}\,d\beta = -\frac{\sin^{23}\beta}{23} + C.$

III. Show that $\displaystyle\int \frac{-1}{(1+10t+25t^2)(1+5t)}\,dt = \frac{1}{10+100t+250t^2} + C.$

IV. Show that $\displaystyle\int (v+3)(v-1)^5\,dv = \frac{1}{21}(v-1)^6(3v+11) + C.$

V. Show that $\displaystyle\int \sqrt{4-\sqrt{\theta}}\,d\theta = -\frac{4}{15}(4-\sqrt{\theta})^{3/2}(8+3\sqrt{\theta}) + C.$

VI. Show that $\displaystyle\int 8x\sqrt{1+4x}\,dx = \frac{2}{15}(1+4x)^{3/2}(6x-1) + C.$

VII. Show that $\displaystyle\int x\sqrt{2-3x}\,dx = -\frac{2}{135}(2-3x)^{3/2}(9x+4) + C.$

VIII. Show that $\displaystyle\int x^3\sqrt{x^2+5}\,dx = \frac{1}{15}(x^2+5)^{3/2}(3x^2-10) + C.$

IX. Show that $\displaystyle\int x^5\sqrt{2x^3+4}\,dx = \frac{2\sqrt{2}}{45}(x^3+2)^{3/2}(3x^3-4) + C.$

Additional exercises

Ex. 3 (§5.5—#261). Evaluate the integral $\int (x+1)^4 \, dx$ by making the substitution $u = x + 1$.

Ex. 4 (§5.5—#261). Evaluate the integral $\int (2x-3)^{-7} \, dx$ by making the substitution $u = 2x - 3$.

Ex. 5 (§5.5—#265). Evaluate the integral $\int \frac{x}{\sqrt{x^2+1}} \, dx$ by making the substitution $u = x^2 + 1$.

Ex. 6 (§5.5—#267). Evaluate the integral $\int (x-1)(x^2-2x)^3 \, dx$ by making the substitution $u = x^2 - 2x$.

Ex. 7 (§5.5—#269). Evaluate the integral $\int \cos^3\theta \, d\theta$ by making the substitution $u = \sin\theta$. (*Hint*: $\cos^2\theta = 1 - \sin^2\theta$.)

Ex. 8 (§5.5—#271, 273, 275, 279, 281, 283). Evaluate the indefinite integral.

(a) $\int x(1-x)^{99} \, dx$

(b) $\int (11x-7)^{-3} \, dx$

(c) $\int \cos^3(\theta)\sin(\theta) \, d\theta$

(d) $\int t\sin(t^2)\cos(t^2) \, dt$

(e) $\int \frac{x^2}{(x^3-3)^2} \, dx$

(f) $\int \frac{y^5}{(1-y^3)^{3/2}} \, dy$

Ex. 8. Verify each equation using the Substitution Rule.

(a) $\int x\sqrt{2-3x}\,dx = -\frac{2}{135}(2-3x)^{3/2}(9x+4)+C$

(b) $\int x^3\sqrt{x^2+5}\,dx = \frac{1}{15}(x^2+5)^{3/2}(3x^2-10)+C$

(c) $\int x^5\sqrt{2x^3+4}\,dx = \frac{2\sqrt{2}}{45}(x^3+2)^{3/2}(3x^3-4)+C$

Workbook Lesson 30

§5.6, Integrals Involving Exponential and Logarithmic Functions

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Objectives

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Recall (Sections 3.9 and 4.10):

Integration rule	Corresponding differentiation rule
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\frac{d}{dx}\big[\ln\lvert x\rvert\big] = \frac{1}{x}$
$\int e^x\,dx = e^x + C$	$\frac{d}{dx}\big[e^x\big] = e^x$
$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$ for $0 < b \neq 1$	$\frac{d}{dx}\big[b^x\big] = b^x\ln(b)$

We now add two integration rules:

$$\int \ln(x)\,dx = x\ln(x) - x + C$$
$$= x\big(\ln(x) - 1\big) + C$$

$$\int \log_b(x)\,dx = \frac{x}{\ln(b)}\big(\ln(x) - 1\big) + C$$

Ex. 1. True/False: The derivative of e^x is xe^{x-1}.

Ex. 2. Find the most general antiderivative of $e^x\sqrt{1+e^x}$.

Solution:

Take $u = 1 + e^x$. Then $du = e^x\,dx$, so

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du$$
$$= \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}(1+e^x)^{3/2} + C$$

Ex. 3. Integrate: $\int 3x^2 e^{2x^3}\,dx$.

Solution:

Take

$$u = 2x^3.$$

Then

$$du = 6x^2\,dx.$$

Since we want to match the expression $3x^2\,dx$ that actually appears in the integral, we rewrite the previous equation as

$$\frac{1}{2}\,du = 3x^2\,dx$$

$$\begin{aligned}\int 3x^2 e^{2x^3}\,dx &= \frac{1}{2}\int e^u\,du\\ &= \frac{1}{2}e^u + C\\ &= \frac{1}{2}e^{2x^3} + C.\end{aligned}$$

Ex. 4. Find $\displaystyle\int_0^2 e^{2x}\,dx$.

Solution:

Take

$$u = 2x.$$

Then

$$du = 2\,dx.$$

Since we want to match the expression dx that actually appears in the integral, we rewrite the previous equation as

$$\frac{1}{2}\,du = dx$$

$$\begin{aligned}\int_0^2 e^{2x}\,dx &= \frac{1}{2}\int e^u\,du\\ &= \frac{1}{2}e^u\Big|_{x=0}^{x=2}\\ &= \frac{1}{2}e^{2x}\Big|_{x=0}^{x=2}\\ &= \frac{1}{2}\left(e^4 - e^0\right)\\ &= \frac{e^4 - 1}{2}\end{aligned}$$

Ex. 5. Find $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$.

Solution:

Let us rewrite the given integral as

$$\int_1^2 e^{x^{-1}} x^{-2}\,dx$$

Take

$$u = x^{-1}.$$

Then

$$du = -x^{-2}\,dx.$$

Since we want to match the expression $x^{-2}\,dx$ that actually appears in the integral, we rewrite the previous equation as

$$-\,du = x^{-2}\,dx$$

Now

$$\begin{aligned}
\int_1^2 \frac{e^{1/x}}{x^2}\,dx &= \int_1^2 e^{x^{-1}} x^{-2}\,dx \\
&= -\int e^{u}\,du \\
&= -\,e^{u}\Big|_{x=1}^{x=2} \\
&= -\,e^{x^{-1}}\Big|_{x=1}^{x=2} \\
&= -\left(e^{1/2} - e^{1}\right) \\
&= e - \sqrt{e}.
\end{aligned}$$

Ex. 6. Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution:

We apply the Net Change Theorem:

$$G(10) = G(0) + \int_0^{10} 2e^{0.02t}\,dt$$

Taking $u = 0.02t$, so that $du = 0.02\,dx$ and $\frac{1}{0.02}dx = du$, we see that the net change is

$$\begin{aligned}\int_0^{10} 2e^{0.02t}\,dt &= \int_0^{10} \frac{2}{0.02}e^u\,du \\ &= \frac{2}{0.02}e^u\Big|_{t=0}^{t=10} \\ &= \frac{2}{0.02}e^{0.02t}\Big|_{t=0}^{t=10} \\ &= 100e^{0.2} - 100.\end{aligned}$$

The initial population is

$$G(0) = 100,$$

so the population after 10 days is

$$G(10) = G(0) + \int_0^{10} 2e^{0.02t}\,dt = 100 + 100e^{0.2} - 100 \approx 122.$$

Ex. 7. Find the antiderivative of $\dfrac{3}{x-10}$.

Solution:

First, let's rewrite the integrand in the form $\dfrac{1}{\square}$:

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx$$

Take $u = x - 10$. Then $du = dx$:

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx = 3\int \frac{1}{u}\,du = 3\ln|u| + C = 3\ln|x-10| + C$$

Is there any value of x that must be excluded for the integrand $\dfrac{3}{x-10}$ and its antiderivative $3\ln|x-10|$ to be defined? Yes—we must require $x \neq 10$.

Answer:

$$\boxed{3\ln|x-10| + C,\ x \neq 10}$$

Ex. 8. Find the antiderivative of $\dfrac{2x^3+3x}{x^4+3x^2}$.

Solution:

$$\begin{aligned} u &= x^4 + 3x^2 \\ du &= (4x^3+6x)\,dx = 2(2x^3+3x)\,dx \\ \frac{1}{2}\,du &= (2x^3+3x)\,dx \end{aligned}$$

$$\begin{aligned} \int \frac{2x^3+3x}{x^4+3x^2}\,dx &= \frac{1}{2}\int \frac{du}{u} \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^4+3x^2| + C \end{aligned}$$

Additional exercises

Ex. 9 (§5.6—#321, 329, 336, 337, 341). Evaluate the indefinite integral.

(a) $\displaystyle\int e^{-3x}\,dx$

(b) $\displaystyle\int \frac{2}{x}\,dx$

(c) $\displaystyle\int \frac{dx}{x(\ln x)^2}$

(d) $\displaystyle\int xe^{-x^2}\,dx$

(e) $\displaystyle\int x^2e^{-x^3}\,dx$

(f) $\displaystyle\int \frac{e^{\ln(1-t)}}{1-t}\,dt$

Ex. 10 (§5.6—#355). Evaluate the indefinite integral.

(a) $\displaystyle\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx$

(b) $\displaystyle\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

Ex. 11 (§5.6—#383). Use the identity

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

to derive the identity

$$\ln\frac{1}{x} = -\ln x.$$