Calculus for Business and Economics

Workbook

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First Edition

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Calculus for Business and Economics: Workbook

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AI Use Disclosure

This workbook was developed with the assistance of artificial intelligence tools, including OpenAI's ChatGPT. These tools were used to support drafting, editing, formatting, and the generation of instructional examples and practice problems. All content was reviewed, revised, and verified by the authors to ensure mathematical accuracy, pedagogical soundness, and alignment with course learning objectives.

The authors retain full responsibility for the content of this workbook.

Preface

This workbook is designed to support a one-semester course in *Calculus for Business and Economics*. It is intended for undergraduate students pursuing degrees in business, economics, and related fields who require a strong conceptual and applied understanding of calculus without unnecessary abstraction.

The primary goal of this workbook is to reinforce key calculus concepts through clear explanations, carefully worked examples, and structured practice problems grounded in business and economic contexts. Emphasis is placed on interpretation, modeling, and problem-solving rather than formal proofs. Topics are selected to align with standard business calculus curricula and to support informed decision-making using mathematical tools.

This workbook assumes a basic background in algebra and functions. A brief review chapter is included to refresh essential prerequisite skills. Trigonometric functions are intentionally excluded, as they are not central to most business calculus applications. Throughout the workbook, students are encouraged to think critically about the meaning of results and their practical implications.

Each section includes learning objectives, key definitions and concepts, fully explained solved examples, and practice problems with space provided for student work. Selected answers are collected in an appendix to allow students to check their understanding while still encouraging independent problem-solving.

This workbook is an Open Educational Resource (OER) created to reduce the cost of instructional materials and increase access to high-quality learning resources. It may be freely used, adapted, and shared in accordance with its open license. Instructors are encouraged to modify and supplement the material to best meet the needs of their students.

We hope this workbook serves as a useful and accessible resource that supports student learning and confidence in applying calculus to real-world business and economic problems.

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Contents

Chapter 1 Review

1.1 Functions and Graphs

Learning Objectives

After completing this section, you should be able to:

- Understand and use function notation.
- Evaluate functions at given input values.
- Identify domains and ranges of basic functions.
- Interpret graphs of functions in business contexts.
- Recognize increasing and decreasing behavior from graphs and formulas.

Key Definitions and Concepts

Function. A function is a rule that assigns exactly one output to each input. If f is a function, we write

$$y = f(x).$$

Function Notation. The symbol $f(x)$ represents the value of the function when the input is x. It does *not* mean multiplication.

Domain. The domain of a function is the set of all input values for which the function is defined.

Range. The range of a function is the set of all possible output values.

Graph of a Function. The graph of $f(x)$ is the set of all points $(x, f(x))$ in the coordinate plane.

Increasing and Decreasing Functions. A function is increasing on an interval if larger inputs produce larger outputs, and decreasing if larger inputs produce smaller outputs.

Solved Examples

Example 1: Evaluating a Function

Problem. Let

$$f(x) = 3x - 5.$$

Find $f(4)$ and interpret the result.

Solution.

$$f(4) = 3(4) - 5 = 7.$$

Interpretation. When the input is 4, the output of the function is 7.

Example 2: Domain of a Function

Problem. Find the domain of

$$g(x) = \frac{1}{x-2}.$$

Solution.

The denominator cannot be zero, so $x \neq 2$. The domain is all real numbers except $x = 2$.

Example 3: Interpreting a Graph

Problem. Suppose $C(x)$ represents the cost (in dollars) of producing x units of a product, and the graph of $C(x)$ is increasing. What does this mean?

Solution.

An increasing cost function means that producing more units leads to higher total cost.

Practice Problems

1. Let $f(x) = 2x + 1$.
   (a). Find $f(5)$.

   (b). Find x such that $f(x) = 9$.

2. Find the domain of the function

$$h(x) = \frac{4}{x+3}.$$

3. A revenue function $R(q)$ depends on the number of units sold, q.

(a). What does $R(0)$ represent?

(b). If the graph of $R(q)$ is decreasing, what does this indicate?

4. Describe the graph of the function $y = -3$.

Section Summary

- Functions assign one output to each input.
- Function notation is used throughout calculus.
- Domains restrict allowable inputs.
- Graphs provide visual insight into function behavior.
- Understanding functions is essential for limits, derivatives, and integrals.

1.2 Algebraic Operations and Simplification

Learning Objectives

After completing this section, you should be able to:

- Simplify algebraic expressions correctly.
- Factor basic polynomial expressions.
- Simplify rational expressions.
- Solve basic linear and quadratic equations.
- Apply algebraic simplification to prepare expressions for calculus.

Key Definitions and Concepts

Simplifying Expressions. Simplifying an expression means rewriting it in an equivalent form that is easier to work with by combining like terms, factoring, or canceling common factors.

Factoring. Factoring is the process of writing an expression as a product of simpler expressions.

Rational Expression. A rational expression is a ratio of two polynomials. The expression is undefined when the denominator equals zero.

Why This Matters for Calculus. Algebraic simplification is essential for:

- evaluating limits,
- finding derivatives,
- computing integrals,
- avoiding unnecessary errors.

Solved Examples

Example 1: Simplifying an Expression

Problem. Simplify:

$$3x - 2x + 5.$$

Solution.

$$3x - 2x + 5 = x + 5.$$

Example 2: Factoring

Problem. Factor the expression

$$x^2 - 9.$$

Solution.

This is a difference of squares:

$$x^2 - 9 = (x - 3)(x + 3).$$

Example 3: Simplifying a Rational Expression

Problem. Simplify

$$\frac{x^2 - 4}{x - 2}, \quad x \neq 2.$$

Solution.

Factor the numerator:

$$\frac{(x - 2)(x + 2)}{x - 2} = x + 2, \quad x \neq 2.$$

Example 4: Solving an Equation

Problem. Solve

$$2x - 7 = 5.$$

Solution.

$$2x = 12 \Rightarrow x = 6.$$

Practice Problems

1. Simplify the expression:

$$4x - 7 + 3x.$$

2. Factor completely:

$$x^2 - 16.$$

3. Simplify the rational expression:

$$\frac{x^2 - 5x}{x}, \quad x \neq 0.$$

4. Solve the equation:

$$3x + 4 = 19.$$

5. Explain why algebraic simplification is important before computing a limit.

Section Summary

- Algebraic simplification is a foundational skill for calculus.
- Factoring allows cancellation in rational expressions.
- Rational expressions are undefined where the denominator is zero.
- Strong algebra skills reduce errors in limits, derivatives, and integrals.

1.3 Exponential and Logarithmic Functions

Learning Objectives

After completing this section, you should be able to:

- Recognize exponential and logarithmic functions.
- Apply laws of exponents and logarithms.
- Evaluate exponential and logarithmic expressions.
- Solve basic exponential and logarithmic equations.
- Interpret exponential models in business contexts.

Key Definitions and Concepts

Exponential Function. An exponential function has the form

$$f(x) = ab^x,$$

where $a \neq 0$ and $b > 0$, $b \neq 1$.

Exponential functions are commonly used to model growth and decay in business, such as revenue growth, inflation, or population change.

Natural Exponential Function. The function

$$f(x) = e^x$$

uses the base $e \approx 2.71828$ and appears frequently in calculus.

Logarithmic Function. A logarithmic function is the inverse of an exponential function. The natural logarithm is written as

$$f(x) = \ln x,$$

and is defined for $x > 0$.

Laws of Logarithms.

$$\ln(ab) = \ln a + \ln b, \qquad \ln\left(\frac{a}{b}\right) = \ln a - \ln b, \qquad \ln(a^r) = r \ln a.$$

Solved Examples

Example 1: Evaluating an Exponential Expression

Problem. Evaluate

$$f(2) = 3e^2.$$

Solution.

$$f(2) = 3e^2.$$

Example 2: Simplifying Logarithms

Problem. Simplify

$$\ln(5x^2).$$

Solution.

Using logarithm laws,

$$\ln(5x^2) = \ln 5 + 2\ln x.$$

Example 3: Solving an Exponential Equation

Problem. Solve

$$e^{2x} = 7.$$

Solution.

Take the natural logarithm of both sides:

$$2x = \ln 7 \Rightarrow x = \frac{1}{2}\ln 7.$$

Example 4: Business Interpretation

Problem. A company's revenue is modeled by

$$R(t) = 500e^{0.04t},$$

where t is measured in years. Interpret the rate 0.04.

Solution.

The revenue grows continuously at a rate of 4% per year.

Practice Problems

1. Evaluate the expression:

$$2e^3.$$

2. Simplify the logarithmic expression:

$$\ln(4x).$$

3. Solve the equation:

$$e^x = 10.$$

4. Solve the equation:

$$\ln x = 3.$$

5. A population grows according to

$$P(t) = 1000e^{0.02t}.$$

Explain the meaning of the growth rate.

Section Summary

- Exponential functions model growth and decay.
- Logarithms are inverses of exponential functions.
- Logarithm laws simplify expressions and equations.
- Exponential models are fundamental in business and economics.

Chapter 2 The Derivative

2.1 Limits and Continuity

Learning Objectives

After completing this section, you should be able to:

- Evaluate limits of functions algebraically.
- Interpret limits in the context of business and economics.
- Determine whether a function is continuous at a given point.
- Identify and classify types of discontinuities.
- Explain the practical meaning of continuity in business models.

Key Definitions and Concepts

Limit. The limit of a function $f(x)$ as x approaches a value a is the number that $f(x)$ approaches as x gets close to a, provided this value exists.

$$\lim_{x \to a} f(x) = L$$

In business applications, limits are often used to analyze behavior *near* a certain level of output, price, or time, rather than exactly at that value.

One-Sided Limits. If a function approaches different values from the left and the right of a, we consider one-sided limits:

$$\lim_{x \to a^-} f(x), \qquad \lim_{x \to a^+} f(x)$$

A two-sided limit exists only if these one-sided limits are equal.

Continuity. A function $f(x)$ is continuous at $x = a$ if all three of the following conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

In business models, continuity means that small changes in input produce small changes in output, which is often a realistic assumption for cost, revenue, and profit functions.

Solved Examples

Example 1: Evaluating a Limit Algebraically

Problem. Suppose the cost (in dollars) of producing x units of a product is given by

$$C(x) = 3x^2 + 5x + 200.$$

Find

$$\lim_{x \to 10} C(x)$$

and interpret the result.

Solution.

Step 1: Recognize that $C(x)$ is a polynomial. Polynomials are continuous for all real values of x.

Step 2: Use direct substitution to evaluate the limit.

$$\lim_{x \to 10} C(x) = C(10)$$

Step 3: Compute the value.

$$C(10) = 3(10)^2 + 5(10) + 200 = 550$$

Interpretation. As production approaches 10 units, the cost approaches \$550. Producing close to 10 units will result in a cost close to \$550.

Example 2: A Limit That Does Not Exist

Problem. A company's pricing model is defined by

$$p(x) = \begin{cases} 50, & x < 100, \\ 65, & x \geq 100, \end{cases}$$

where x is the number of units ordered. Determine whether

$$\lim_{x \to 100} p(x)$$

exists.

Solution.

Step 1: Evaluate the left-hand limit.

$$\lim_{x \to 100^-} p(x) = 50$$

Step 2: Evaluate the right-hand limit.

$$\lim_{x \to 100^+} p(x) = 65$$

Step 3: Compare the one-sided limits. Since $50 \neq 65$, the two-sided limit does not exist.

Interpretation. The price jumps at 100 units, indicating a quantity discount threshold. The pricing function is not continuous at $x = 100$.

Example 3: Continuity at a Point

Problem. Let profit (in dollars) be given by

$$P(x) = \begin{cases} 4x - 100, & x \leq 50, \\ 3x + 50, & x > 50. \end{cases}$$

Determine whether $P(x)$ is continuous at $x = 50$.

Solution.

Step 1: Evaluate $P(50)$.

$$P(50) = 4(50) - 100 = 100$$

Step 2: Compute the left-hand limit.

$$\lim_{x \to 50^-} P(x) = 100$$

Step 3: Compute the right-hand limit.

$$\lim_{x \to 50^+} P(x) = 200$$

Step 4: Compare the results. Since the left-hand and right-hand limits are not equal, the limit does not exist.

Conclusion. The profit function is not continuous at $x = 50$, indicating an abrupt change in profit behavior at that production level.

Practice Problems

1. A revenue function is given by $R(x) = 12x + 80$.

   (a). Evaluate $\lim_{x \to 25} R(x)$.

   (b). Explain the meaning of the result in context.

2. Consider the function

$$C(x) = \begin{cases} 500, & x < 40, \\ 500 + 10(x - 40), & x \geq 40. \end{cases}$$

(a). Find $\lim_{x \to 40^-} C(x)$.

(b). Find $\lim_{x \to 40^+} C(x)$.

(c). Determine whether $C(x)$ is continuous at $x = 40$. Justify using the definition of continuity.

3. A demand function is defined for all $x \neq 60$.

(a). Explain why $\lim_{x \to 60} f(x)$ may exist even if $f(60)$ is not defined.

Section Summary

- Limits describe how a function behaves as the input approaches a particular value.
- A limit may exist even if the function is not defined at that value.
- Continuity requires that the function value and the limit agree at a point.
- Discontinuities often represent abrupt changes in business policies, pricing, or production.
- Understanding limits and continuity is essential for modeling smooth changes in cost, revenue, and profit.

2.2 The Derivative

Learning Objectives

After completing this section, you should be able to:

- Explain the derivative as an instantaneous rate of change and as the slope of a tangent line.
- Compute a derivative at a point using the limit definition.
- Interpret derivatives in business and economics contexts (e.g., marginal cost, marginal revenue, marginal profit).
- Approximate derivatives from tables or graphs using nearby average rates of change.
- Determine whether a function is differentiable at a point based on its behavior.

Key Definitions and Concepts

Derivative at a Point. The derivative of a function f at $x = a$ (if it exists) is defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

It represents the *instantaneous rate of change* of f with respect to x at $x = a$.

Geometric Meaning. If $y = f(x)$, then $f'(a)$ is the slope of the line tangent to the graph of f at the point $(a, f(a))$.

Derivative Function. If the limit exists for each x in an interval, then the derivative function is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notation. If $y = f(x)$, the derivative may be written as

$$f'(x), \qquad y', \qquad \frac{dy}{dx}, \qquad \frac{df}{dx}.$$

The notation $\frac{dy}{dx}$ is called *Leibniz notation*; it reminds us the derivative measures how y changes as x changes.

Differentiable. A function is *differentiable* at $x = a$ if $f'(a)$ exists. Differentiability implies continuity, but a continuous function may fail to be differentiable at a sharp corner, cusp, or vertical tangent.

Practical Approximation. When h is very small,

$$f'(a) \approx \frac{f(a+h) - f(a)}{h},$$

which is the slope of a secant line over a tiny interval.

Solved Examples

Example 1: Derivative at a Point (Limit Definition)

Problem. Suppose the cost (in dollars) of producing x units is

$$C(x) = 0.02x^2 + 5x + 1000.$$

Find $C'(50)$ using the limit definition and interpret the meaning.

Solution.

Step 1. Start with the definition:

$$C'(50) = \lim_{h\to 0} \frac{C(50+h) - C(50)}{h}.$$

Step 2. Compute $C(50+h)$:

$$C(50+h) = 0.02(50+h)^2 + 5(50+h) + 1000.$$

Expand $(50+h)^2 = 2500 + 100h + h^2$:

$$C(50+h) = 0.02(2500 + 100h + h^2) + 250 + 5h + 1000.$$

$$C(50+h) = 50 + 2h + 0.02h^2 + 250 + 5h + 1000 = 1300 + 7h + 0.02h^2.$$

Step 3. Compute $C(50)$:

$$C(50) = 0.02(2500) + 5(50) + 1000 = 50 + 250 + 1000 = 1300.$$

Step 4. Form the difference quotient:

$$\frac{C(50+h) - C(50)}{h} = \frac{(1300 + 7h + 0.02h^2) - 1300}{h} = \frac{7h + 0.02h^2}{h} = 7 + 0.02h.$$

Step 5. Take the limit:

$$C'(50) = \lim_{h\to 0}(7 + 0.02h) = 7.$$

Interpretation. At $x = 50$ units, the cost is increasing at about \$7 per additional unit. In business language, \$7 is the *marginal cost* at 50 units.

Example 2: Marginal Revenue from a Revenue Model

Problem. Revenue (in dollars) from selling x units is

$$R(x) = 200x - 0.5x^2.$$

Use the limit definition to find $R'(x)$, then evaluate $R'(100)$ and interpret.

Solution.

Step 1. Use the definition:

$$R'(x) = \lim_{h\to 0} \frac{R(x+h) - R(x)}{h}.$$

Step 2. Compute $R(x+h)$:

$$R(x+h) = 200(x+h) - 0.5(x+h)^2 = 200x + 200h - 0.5(x^2 + 2xh + h^2).$$

$$R(x+h) = 200x + 200h - 0.5x^2 - xh - 0.5h^2.$$

Step 3. Subtract $R(x) = 200x - 0.5x^2$:

$$R(x+h) - R(x) = \left(200x + 200h - 0.5x^2 - xh - 0.5h^2\right) - \left(200x - 0.5x^2\right) = 200h - xh - 0.5h^2.$$

Step 4. Divide by h and take the limit:

$$\frac{R(x+h) - R(x)}{h} = 200 - x - 0.5h \quad \Rightarrow \quad R'(x) = \lim_{h \to 0}(200 - x - 0.5h) = 200 - x.$$

Step 5. Evaluate at $x = 100$:

$$R'(100) = 200 - 100 = 100.$$

Interpretation. At 100 units, revenue is increasing at about \$100 per additional unit sold. This is the *marginal revenue* at $x = 100$.

Example 3: Approximating a Derivative from a Table

Problem. Let $S(t)$ be the total sales revenue (in thousands of dollars) after t months. The table shows values of $S(t)$:

t	1	2	3	4	5
$S(t)$	42	47	55	66	80

Approximate $S'(3)$ and interpret the result.

Solution.

Step 1. Use nearby average rates of change (secant slopes).

Using the interval from $t = 2$ to $t = 3$:

$$\frac{S(3) - S(2)}{3 - 2} = \frac{55 - 47}{1} = 8.$$

Using the interval from $t = 3$ to $t = 4$:

$$\frac{S(4) - S(3)}{4 - 3} = \frac{66 - 55}{1} = 11.$$

Step 2. Combine the information. A reasonable estimate is to average these two:

$$S'(3) \approx \frac{8 + 11}{2} = 9.5.$$

Interpretation. At month 3, sales revenue is increasing at about 9.5 thousand dollars per month (about \$9,500 per month).

Practice Problems

1. Let the cost (in dollars) be $C(x) = 0.01x^2 + 4x + 500$.
   (a). Use the limit definition to find $C'(x)$.

Hint: Start with $C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$.

(b). Evaluate $C'(80)$.

(c). Interpret $C'(80)$ in context.

2. Revenue is modeled by $R(x) = 150x - 0.25x^2$ (dollars).

(a). Find $R'(x)$ using the limit definition.

Hint: Expand $(x+h)^2$ *and simplify before dividing by* h.

(b). Compute $R'(60)$.

(c). Interpret $R'(60)$ in context.

3. Total profit $P(t)$ (in thousands of dollars) after t weeks is estimated by:

t	2	3	4	5	6
$P(t)$	18	22	27	31	34

(a). Approximate $P'(4)$ using secant slopes on both sides of $t = 4$.

Hint: Use $\frac{P(4)-P(3)}{4-3}$ *and* $\frac{P(5)-P(4)}{5-4}$*, then average.*

(b). Interpret your result.

Section Summary

- The derivative measures an instantaneous rate of change and equals the slope of a tangent line.

- The derivative at $x = a$ is defined by the limit $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
- Derivatives can be interpreted as marginal quantities in business (marginal cost, marginal revenue, marginal profit).
- When only tables or graphs are available, derivatives can be approximated using nearby average rates of change.
- A function must be differentiable at a point for the derivative to exist there.

2.3 Power and Sum Rules for Derivatives

Learning Objectives

After completing this section, you should be able to:

- Use the power rule to differentiate polynomial functions.
- Apply the constant multiple rule and sum rule for derivatives.
- Differentiate cost, revenue, and profit functions efficiently.
- Interpret derivatives obtained using rules as marginal quantities.
- Combine differentiation rules to handle realistic business models.

Key Definitions and Concepts

Power Rule. If $f(x) = x^n$, where n is a real number, then the derivative of f is

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

This rule allows derivatives of polynomial terms to be computed quickly without using limits.

Constant Multiple Rule. If $f(x)$ is differentiable and c is a constant, then

$$\frac{d}{dx}\big[c\,f(x)\big] = c\,f'(x).$$

Sum and Difference Rules. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x).$$

Derivative of a Constant. If $f(x) = c$, where c is a constant, then

$$\frac{d}{dx}(c) = 0.$$

Business Interpretation. When cost, revenue, or profit functions are differentiated using these rules, the resulting derivative represents a *marginal* quantity, such as marginal cost or marginal revenue.

Solved Examples

Example 1: Using the Power Rule

Problem. Find the derivative of

$$f(x) = 7x^4.$$

Solution.

Using the power rule,

$$f'(x) = 7(4)x^3 = 28x^3.$$

Example 2: Applying the Sum Rule

Problem. Find the derivative of

$$g(x) = 3x^3 - 5x^2 + 12x.$$

Solution.

Differentiate each term separately:

$$g'(x) = 9x^2 - 10x + 12.$$

Example 3: Marginal Cost

Problem. Suppose the cost (in dollars) of producing x units is

$$C(x) = 0.01x^3 + 4x^2 + 200x + 500.$$

Find the marginal cost function and evaluate it at $x = 50$.

Solution.

Step 1. Differentiate using the power and sum rules:

$$C'(x) = 0.03x^2 + 8x + 200.$$

Step 2. Evaluate at $x = 50$:

$$C'(50) = 0.03(2500) + 8(50) + 200 = 75 + 400 + 200 = 675.$$

Interpretation. At 50 units, the cost is increasing at about $675 per additional unit.

Example 4: Marginal Profit

Problem. Profit (in dollars) from selling x units is given by

$$P(x) = -0.02x^2 + 40x - 500.$$

Find the marginal profit function.

Solution.

Differentiate term by term:

$$P'(x) = -0.04x + 40.$$

Practice Problems

1. Find the derivative of

$$f(x) = 5x^6.$$

2. Find the derivative of

$$g(x) = 4x^3 - 7x^2 + 9.$$

3. The cost (in dollars) of producing x units is

$$C(x) = 0.02x^3 + 6x^2 + 150x + 800.$$

(a). Find the marginal cost function.

(b). Find the marginal cost when $x = 40$.

4. Revenue (in dollars) from selling x units is

$$R(x) = 120x - 0.3x^2.$$

(a). Find the marginal revenue function.

(b). Interpret $R'(50)$.

Section Summary

- The power rule provides a fast method for differentiating powers of x.
- Constants differentiate to zero, and constant multiples factor out of derivatives.
- The derivative of a sum or difference is the sum or difference of the derivatives.
- These rules simplify the process of finding marginal cost, revenue, and profit.
- Differentiation rules allow complex business models to be analyzed efficiently.

2.4 Product and Quotient Rules

Learning Objectives

After completing this section, you should be able to:

- Use the product rule to differentiate products of functions.
- Use the quotient rule to differentiate ratios of functions.
- Apply product and quotient rules to business-related functions.
- Combine differentiation rules to handle realistic cost, revenue, and profit models.
- Interpret derivatives obtained using these rules as marginal quantities.

Key Definitions and Concepts

Product Rule. If $f(x)$ and $g(x)$ are differentiable functions, then the derivative of their product is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule. If $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

When to Use These Rules.

- Use the product rule when a function is written as a *product* of two functions.
- Use the quotient rule when a function is written as a *ratio* of two functions.
- Do *not* distribute derivatives across products or quotients.

Business Interpretation. Product and quotient rules often arise when costs, revenues, or profits depend on multiple interacting variables, such as price times quantity or averages per unit.

Solved Examples

Example 1: Using the Product Rule

Problem. Suppose revenue (in dollars) is given by

$$R(x) = (50x)(100 - 2x),$$

where x is the number of units sold. Find $R'(x)$.

Solution.

Let

$$f(x) = 50x, \qquad g(x) = 100 - 2x.$$

Step 1. Compute the derivatives:

$$f'(x) = 50, \qquad g'(x) = -2.$$

Step 2. Apply the product rule:

$$R'(x) = f'(x)g(x) + f(x)g'(x) = 50(100 - 2x) + 50x(-2).$$

Step 3. Simplify:

$$R'(x) = 5000 - 100x - 100x = 5000 - 200x.$$

Example 2: Interpreting Marginal Revenue

Problem. Using the result from Example 1, evaluate $R'(20)$ and interpret.

Solution.

$$R'(20) = 5000 - 200(20) = 5000 - 4000 = 1000.$$

Interpretation. At 20 units sold, revenue is increasing at about $1,000 per additional unit.

Example 3: Using the Quotient Rule

Problem. Suppose the average cost (in dollars per unit) of producing x units is

$$A(x) = \frac{0.02x^2 + 200x + 1000}{x}.$$

Find $A'(x)$.

Solution.

Let

$$f(x) = 0.02x^2 + 200x + 1000, \qquad g(x) = x.$$

Step 1. Compute derivatives:

$$f'(x) = 0.04x + 200, \qquad g'(x) = 1.$$

Step 2. Apply the quotient rule:

$$A'(x) = \frac{(0.04x + 200)(x) - (0.02x^2 + 200x + 1000)(1)}{x^2}.$$

Step 3. Simplify the numerator:

$$A'(x) = \frac{0.04x^2 + 200x - 0.02x^2 - 200x - 1000}{x^2} = \frac{0.02x^2 - 1000}{x^2}.$$

Example 4: Simplifying Before Differentiation

Problem. Simplify $A(x)$ from Example 3 before differentiating, then find $A'(x)$.

Solution.

First simplify:

$$A(x) = 0.02x + 200 + \frac{1000}{x}.$$

Differentiate term by term:

$$A'(x) = 0.02 - \frac{1000}{x^2}.$$

This result matches the simplified form of the quotient rule result.

Practice Problems

1. Find the derivative of
$$f(x) = (4x)(x^2 + 3).$$

2. Find the derivative of
$$g(x) = \frac{6x^2 + 50x}{x}.$$

3. Revenue (in dollars) is given by
$$R(x) = x(120 - 3x).$$

   (a). Find $R'(x)$.

   (b). Evaluate $R'(10)$ and interpret the result.

4. The average cost (in dollars per unit) of producing x units is

$$A(x) = \frac{0.05x^2 + 300x + 2000}{x}.$$

(a). Find $A'(x)$.

(b). Determine whether average cost is increasing or decreasing at $x = 50$.

Section Summary

- The product rule is used to differentiate products of functions.
- The quotient rule is used to differentiate ratios of functions.
- Simplifying expressions before differentiating can reduce algebraic complexity.
- Product and quotient rules frequently arise in business and economic models.
- Derivatives obtained using these rules often represent marginal or average rates of change.

2.5 Chain Rule

Learning Objectives

After completing this section, you should be able to:

- Recognize composite functions.
- Apply the chain rule to differentiate composite functions.
- Differentiate business functions involving powers and nested expressions.
- Combine the chain rule with power, product, and quotient rules.
- Interpret derivatives obtained using the chain rule in business contexts.

Key Definitions and Concepts

Composite Function. A function is called a *composite function* if it can be written in the form

$$y = f(g(x)),$$

where $g(x)$ is an inner function and $f(u)$ is an outer function.

Chain Rule. If $y = f(g(x))$ and both f and g are differentiable, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

Alternative Notation. If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

When the Chain Rule Is Needed. The chain rule is used whenever a function involves a power or expression applied to another function, such as

$$(x^2 + 5x + 1)^4 \quad \text{or} \quad \sqrt{3x^2 + 10}.$$

Business Interpretation. In business and economics, the chain rule allows us to compute marginal changes when one quantity depends on another, which in turn depends on the input variable.

Solved Examples

Example 1: Basic Use of the Chain Rule

Problem. Find the derivative of

$$f(x) = (3x^2 + 5)^4.$$

Solution.

Let $u = 3x^2 + 5$, so that $f(x) = u^4$.

Differentiate:

$$\frac{df}{du} = 4u^3, \qquad \frac{du}{dx} = 6x.$$

Apply the chain rule:

$$f'(x) = 4(3x^2 + 5)^3(6x) = 24x(3x^2 + 5)^3.$$

Example 2: Chain Rule with a Power

Problem. Find the derivative of

$$g(x) = \sqrt{5x^2 + 20}.$$

Solution.

Rewrite the function:

$$g(x) = (5x^2 + 20)^{1/2}.$$

Differentiate using the chain rule:

$$g'(x) = \frac{1}{2}(5x^2 + 20)^{-1/2}(10x) = \frac{5x}{\sqrt{5x^2 + 20}}.$$

Example 3: Marginal Cost with the Chain Rule

Problem. Suppose the cost (in dollars) of producing x units is

$$C(x) = (0.01x^2 + 4x + 500)^2.$$

Find the marginal cost function.

Solution.

Let $u = 0.01x^2 + 4x + 500$, so $C(x) = u^2$.

Differentiate:

$$\frac{dC}{du} = 2u, \qquad \frac{du}{dx} = 0.02x + 4.$$

Apply the chain rule:

$$C'(x) = 2(0.01x^2 + 4x + 500)(0.02x + 4).$$

Interpretation. The marginal cost depends on both the cost level and how rapidly the cost level is changing.

Practice Problems

1. Find the derivative of

$$f(x) = (x^2 + 3x + 1)^5.$$

2. Find the derivative of

$$g(x) = \sqrt{2x^2 + 10x}.$$

3. The revenue (in dollars) from selling x units is

$$R(x) = (100 - 2x)^3.$$

(a). Find $R'(x)$.

(b). Interpret $R'(20)$.

4. The profit (in dollars) from selling x units is

$$P(x) = (x^2 + 10x + 25)^{1/2}.$$

(a). Find $P'(x)$.

(b). Determine $P'(25)$.

Section Summary

- The chain rule is used to differentiate composite functions.
- It allows differentiation of powers applied to inner functions.
- The chain rule frequently appears in business models involving nested relationships.
- It is often combined with other differentiation rules.
- Chain rule derivatives often represent marginal changes in complex systems.

2.6 Second Derivative and Concavity

Learning Objectives

After completing this section, you should be able to:

- Compute second derivatives of functions.
- Interpret the second derivative as a rate of change of a rate of change.
- Determine intervals where a function is concave up or concave down.
- Identify points of inflection using the second derivative.
- Apply concavity concepts to business and economic models.

Key Definitions and Concepts

Second Derivative. If $f(x)$ is a differentiable function and its derivative $f'(x)$ is also differentiable, then the *second derivative* of f is

$$f''(x) = \frac{d}{dx}\left[f'(x)\right].$$

The second derivative measures how the rate of change of f itself is changing.

Interpretation.

- $f'(x)$ measures the rate of change of $f(x)$.
- $f''(x)$ measures how fast that rate of change is increasing or decreasing.

Concavity.

- A function is *concave up* on an interval if $f''(x) > 0$ on that interval.
- A function is *concave down* on an interval if $f''(x) < 0$ on that interval.

Point of Inflection. A point $x = c$ is called a *point of inflection* if the concavity of $f(x)$ changes at $x = c$. This typically occurs where $f''(c) = 0$ or $f''(c)$ is undefined and concavity changes.

Business Interpretation. In business and economics:

- A positive second derivative may indicate increasing marginal cost or accelerating growth.
- A negative second derivative may indicate decreasing marginal profit or slowing growth.

Solved Examples

Example 1: Computing a Second Derivative

Problem. Let

$$f(x) = 2x^3 - 15x^2 + 36x + 10.$$

Find $f'(x)$ and $f''(x)$.

Solution.

Differentiate once:

$$f'(x) = 6x^2 - 30x + 36.$$

Differentiate again:

$$f''(x) = 12x - 30.$$

Example 2: Concavity of a Cost Function

Problem. Suppose the cost (in dollars) of producing x units is

$$C(x) = 0.5x^3 - 6x^2 + 40x + 100.$$

Determine where the cost function is concave up or concave down.

Solution.

Step 1. Find the first derivative:

$$C'(x) = 1.5x^2 - 12x + 40.$$

Step 2. Find the second derivative:

$$C''(x) = 3x - 12.$$

Step 3. Determine where $C''(x)$ is positive or negative.

Set $C''(x) = 0$:

$$3x - 12 = 0 \quad \Rightarrow \quad x = 4.$$

Step 4. Test intervals:

- For $x < 4$, $C''(x) < 0$ (concave down).
- For $x > 4$, $C''(x) > 0$ (concave up).

Example 3: Point of Inflection

Problem. Let profit (in dollars) be given by

$$P(x) = -x^3 + 12x^2 - 36x + 20.$$

Find the point of inflection.

Solution.

Step 1. Compute the derivatives:

$$P'(x) = -3x^2 + 24x - 36,$$

$$P''(x) = -6x + 24.$$

Step 2. Set the second derivative equal to zero:

$$-6x + 24 = 0 \quad \Rightarrow \quad x = 4.$$

Step 3. Verify concavity change:

- For $x < 4$, $P''(x) > 0$ (concave up).
- For $x > 4$, $P''(x) < 0$ (concave down).

Conclusion. The profit function has a point of inflection at $x = 4$.

Practice Problems

1. Find the first and second derivatives of

$$f(x) = x^4 - 8x^3 + 18x^2.$$

2. The revenue (in dollars) from selling x units is

$$R(x) = 3x^3 - 30x^2 + 90x.$$

(a). Find $R'(x)$ and $R''(x)$.

(b). Determine where the revenue function is concave up or concave down.

3. The cost (in dollars) of producing x units is

$$C(x) = 0.25x^3 - 6x^2 + 48x + 200.$$

(a). Find $C''(x)$.

(b). Find any point(s) of inflection.

Section Summary

- The second derivative measures how a rate of change is itself changing.
- Concavity describes the overall shape of a graph.
- A positive second derivative indicates concave up behavior.
- A negative second derivative indicates concave down behavior.
- Points of inflection occur where concavity changes.

2.7 Optimization

Learning Objectives

After completing this section, you should be able to:

- Explain the goal of optimization problems in business and economics.
- Use derivatives to find critical points of a function.
- Apply the first derivative test to identify maximum and minimum values.
- Solve optimization problems involving cost, revenue, and profit.
- Interpret optimal solutions in real-world business contexts.

Key Definitions and Concepts

Optimization. Optimization involves finding the *maximum* or *minimum* value of a function subject to given constraints.

Critical Point. A critical point of a function $f(x)$ is a value $x = c$ where

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined.}$$

Absolute vs. Relative Extrema.

- A *relative (local) maximum or minimum* occurs near a point.
- An *absolute (global) maximum or minimum* is the highest or lowest value over an entire interval.

First Derivative Test. If $f'(x)$ changes sign at a critical point c:

- from positive to negative, f has a local maximum at c;
- from negative to positive, f has a local minimum at c.

Business Interpretation. Optimization techniques are used to determine optimal production levels, pricing strategies, and resource allocation that maximize profit or minimize cost.

Solved Examples

Example 1: Maximizing Profit

Problem. Suppose profit (in dollars) from selling x units is

$$P(x) = -2x^2 + 200x - 1000.$$

Find the production level that maximizes profit and determine the maximum profit.

Solution.

Step 1. Compute the first derivative:

$$P'(x) = -4x + 200.$$

Step 2. Find critical points by setting $P'(x) = 0$:

$$-4x + 200 = 0 \quad \Rightarrow \quad x = 50.$$

Step 3. Use the first derivative test.

For $x < 50$, $P'(x) > 0$ (profit increasing). For $x > 50$, $P'(x) < 0$ (profit decreasing).

Conclusion. Profit is maximized at $x = 50$ units.

Step 4. Find the maximum profit:

$$P(50) = -2(50)^2 + 200(50) - 1000 = 4000.$$

Example 2: Minimizing Average Cost

Problem. The average cost (in dollars per unit) of producing x units is

$$A(x) = 0.5x + \frac{800}{x}.$$

Find the production level that minimizes average cost.

Solution.

Step 1. Compute the derivative:

$$A'(x) = 0.5 - \frac{800}{x^2}.$$

Step 2. Set $A'(x) = 0$:

$$0.5 - \frac{800}{x^2} = 0 \quad \Rightarrow \quad x^2 = 1600 \quad \Rightarrow \quad x = 40.$$

Step 3. Interpret. Average cost is minimized when 40 units are produced.

Example 3: Optimization with Constraints

Problem. A company can sell its product for \$60 per unit. The cost of producing x units is

$$C(x) = 0.5x^2 + 20x + 500.$$

Find the production level that maximizes profit.

Solution.

Step 1. Write the revenue function:

$$R(x) = 60x.$$

Step 2. Write the profit function:

$$P(x) = R(x) - C(x) = -0.5x^2 + 40x - 500.$$

Step 3. Differentiate:

$$P'(x) = -x + 40.$$

Step 4. Find the critical point:

$$-x + 40 = 0 \quad \Rightarrow \quad x = 40.$$

Conclusion. Profit is maximized when 40 units are produced.

Practice Problems

1. Profit (in dollars) is given by

$$P(x) = -x^2 + 120x - 900.$$

(a). Find the production level that maximizes profit.

(b). Find the maximum profit.

2. The average cost (in dollars per unit) is

$$A(x) = x + \frac{1000}{x}.$$

(a). Find the value of x that minimizes average cost.

(b). Interpret the result.

3. Revenue (in dollars) from selling x units is

$$R(x) = -3x^2 + 180x.$$

(a). Find the value of x that maximizes revenue.

(b). Find the maximum revenue.

4. A firm produces a product with cost

$$C(x) = x^2 + 40x + 600.$$

The product sells for \$80 per unit.

(a). Write the profit function.

(b). Find the production level that maximizes profit.

Section Summary

- Optimization problems involve finding maximum or minimum values.
- Critical points occur where the derivative is zero or undefined.
- The first derivative test helps classify extrema.
- Optimization is widely used in pricing, production, and cost analysis.
- Calculus provides a systematic approach to decision-making in business.

2.8 Curve Sketching

Learning Objectives

After completing this section, you should be able to:

- Identify key features of a function needed for sketching its graph.
- Use first derivatives to determine intervals of increase and decrease.
- Use second derivatives to analyze concavity.
- Identify critical points and points of inflection.
- Sketch graphs of business-related functions using calculus-based analysis.

Key Definitions and Concepts

Curve Sketching. Curve sketching is the process of drawing a graph of a function by analyzing its key characteristics rather than plotting many individual points.

Critical Points. A critical point occurs at a value $x = c$ where

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined.}$$

Critical points are candidates for local maxima or minima.

Increasing and Decreasing Intervals.

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Concavity.

- If $f''(x) > 0$, the graph is concave up.
- If $f''(x) < 0$, the graph is concave down.

Point of Inflection. A point of inflection occurs where the concavity of a function changes, typically where

$$f''(x) = 0$$

and the concavity changes sign.

Business Interpretation. Curve sketching helps visualize how cost, revenue, or profit behaves over time or levels of production, revealing trends such as increasing returns, diminishing returns, or saturation.

Solved Examples

Example 1: Curve Sketching a Profit Function

Problem. Suppose profit (in dollars) is given by

$$P(x) = -x^3 + 12x^2 - 36x + 20.$$

Use calculus to analyze the graph of $P(x)$.

Solution.

Step 1. Compute the first derivative:

$$P'(x) = -3x^2 + 24x - 36.$$

Step 2. Find critical points by setting $P'(x) = 0$:

$$-3x^2 + 24x - 36 = 0 \quad \Rightarrow \quad x^2 - 8x + 12 = 0 \quad \Rightarrow \quad x = 2,\ 6.$$

Step 3. Determine increasing/decreasing intervals.

Test intervals:

- For $x < 2$, $P'(x) < 0$ (decreasing).
- For $2 < x < 6$, $P'(x) > 0$ (increasing).
- For $x > 6$, $P'(x) < 0$ (decreasing).

Thus, $x = 2$ is a local minimum and $x = 6$ is a local maximum.

Step 4. Compute the second derivative:

$$P''(x) = -6x + 24.$$

Step 5. Find points of inflection:

$$-6x + 24 = 0 \Rightarrow x = 4.$$

Step 6. Determine concavity:

- For $x < 4$, $P''(x) > 0$ (concave up).
- For $x > 4$, $P''(x) < 0$ (concave down).

Conclusion. The profit function decreases, then increases, then decreases, with a point of inflection at $x = 4$.

Example 2: Curve Sketching a Cost Function

Problem. Let the cost (in dollars) of producing x units be

$$C(x) = 0.25x^3 - 3x^2 + 12x + 100.$$

Analyze the graph of $C(x)$.

Solution.

Step 1. First derivative:

$$C'(x) = 0.75x^2 - 6x + 12.$$

Step 2. Critical points:

$$0.75x^2 - 6x + 12 = 0 \quad \Rightarrow \quad x^2 - 8x + 16 = 0 \quad \Rightarrow \quad x = 4.$$

Step 3. Second derivative:

$$C''(x) = 1.5x - 6.$$

Step 4. Inflection point:

$$1.5x - 6 = 0 \Rightarrow x = 4.$$

Interpretation. The cost function changes concavity at $x = 4$, indicating a change in how marginal cost is increasing.

Practice Problems

1. Let

$$f(x) = x^3 - 9x^2 + 24x.$$

(a). Find all critical points.

(b). Determine intervals of increase and decrease.

(c). Find any points of inflection.

2. The revenue (in dollars) from selling x units is

$$R(x) = -x^3 + 15x^2 - 50x.$$

(a). Find all critical points of $R(x)$.

Note: Critical points are not necessarily integers.

(b). Determine the intervals on which revenue is increasing and decreasing.

(c). Identify the intervals of concavity.

Section Summary

- Curve sketching uses derivatives to analyze the shape of a graph.
- First derivatives determine increasing and decreasing behavior.
- Second derivatives determine concavity and inflection points.

- Curve sketching provides a visual understanding of business models.
- Calculus-based sketches reveal trends not obvious from formulas alone.

2.9 Applied Optimization

Learning Objectives

After completing this section, you should be able to:

- Translate real-world business problems into optimization models.
- Identify objective functions and constraints.
- Use derivatives to solve applied optimization problems.
- Interpret optimal solutions in realistic business contexts.
- Distinguish between mathematical solutions and practical feasibility.

Key Definitions and Concepts

Objective Function. The objective function represents the quantity to be maximized or minimized, such as profit, revenue, cost, or output.

Constraint. A constraint is an equation that limits the possible values of the variables in a problem.

Applied Optimization Process.

1. Define variables clearly.
2. Write the objective function.
3. Use the constraint to express the objective function in one variable.
4. Differentiate and find critical points.
5. Interpret the solution in the business context.

Business Interpretation. Applied optimization helps firms allocate resources efficiently and make informed strategic decisions.

Solved Examples

Example 1: Maximizing Profit with a Linear Constraint

Problem. A company sells two products. Product A yields a profit of \$30 per unit and Product B yields a profit of \$20 per unit. The company can produce at most 100 units total. How many units of each product should be produced to maximize profit?

Solution.

Let x be the number of units of Product A. Then $100 - x$ units of Product B are produced.

Step 1. Write the profit function:

$$P(x) = 30x + 20(100 - x) = 10x + 2000.$$

Step 2. Analyze the function.

Since $P(x)$ is increasing for all x, profit is maximized at the largest feasible value of x.

Conclusion. Produce 100 units of Product A and 0 units of Product B.

Example 2: Minimizing Cost with a Packaging Constraint

Problem. A company wants to design an open-top rectangular box with a square base that holds 500 cubic inches. Find the dimensions that minimize the amount of material used.

Solution.

Let x be the side length of the square base and h the height.

Step 1. Write the volume constraint:

$$x^2h = 500 \quad \Rightarrow \quad h = \frac{500}{x^2}.$$

Step 2. Write the surface area function:

$$S = x^2 + 4xh.$$

Substitute for h:

$$S(x) = x^2 + \frac{2000}{x}.$$

Step 3. Differentiate:

$$S'(x) = 2x - \frac{2000}{x^2}.$$

Step 4. Find the critical point:

$$2x = \frac{2000}{x^2} \Rightarrow x^3 = 1000 \Rightarrow x = 10.$$

Step 5. Find the height:

$$h = \frac{500}{10^2} = 5.$$

Conclusion. The optimal box has base 10×10 inches and height 5 inches.

Practice Problems

1. A company produces two products. Product X yields \$40 profit per unit and Product Y yields \$25 profit per unit. The company can produce at most 120 units total.
   (a). Write the profit function.

(b). Determine the production plan that maximizes profit.

2. A rectangular storage area is to be built along a straight wall, using fencing on only three sides. The total fencing available is 200 meters.

(a). Express the area as a function of one variable.

(b). Find the dimensions that maximize the area.

3. A product has a demand function

$$q = 500 - 5p,$$

where p is the price (in dollars) and q is the number of units sold.

(a). Write the revenue function in terms of p.

(b). Find the price that maximizes revenue.

Section Summary

- Applied optimization models real business decisions.
- Constraints reduce problems to single-variable functions.
- Derivatives identify optimal values.
- Interpretation ensures solutions are meaningful in practice.
- Optimization supports efficient use of resources and pricing strategies.

2.10 Other Applications

Learning Objectives

After completing this section, you should be able to:

- Use tangent line approximation to estimate function values.
- Interpret tangent line approximation in business contexts.
- Compute and interpret elasticity of demand.
- Classify demand as elastic, inelastic, or unitary.
- Explain how elasticity affects revenue decisions.

Key Definitions and Concepts

Tangent Line Approximation (TLA). Tangent line approximation uses the tangent line to a function at a known point to estimate the value of the function at a nearby point.

To approximate $f(x)$ using TLA:

1. Choose a value a such that a is close to x.
2. The exact values of $f(a)$ and $f'(a)$ are known.

The tangent line approximation formula is

$$f(x) \approx f(a) + f'(a)(x - a).$$

Another way to express the same idea is

$$\Delta y \approx f'(a)\Delta x.$$

The accuracy of the approximation depends on how close x is to a and on the shape of the graph of f.

Elasticity of Demand. If a demand function gives quantity q in terms of price p, the elasticity of demand is defined by

$$E = \left| \frac{p}{q} \frac{dq}{dp} \right|.$$

Since demand typically decreases as price increases, $\frac{dq}{dp}$ is negative. The absolute value ensures elasticity is positive.

Classification of Demand.

- If $E < 1$, demand is *inelastic*.
- If $E > 1$, demand is *elastic*.
- If $E = 1$, demand is *unitary*.

Interpretation of Elasticity. If the price increases by 1%, the demand will decrease by approximately $E\%$.

Solved Examples

Example 1: Tangent Line Approximation

Problem. Let

$$f(x) = \sqrt{x}.$$

Use tangent line approximation at $x = 16$ to estimate $\sqrt{15}$.

Solution.

Step 1. Compute $f(16)$ and $f'(x)$.

$$f(16) = 4, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(16) = \frac{1}{8}.$$

Step 2. Apply the approximation formula.

$$f(15) \approx f(16) + f'(16)(15 - 16) = 4 - \frac{1}{8} = 3.875.$$

Interpretation. $\sqrt{15}$ is approximately 3.875.

Example 2: Computing Elasticity of Demand

Problem. Suppose the demand function is

$$q = 200 - 4p.$$

Find the elasticity of demand when $p = 20$ and classify the demand.

Solution.

Step 1. Compute $\frac{dq}{dp}$.

$$\frac{dq}{dp} = -4.$$

Step 2. Evaluate q at $p = 20$.

$$q = 200 - 4(20) = 120.$$

Step 3. Compute elasticity.

$$E = \left| \frac{20}{120}(-4) \right| = \frac{2}{3}.$$

Conclusion. Since $E < 1$, demand is inelastic. Increasing price will increase revenue.

Practice Problems

1. Use tangent line approximation to estimate $\sqrt{24}$ using $a = 25$.

2. A demand function is given by

$$q = 300 - 6p.$$

(a). Find the elasticity of demand.

(b). Evaluate elasticity at $p = 30$.

(c). Classify the demand.

3. Explain what it means if demand is unitary.

Section Summary

- Tangent line approximation estimates function values near known points.
- The derivative provides the linear rate of change.
- Elasticity measures responsiveness of demand to price changes.
- Elasticity guides pricing and revenue decisions.
- Calculus connects mathematical models to real business behavior.

2.11 Implicit Differentiation and Related Rates

Learning Objectives

After completing this section, you should be able to:

- Differentiate equations defined implicitly.
- Solve for $\frac{dy}{dx}$ using implicit differentiation.
- Apply related rates to business and applied problems.
- Interpret rates of change in context.
- Translate word problems into mathematical relationships.

Key Definitions and Concepts

Implicit Differentiation. Implicit differentiation is used when a function is not explicitly solved for one variable in terms of another. Instead of rewriting the equation, both sides are differentiated with respect to x.

When differentiating terms involving y, the chain rule is applied:

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}.$$

Related Rates. Related rates problems involve finding the rate of change of one quantity by relating it to other changing quantities using derivatives.

General Strategy for Related Rates.

1. Write an equation relating the variables.
2. Differentiate both sides with respect to time.
3. Substitute known values.
4. Solve for the desired rate.

Business Interpretation. Related rates help model how changes in production, revenue, cost, or resources affect one another over time.

Solved Examples

Example 1: Implicit Differentiation

Problem. Differentiate the equation

$$x^2 + xy + y^2 = 25$$

with respect to x.

Solution.

Differentiate both sides with respect to x:

$$2x + (x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 0.$$

Group terms involving $\frac{dy}{dx}$:

$$(x + 2y)\frac{dy}{dx} = -(2x + y).$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

Example 2: Related Rates in Production

Problem. A company produces square metal sheets. Let x be the length of a side (in meters) and A the area (in square meters). If the side length is increasing at a rate of 0.5 m/min, find the rate at which the area is increasing when $x = 10$.

Solution.

Step 1. Write the equation:

$$A = x^2.$$

Step 2. Differentiate with respect to time t:

$$\frac{dA}{dt} = 2x\frac{dx}{dt}.$$

Step 3. Substitute values:

$$\frac{dA}{dt} = 2(10)(0.5) = 10.$$

Conclusion. The area is increasing at 10 m^2/min.

Example 3: Related Rates in Revenue

Problem. Revenue is given by

$$R = pq,$$

where p is price and q is quantity sold. Suppose price is decreasing at \$2 per unit per month and sales volume is increasing at 50 units per month. Find the rate of change of revenue when $p = 40$ and $q = 500$.

Solution.

Differentiate with respect to time:

$$\frac{dR}{dt} = p\frac{dq}{dt} + q\frac{dp}{dt}.$$

Substitute values:

$$\frac{dR}{dt} = 40(50) + 500(-2) = 2000 - 1000 = 1000.$$

Interpretation. Revenue is increasing at \$1,000 per month.

Practice Problems

1. Use implicit differentiation to find $\frac{dy}{dx}$:
$$x^2 + y^2 + 4xy = 16.$$

2. Suppose the area of a rectangular warehouse is
$$A = lw,$$
where l and w are changing over time. If l is increasing at 3 m/month and w is decreasing at 1 m/month, find $\frac{dA}{dt}$ when $l = 20$ and $w = 10$.

3. A company's inventory value is given by
$$V = pq.$$
If price is increasing at \$1 per unit per month and quantity is decreasing at 30 units per month, find $\frac{dV}{dt}$ when $p = 25$ and $q = 400$.

Section Summary

- Implicit differentiation is used when variables are interdependent.
- The chain rule accounts for variables changing with respect to another.
- Related rates problems involve differentiating with respect to time.
- Business applications include production, revenue, and inventory changes.
- Calculus connects multiple changing quantities in real-world systems.

Chapter 3 The Integral

3.1 The Definite Integral

Learning Objectives

After completing this section, you should be able to:

- Explain the meaning of the definite integral as an accumulation process.
- Interpret definite integrals in business and economic contexts.
- Understand the role of partitions and sums in defining an integral.
- Use definite-integral notation correctly.
- Distinguish between signed area and total accumulation.

Key Definitions and Concepts

Definite Integral. The definite integral of a function $f(x)$ from a to b is a number that represents the total accumulation of the values of $f(x)$ over the interval $[a, b]$. It is denoted by

$$\int_a^b f(x)\,dx.$$

In business and economics, definite integrals often represent total cost, total revenue, total profit, or total change over a given interval.

Partition of an Interval. A partition of $[a, b]$ divides the interval into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Each subinterval has width Δx.

Riemann Sum. A Riemann sum approximates the definite integral by summing function values multiplied by subinterval widths:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

where x_i^* is a sample point in the ith subinterval.

Limit of Riemann Sums. The definite integral is defined as the limit of Riemann sums as the number of subintervals increases:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Signed Area. If $f(x)$ is positive on an interval, the definite integral is positive. If $f(x)$ is negative, the definite integral is negative. Thus, the definite integral represents *net accumulation*, not just geometric area.

Solved Examples

Example 1: Interpreting a Definite Integral

Problem. Suppose $C'(x)$ represents the marginal cost (in dollars per unit) of producing x units of a product. Explain the meaning of

$$\int_{20}^{50} C'(x)\,dx.$$

Solution.

The definite integral represents the total change in cost as production increases from 20 units to 50 units.

Interpretation. $\int_{20}^{50} C'(x)\,dx$ is the total additional cost incurred when production increases from 20 to 50 units.

Example 2: Accumulation of Revenue

Problem. Let $R'(x)$ denote marginal revenue (in dollars per unit). What does

$$\int_{0}^{100} R'(x)\,dx$$

represent?

Solution.

This definite integral measures the accumulated revenue generated from selling the first 100 units.

Interpretation. The integral gives the total revenue earned from unit 0 through unit 100.

Example 3: Signed Area and Net Change

Problem. Suppose $f(x)$ represents net cash flow (in dollars per day), where positive values indicate profit and negative values indicate loss. Interpret

$$\int_{0}^{30} f(x)\,dx.$$

Solution.

The integral represents the net cash flow over 30 days, accounting for both profits and losses.

Interpretation. Positive portions add to the total, while negative portions subtract, giving the overall net result.

Practice Problems

1. Let $P'(x)$ represent marginal profit (in dollars per unit).
   (a). Explain the meaning of $\int_{10}^{40} P'(x)\,dx$.

(b). What does a negative value of this integral indicate?

2. Suppose $C'(x)$ is marginal cost.

(a). Interpret $\int_a^b C'(x)\,dx$ in words.

(b). How does this relate to total cost?

3. A function $f(x)$ represents the rate of change of inventory (units per week).

(a). Explain the meaning of $\int_0^8 f(x)\,dx$.

Section Summary

- The definite integral measures total accumulation over an interval.
- It is defined as the limit of Riemann sums.
- Definite integrals represent net change, not just geometric area.
- In business, integrals model total cost, revenue, profit, and inventory change.
- Understanding accumulation prepares for the Fundamental Theorem of Calculus.

3.2 The Fundamental Theorem of Calculus and Antidifferentiation

Learning Objectives

After completing this section, you should be able to:

- State and explain the Fundamental Theorem of Calculus.
- Evaluate definite integrals using antiderivatives.
- Understand the relationship between differentiation and integration.
- Apply the Fundamental Theorem of Calculus in business contexts.
- Interpret definite integrals as net change.

Key Definitions and Concepts

Antiderivative. A function $F(x)$ is an antiderivative of $f(x)$ if

$$F'(x) = f(x).$$

The collection of all antiderivatives of $f(x)$ is denoted by

$$\int f(x)\,dx = F(x) + C,$$

where C is a constant.

Fundamental Theorem of Calculus (FTC). If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Interpretation. The Fundamental Theorem of Calculus shows that differentiation and integration are inverse processes. It allows us to compute accumulated change using antiderivatives.

Business Interpretation. If $f(x)$ represents a marginal quantity (such as marginal cost or marginal revenue), then the definite integral gives the total change over an interval.

Solved Examples

Example 1: Evaluating a Definite Integral Using FTC

Problem. Evaluate

$$\int_2^5 (3x^2 - 4x + 1)\,dx.$$

Solution.

Step 1. Find an antiderivative:

$$F(x) = x^3 - 2x^2 + x.$$

Step 2. Apply the Fundamental Theorem of Calculus:

$$\int_2^5 (3x^2 - 4x + 1)\,dx = F(5) - F(2).$$

Step 3. Evaluate:

$$F(5) = 125 - 50 + 5 = 80, \quad F(2) = 8 - 8 + 2 = 2.$$

Conclusion.

$$\int_2^5 (3x^2 - 4x + 1)\,dx = 78.$$

Example 2: Total Cost from Marginal Cost

Problem. Suppose marginal cost (in dollars per unit) is given by

$$C'(x) = 4x + 20.$$

Find the increase in total cost when production increases from 10 units to 25 units.

Solution.

Step 1. Integrate marginal cost:

$$\int_{10}^{25} (4x + 20)\,dx.$$

Step 2. Find an antiderivative:

$$F(x) = 2x^2 + 20x.$$

Step 3. Apply FTC:

$$F(25) - F(10) = \big(2(25)^2 + 20(25)\big) - \big(2(10)^2 + 20(10)\big).$$

Step 4. Evaluate:

$$(1250 + 500) - (200 + 200) = 1750 - 400 = 1350.$$

Interpretation. The total cost increases by \$1,350 when production increases from 10 to 25 units.

Example 3: Net Change in Revenue

Problem. Marginal revenue is given by

$$R'(x) = 60 - 3x.$$

Find the change in revenue from selling 5 units to selling 15 units.

Solution.

Step 1. Set up the integral:

$$\int_5^{15} (60 - 3x)\,dx.$$

Step 2. Find an antiderivative:

$$F(x) = 60x - \frac{3}{2}x^2.$$

Step 3. Apply FTC:

$$F(15) - F(5).$$

Step 4. Evaluate:

$$\big(900 - 337.5\big) - \big(300 - 37.5\big) = 562.5 - 262.5 = 300.$$

Interpretation. Revenue increases by \$300 when sales increase from 5 to 15 units.

Practice Problems

1. Evaluate the definite integral:

$$\int_1^4 (2x^2 - x)\,dx.$$

2. Marginal cost is given by

$$C'(x) = 6x + 10.$$

Find the increase in total cost when production increases from 5 to 20 units.

3. Marginal profit is given by

$$P'(x) = 50 - 2x.$$

Find the change in profit when sales increase from 10 units to 30 units.

Section Summary

- Antiderivatives reverse differentiation.
- The Fundamental Theorem of Calculus links derivatives and integrals.
- FTC allows efficient computation of definite integrals.
- In business, marginal functions integrate to total change.
- FTC is foundational for all applied integration techniques.

3.3 Antiderivatives of Formulas

Learning Objectives

After completing this section, you should be able to:

- Use basic antiderivative rules to evaluate indefinite integrals.
- Apply the constant multiple and sum rules correctly.
- Use the power rule for antiderivatives.
- Integrate exponential and logarithmic functions.
- Include the constant of integration in all indefinite integrals.

Key Definitions and Concepts

Indefinite Integral. An indefinite integral represents the family of all antiderivatives of a function and is written as

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

Antiderivative Rules (Building Blocks).

Let f and g be functions of x, and let k and n be constants.

Constant Multiple Rule.

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Sum and Difference Rule.

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Power Rule. For $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Special Case: Constant Function.

$$\int k\,dx = kx + C.$$

Exponential Functions.

$$\int e^x\,dx = e^x + C, \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1).$$

Natural Logarithm.

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Solved Examples

Example 1: Using the Power Rule

Problem. Evaluate

$$\int (4x^3 - 6x + 5)\,dx.$$

Solution.

Apply the sum rule and power rule:

$$\int 4x^3\,dx = x^4, \quad \int -6x\,dx = -3x^2, \quad \int 5\,dx = 5x.$$

Final Answer.

$$x^4 - 3x^2 + 5x + C.$$

Example 2: Exponential Functions

Problem. Evaluate

$$\int \left(3e^x + 2 \cdot 5^x\right) dx.$$

Solution.

Apply the constant multiple and exponential rules:

$$\int 3e^x\,dx = 3e^x, \qquad \int 2 \cdot 5^x\,dx = \frac{2 \cdot 5^x}{\ln 5}.$$

Final Answer.

$$3e^x + \frac{2 \cdot 5^x}{\ln 5} + C.$$

Example 3: Logarithmic Antiderivative

Problem. Evaluate

$$\int \frac{7}{x}\,dx.$$

Solution.

Use the constant multiple rule:

$$\int \frac{7}{x}\,dx = 7\ln|x| + C.$$

Practice Problems

1. Evaluate:

$$\int (6x^2 - 4x + 9)\,dx.$$

2. Evaluate:

$$\int (2e^x - 3 \cdot 4^x)\, dx.$$

3. Evaluate:

$$\int \frac{5}{x}\, dx.$$

4. Evaluate:

$$\int (x^{-3} + 2x^{1/2})\, dx.$$

Section Summary

- Antiderivative rules allow efficient computation of indefinite integrals.
- The power rule applies to most algebraic functions.
- Exponential and logarithmic functions have special antiderivative formulas.
- The constant of integration represents a family of functions.
- These rules form the foundation for all integration techniques.

3.4 Substitution

Learning Objectives

After completing this section, you should be able to:

- Recognize integrals that require substitution.
- Apply substitution to evaluate indefinite integrals.
- Correctly change variables and differentials.
- Reverse the chain rule through integration.
- Use substitution in simple business and economic models.

Key Definitions and Concepts

Substitution Method. Substitution is a technique for evaluating integrals by changing variables to simplify the integrand. It is based on reversing the chain rule for differentiation.

Basic Idea. If an integrand contains a function and its derivative (or something close to it), substitution may simplify the integral.

Procedure for Substitution.

1. Choose a substitution $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral entirely in terms of u.
4. Integrate with respect to u.
5. Substitute back in terms of x.

Why Substitution Works. Substitution reverses the chain rule:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

Integration undoes this process.

Solved Examples

Example 1: Basic Substitution

Problem. Evaluate

$$\int 2x(3x^2+1)^4\,dx.$$

Solution.

Step 1. Let $u = 3x^2 + 1$.

Step 2. Compute $du = 6x\,dx$, so $\frac{1}{3}du = 2x\,dx$.

Step 3. Rewrite the integral:

$$\int (3x^2+1)^4(2x\,dx) = \frac{1}{3}\int u^4\,du.$$

Step 4. Integrate:

$$\frac{1}{3}\cdot\frac{u^5}{5}+C = \frac{u^5}{15}+C.$$

Step 5. Substitute back:

$$\frac{(3x^2+1)^5}{15}+C.$$

Example 2: Exponential Function

Problem. Evaluate

$$\int e^{5x-2}\,dx.$$

Solution.

Step 1. Let $u = 5x-2$.

Step 2. Then $du = 5\,dx$, so $\frac{1}{5}du = dx$.

Step 3. Rewrite and integrate:

$$\frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C.$$

Final Answer.

$$\frac{1}{5}e^{5x-2}+C.$$

Example 3: Logarithmic Form

Problem. Evaluate

$$\int \frac{4x}{x^2+9}\,dx.$$

Solution.

Step 1. Let $u = x^2+9$.

Step 2. Then $du = 2x\,dx$, so $2\,du = 4x\,dx$.

Step 3. Rewrite and integrate:

$$\int \frac{4x}{x^2+9}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u|+C.$$

Final Answer.

$$2\ln(x^2+9)+C.$$

Practice Problems

1. Evaluate:

$$\int x(2x^2+5)^3\,dx.$$

2. Evaluate:

$$\int e^{4x+1}\,dx.$$

3. Evaluate:

$$\int \frac{6x}{x^2+4}\,dx.$$

Section Summary

- Substitution simplifies integrals by changing variables.
- It reverses the chain rule for differentiation.
- Successful substitution identifies a function and its derivative.
- Algebraic substitution prepares for applied integration problems.
- Substitution is a core technique for evaluating complex integrals.

3.5 Additional Integration Techniques

Learning Objectives

After completing this section, you should be able to:

- Apply integration by parts to evaluate integrals of products.
- Choose appropriate functions for u and dv.
- Use the integration by parts formula for indefinite integrals.
- Recognize integrals that match standard integral formulas.
- Evaluate selected algebraic integrals using known results.

Key Definitions and Concepts

Integration by Parts. Integration by parts is a technique used to evaluate integrals of products of functions. It is based on the product rule for differentiation.

Formula for Integration by Parts. If $u = u(x)$ and $v = v(x)$ are differentiable functions, then

$$\int u\,dv = uv - \int v\,du.$$

Choosing u and dv. The goal is to choose u so that its derivative du is simpler than u, while dv is easy to integrate.

Definite Integrals. For definite integrals, integration by parts becomes

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du.$$

Standard Integral Forms. Some integrals occur frequently and have known formulas, such as:

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C.$$

Solved Examples

Example 1: Integration by Parts

Problem. Evaluate

$$\int x \ln x\,dx.$$

Solution.

Step 1. Choose

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx,$$

$$dv = x\,dx \quad \Rightarrow \quad v = \frac{x^2}{2}.$$

Step 2. Apply the formula:

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx.$$

Step 3. Simplify and integrate:

$$\frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Example 2: Definite Integral Using Integration by Parts

Problem. Evaluate

$$\int_1^e \ln x \, dx.$$

Solution.

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$.

$$\int_1^e \ln x \, dx = x \ln x|_1^e - \int_1^e 1 \, dx.$$

$$= (e \cdot 1 - 0) - (e - 1) = 1.$$

Conclusion.

$$\int_1^e \ln x \, dx = 1.$$

Example 3: Using a Standard Integral Formula

Problem. Evaluate

$$\int \frac{1}{x^2 - 9} \, dx.$$

Solution.

This matches the standard form with $a = 3$:

$$\int \frac{1}{x^2 - 9} \, dx = \frac{1}{6} \ln \left| \frac{x - 3}{x + 3} \right| + C.$$

Practice Problems

1. Evaluate:

$$\int x \ln x \, dx.$$

2. Evaluate:

$$\int_1^e \ln x \, dx.$$

3. Evaluate:

$$\int \frac{1}{x^2 - 16} \, dx.$$

Section Summary

- Integration by parts evaluates integrals of products.
- It is based on reversing the product rule.
- Choosing u appropriately simplifies the problem.
- Some integrals follow known algebraic formulas.
- These techniques extend the range of integrals we can evaluate.

3.6 Area, Volume, and Average Value

Learning Objectives

After completing this section, you should be able to:

- Compute the area under a curve using definite integrals.
- Find the area between two curves.
- Compute volumes of solids formed by rotating a region about the x-axis.
- Find the average value of a function on an interval.
- Interpret area, volume, and average value in applied contexts.

Key Definitions and Concepts

Area Under a Curve. If $f(x) \geq 0$ on $[a, b]$, the area between the graph of $f(x)$ and the x-axis is

$$\text{Area} = \int_a^b f(x)\,dx.$$

Area Between Two Curves. If $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is

$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

Volume of a Solid of Revolution. Consider the region bounded by the graph of $f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, where $f(x) \geq 0$ on $[a, b]$.

If this region is rotated about the x-axis, it forms a three-dimensional solid.

Partition the interval $[a, b]$ into subintervals of width Δx. Each subinterval generates a thin cylindrical disk with:

- radius $f(x)$,
- height Δx.

The volume of one disk is approximately

$$\pi\big(f(x)\big)^2\Delta x.$$

Adding the volumes of all disks gives a Riemann sum:

$$\sum \pi\big(f(x)\big)^2\Delta x.$$

Taking the limit as $\Delta x \to 0$ yields the exact volume.

Disk Method Formula.

$$V = \int_a^b \pi\big(f(x)\big)^2\,dx.$$

Average Value of a Function. The average value of $f(x)$ on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Solved Examples

Example 1: Area Under a Curve

Problem. Find the area under the curve $f(x) = 2x$ from $x = 0$ to $x = 5$.

Solution.

$$\text{Area} = \int_0^5 2x\, dx = \left[x^2\right]_0^5 = 25.$$

Example 2: Area Between Two Curves

Problem. Find the area between $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$.

Solution.

Since $x \geq x^2$ on $[0, 1]$,

$$\text{Area} = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Example 3: Volume Using the Disk Method

Problem. Find the volume of the solid obtained by rotating $f(x) = x$ about the x-axis on $[0, 2]$.

Solution.

Using the disk method,

$$V = \int_0^2 \pi x^2\, dx = \pi \left[\frac{x^3}{3}\right]_0^2 = \frac{8\pi}{3}.$$

Example 4: Average Value

Problem. Find the average value of $f(x) = 3x^2$ on $[0, 2]$.

Solution.

$$f_{\text{avg}} = \frac{1}{2-0} \int_0^2 3x^2\, dx = \frac{1}{2}\left[x^3\right]_0^2 = 4.$$

Practice Problems

1. Find the area under $f(x) = 4x$ from $x = 0$ to $x = 3$.

2. Find the area between $f(x) = 2x$ and $g(x) = x^2$ on $[0, 2]$.

3. Find the volume of the solid obtained by rotating $f(x) = x^2$ about the x-axis on $[0, 1]$.

4. Find the average value of $f(x) = x^2 + 2$ on $[1, 3]$.

Section Summary

- Definite integrals compute areas and accumulated quantities.
- Area between curves is found by integrating the difference.
- Volumes of solids of revolution are computed using the disk method.
- Average value represents the mean value of a function on an interval.
- These concepts have broad applications in business and economics.

3.7 Applications to Business

Learning Objectives

After completing this section, you should be able to:

- Identify demand and supply functions.
- Determine equilibrium price and quantity.
- Compute consumer and producer surplus using definite integrals.
- Interpret surplus values as gains from trade.
- Compute the present value of a continuous income stream.
- Compare investments using present value.

Key Definitions and Concepts

Demand and Supply. A demand function $p = d(q)$ gives the price consumers are willing to pay for a given quantity q and is typically decreasing.

A supply function $p = s(q)$ gives the price producers are willing to accept for a given quantity q and is typically increasing.

Equilibrium. The equilibrium point (q^*, p^*) occurs where demand equals supply:

$$d(q^*) = s(q^*) = p^*.$$

Consumer Surplus. Consumer surplus is the total benefit consumers receive by paying less than the maximum price they are willing to pay.

Producer Surplus. Producer surplus is the total benefit producers receive by selling at a price higher than the minimum price they are willing to accept.

Surplus Formulas. Given demand $p = d(q)$, supply $p = s(q)$, and equilibrium (q^*, p^*):

$$\text{Consumer Surplus} = \int_0^{q^*} d(q)\,dq - p^*q^*,$$

$$\text{Producer Surplus} = p^*q^* - \int_0^{q^*} s(q)\,dq.$$

The sum of consumer and producer surplus represents the **total gains from trade**.

Solved Examples

Example 1: Finding Equilibrium

Problem. Suppose demand and supply are given by

$$d(q) = 100 - 2q, \qquad s(q) = 20 + q.$$

Find the equilibrium price and quantity.

Solution.

Set demand equal to supply:

$$100 - 2q = 20 + q.$$

Solving,

$$3q = 80 \quad \Rightarrow \quad q^* = \frac{80}{3}.$$

Substitute into either function:

$$p^* = 100 - 2\left(\frac{80}{3}\right) = \frac{140}{3}.$$

Example 2: Consumer Surplus

Problem. Using the functions from Example 1, find the consumer surplus.

Solution.

$$\text{CS} = \int_0^{80/3} (100 - 2q)\, dq - \frac{140}{3} \cdot \frac{80}{3}.$$

$$= \left[100q - q^2\right]_0^{80/3} - \frac{11200}{9} = \frac{17600}{9} - \frac{11200}{9} = \frac{6400}{9}.$$

Example 3: Producer Surplus

Problem. Using the same functions, find the producer surplus.

Solution.

$$\text{PS} = \frac{140}{3} \cdot \frac{80}{3} - \int_0^{80/3} (20 + q)\, dq.$$

$$= \frac{11200}{9} - \left[20q + \frac{q^2}{2}\right]_0^{80/3} = \frac{11200}{9} - \frac{8000}{9} = \frac{3200}{9}.$$

Continuous Income Stream

Compound Interest Review. Let P be the principal, r the annual interest rate, and t the time in years.

- Compounded n times per year:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

- Compounded continuously:

$$A(t) = Pe^{rt}.$$

Continuous Income Stream. Suppose money earns interest at an annual rate r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) received between time 0 and time T.

Each small payment must be discounted back to the present using the exponential factor.

Present Value Formula.

$$\text{PV} = \int_0^T F(t)e^{-rt}\,dt.$$

Future Value. Once the present value is found, the future value at time T is

$$\text{FV} = \text{PV}\,e^{rT}.$$

Interpretation. Present value allows us to compare different investments by measuring their worth in today's dollars.

Practice Problems

1. Demand and supply are given by

$$d(q) = 60 - q, \qquad s(q) = q + 10.$$

(a). Find the equilibrium quantity and price.

(b). Compute the consumer surplus.

(c). Compute the producer surplus.

2. A continuous income stream pays \$1000 per year for 5 years. The interest rate is 6% per year, compounded continuously.

(a). Find the present value of the income stream.

(b). Find the future value at the end of 5 years.

Section Summary

- Equilibrium occurs where demand equals supply.
- Consumer and producer surplus measure gains from trade.
- Definite integrals compute surplus as areas.
- Continuous income streams are valued using discounted integrals.
- Present value compares investments in today's dollars.

3.8 Differential Equations

Learning Objectives

After completing this section, you should be able to:

- Understand what a differential equation represents.
- Model real-world situations using differential equations.
- Check whether a given function satisfies a differential equation.
- Solve separable differential equations.
- Apply differential equations to growth models.

Key Definitions and Concepts

Differential Equation. A differential equation is an equation involving a function and one or more of its derivatives. Differential equations model relationships between quantities and their rates of change.

Solution of a Differential Equation. A solution of a differential equation is a function that satisfies the equation when substituted into it.

Initial Condition. An initial condition specifies the value of the function at a particular point and allows us to determine a unique solution.

Modeling with Differential Equations

Example 1: Bank Balance Model

Problem. A bank pays 2% interest on a certificate of deposit but charges a \$20 annual fee. Write a differential equation for the balance $B(t)$.

Solution.

The balance changes due to:

- interest earned: $0.02B(t)$ dollars per year,
- fee charged: \$20 per year.

Thus,

$$B'(t) = 0.02B(t) - 20.$$

Interpretation. The rate of change of the balance depends on the current balance and a fixed annual fee.

Checking Solutions of Differential Equations

Example 2: Verifying a Solution

Problem. Check whether $y = x^2 + 5$ is a solution of

$$y' + y = x^2 + 7.$$

Solution.

Compute the derivative:

$$y' = 2x.$$

Substitute into the equation:

$$y' + y = 2x + (x^2 + 5) = x^2 + 2x + 5.$$

Since this expression is not equal to $x^2 + 7$ for all x, $y = x^2 + 5$ is *not* a solution.

Example 3: Checking Another Solution

Problem. Check whether $y = x + \frac{5}{x}$ satisfies

$$y' + \frac{y}{x} = 2.$$

Solution.

Compute the derivative:

$$y' = 1 - \frac{5}{x^2}.$$

Substitute:

$$y' + \frac{y}{x} = \left(1 - \frac{5}{x^2}\right) + \frac{1}{x}\left(x + \frac{5}{x}\right) = 1 - \frac{5}{x^2} + 1 + \frac{5}{x^2} = 2.$$

Thus, $y = x + \frac{5}{x}$ is a solution.

Separable Differential Equations

Definition. A differential equation is separable if it can be written in the form

$$g(y)\,dy = f(x)\,dx.$$

Example 4: Solving a Separable Equation

Problem. Find the general solution of

$$\frac{dy}{dx} = \frac{6x + 1}{2y}.$$

Solution.

Rewrite:

$$2y\,dy = (6x + 1)\,dx.$$

Integrate both sides:

$$\int 2y\,dy = \int (6x + 1)\,dx.$$

$$y^2 = 3x^2 + x + C.$$

Models of Growth

Unlimited Growth. If a quantity grows at a rate proportional to its size, it can be modeled by

$$\frac{dy}{dt} = ry,$$

where r is a constant.

Example 5: Population Growth

Problem. A population grows at 8% per year. If the current population is 5,000, find a formula for the population after t years.

Solution.

$$\frac{dy}{dt} = 0.08y.$$

Separate variables:

$$\frac{1}{y}dy = 0.08\,dt.$$

Integrate:

$$\ln|y| = 0.08t + C.$$

Exponentiate:

$$y = Ae^{0.08t}.$$

Apply the initial condition $y(0) = 5000$:

$$5000 = A.$$

Thus,

$$y = 5000e^{0.08t}.$$

Limited Growth. If growth slows as a quantity approaches a maximum value M, it can be modeled by

$$\frac{dy}{dt} = k(M - y),$$

where k is a constant.

Logistic Growth. If growth depends on both the current size and the distance from a maximum value M, the model is

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{M}\right).$$

The solution has the form

$$y = \frac{M}{1 + Ae^{-rt}}.$$

Practice Problems

1. Write a differential equation for a bank account that earns 3% interest per year and charges a $15 annual fee.

2. Check whether $y = x^2 - 4$ is a solution of $y' + y = x^2 - 2$.

3. Solve the separable differential equation:

$$\frac{dy}{dx} = \frac{4x}{y}.$$

4. A population grows at a rate proportional to its size with growth rate 5%. Write the differential equation and general solution.

5. A population has limited growth with maximum size 10,000 and growth constant $k = 0.02$. Write the differential equation.

Section Summary

- Differential equations model relationships involving rates of change.
- Solutions are functions that satisfy the equation.
- Separable equations can be solved by integration.
- Growth models describe population and financial behavior.
- Differential equations connect calculus to real-world dynamics.

Chapter 4 Functions of Two Variables

4.1 Functions of Two Variables

Learning Objectives

After completing this section, you should be able to:

- Understand the concept of a function of two variables.
- Interpret functions of two variables using formulas, tables, and graphs.
- Evaluate functions of two variables at given input values.
- Identify the domain and range of a function of two variables.
- Apply functions of two variables in business and economic contexts.

Key Definitions and Concepts

Function of Two Variables. A function of two variables assigns a single output value to each ordered pair (x, y) in its domain. We write

$$z = f(x, y).$$

Inputs and Output. The variables x and y are the independent variables, and z is the dependent variable.

Domain. The domain of a function of two variables is the set of all ordered pairs (x, y) for which the function is defined.

Range. The range is the set of all possible output values of the function.

Business Interpretation. In business applications, functions of two variables are often used to model cost, revenue, demand, and profit depending on two inputs such as time and quantity or price and advertising.

Solved Examples

Example 1: Cost as a Function of Two Variables

Problem. Suppose the cost of renting a car depends on the number of days rented, d, and the number of miles driven, m. Explain what the notation $C(d, m)$ represents.

Solution.

The function $C(d, m)$ represents the total cost of renting the car when it is rented for d days and driven m miles.

Example 2: Evaluating a Function from a Formula

Problem. Suppose the cost function is

$$C(d, m) = 40d + 0.15m.$$

Find $C(3, 200)$ and interpret the result.

Solution.

$$C(3, 200) = 40(3) + 0.15(200) = 120 + 30 = 150.$$

Interpretation. The cost of renting the car for 3 days and driving 200 miles is $150.

Example 3: Order Matters

Problem. Using the same function, compare $C(100, 4)$ and $C(4, 100)$.

Solution.

$$C(100, 4) = 40(100) + 0.15(4) = 4000.60,$$

$$C(4, 100) = 40(4) + 0.15(100) = 175.$$

Since the inputs represent different quantities, the order of the inputs matters.

Example 4: Graphing a Function of Two Variables

Problem. Describe the graph of the function $z = 2$.

Solution.

The graph of $z = 2$ is a plane parallel to the xy-plane, located 2 units above it.

Example 5: Distance in Three Dimensions

Problem. Find the distance between the points $A = (1, 2, 3)$ and $B = (7, 5, -3)$.

Solution.

$$\text{Distance} = \sqrt{(7-1)^2 + (5-2)^2 + (-3-3)^2} = \sqrt{36 + 9 + 36} = 9.$$

Practice Problems

1. A company's revenue depends on the number of units sold, q, and the price per unit, p. Explain what the notation $R(q, p)$ represents.

2. Suppose the cost function is

$$C(x, y) = 5x + 2y.$$

(a). Find $C(4, 10)$.

(b). Explain the meaning of this value in context.

3. Determine whether $C(2, 5)$ and $C(5, 2)$ are equal for the function

$$C(x, y) = 3x + 4y.$$

Explain why or why not.

4. Describe the graph of the function $z = -1$.

5. Find the distance between the points $(2, 0, 1)$ and $(5, 4, 5)$.

Section Summary

- Functions of two variables assign one output to each ordered pair of inputs.
- Order matters when evaluating functions of multiple variables.
- Graphs of functions of two variables are surfaces in three dimensions.
- These functions are widely used in business and economics.

- Understanding functions of two variables prepares us for partial derivatives.

4.2 Calculus of Functions of Two Variables

Learning Objectives

After completing this section, you should be able to:

- Compute partial derivatives of functions of two variables.
- Interpret partial derivatives in practical and business contexts.
- Estimate partial derivatives from tables and contour diagrams.
- Use partial derivatives to estimate function values.
- Construct and interpret linear (tangent plane) approximations.

Key Definitions and Concepts

Partial Derivative. The partial derivative of a function $f(x, y)$ with respect to x measures how f changes as x changes while y is held constant. It is denoted by

$$f_x(x, y).$$

Similarly, the partial derivative with respect to y is denoted by

$$f_y(x, y).$$

Interpretation. Partial derivatives represent rates of change. In business applications, they often describe how a quantity such as cost, revenue, or demand responds to changes in one variable while other variables remain fixed.

Linear Approximation (Tangent Plane). Near a point (a, b), a function $f(x, y)$ can be approximated by

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This approximation is useful for estimating values when exact computation is difficult.

Solved Examples

Example 1: Computing Partial Derivatives

Problem. Let

$$f(x, y) = x^2 - 4xy + 4y^2.$$

Find f_x and f_y.

Solution.

Treat y as a constant when differentiating with respect to x:

$$f_x(x, y) = 2x - 4y.$$

Treat x as a constant when differentiating with respect to y:

$$f_y(x, y) = -4x + 8y.$$

Example 2: Evaluating Partial Derivatives

Problem. Using the same function, find $f_x(1,1)$ and $f_y(1,1)$.

Solution.

$$f_x(1,1) = 2(1) - 4(1) = -2, \qquad f_y(1,1) = -4(1) + 8(1) = 4.$$

Example 3: Interpretation

Problem. If $f(x,y)$ represents revenue depending on price x and advertising spending y, explain the meaning of f_x.

Solution.

The partial derivative f_x represents the rate at which revenue changes as price changes, assuming advertising spending remains constant.

Example 4: Linear Approximation

Problem. Suppose $f(1,1) = 2$, $f_x(1,1) = 0.5$, and $f_y(1,1) = 1$. Estimate $f(1.1, 0.9)$.

Solution.

$$f(1.1, 0.9) \approx 2 + 0.5(0.1) + 1(-0.1) = 2.$$

Practice Problems

1. Let $f(x,y) = x^2 + 3y^2$.
   (a). Find f_x and f_y.

   (b). Evaluate $f_x(2,1)$ and $f_y(2,1)$.

2. Suppose $C(x, y)$ represents cost depending on labor x and materials y. Explain the meaning of $C_y(x, y)$.

3. Given

$$g(x, y) = \frac{e^{x+y}}{3} + \ln(y),$$

find g_x and g_y.

4. The partial derivatives of f at $(2, 3)$ are $f_x = 1.2$ and $f_y = -0.5$. Use linear approximation to estimate $f(2.1, 2.9)$.

5. Explain why partial derivatives are useful for estimating values of multivariable functions.

Section Summary

- Partial derivatives measure rates of change in one variable at a time.
- They play a key role in business modeling and economic interpretation.
- Linear approximations allow estimation near known points.
- These ideas extend single-variable calculus to more realistic models.

4.3 Optimization

Learning Objectives

After completing this section, you should be able to:

- Compute second partial derivatives of functions of two variables.
- Find and classify critical points.
- Identify local maxima, local minima, and saddle points.
- Apply the second derivative test for functions of two variables.
- Solve applied optimization problems involving business models.

Key Definitions and Concepts

Critical Point. A point (a, b) is a critical point of $f(x, y)$ if

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0,$$

or if one or both partial derivatives do not exist.

Second Partial Derivatives. The second partial derivatives of $f(x, y)$ are

$$f_{xx}, \quad f_{yy}, \quad f_{xy}, \quad f_{yx}.$$

If these derivatives are continuous, then $f_{xy} = f_{yx}$.

Discriminant. At a critical point (a, b), define

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \big(f_{xy}(a, b)\big)^2.$$

Second Derivative Test.

- If $D > 0$ and $f_{xx} > 0$, f has a **local minimum**.
- If $D > 0$ and $f_{xx} < 0$, f has a **local maximum**.
- If $D < 0$, f has a **saddle point**.
- If $D = 0$, the test is **inconclusive**.

Solved Examples

Example 1: Classifying a Critical Point

Problem. Find and classify the critical point of

$$f(x, y) = x^2 + y^2 - 4x - 6y.$$

Solution.

First partial derivatives:

$$f_x = 2x - 4, \qquad f_y = 2y - 6.$$

Setting both equal to zero gives $(x, y) = (2, 3)$.

Second partial derivatives:

$$f_{xx} = 2, \qquad f_{yy} = 2, \qquad f_{xy} = 0.$$

Thus

$$D = 2 \cdot 2 - 0 = 4 > 0 \quad \text{and} \quad f_{xx} > 0.$$

Conclusion. The function has a local minimum at $(2, 3)$.

Example 2: Saddle Point

Problem. Classify the critical point of

$$f(x, y) = x^2 - y^2.$$

Solution.

$$f_x = 2x, \qquad f_y = -2y.$$

The only critical point is $(0, 0)$.

Second partial derivatives:

$$f_{xx} = 2, \qquad f_{yy} = -2, \qquad f_{xy} = 0.$$

Thus

$$D = 2(-2) - 0 = -4 < 0.$$

Conclusion. The point $(0, 0)$ is a saddle point.

Applied Optimization

Example 3: Maximizing Revenue

Problem. A company sells two products. Revenue (in dollars) is modeled by

$$R(p_1, p_2) = 180p_1 - 4p_1^2 - 2p_1p_2 + 120p_2 - 3p_2^2.$$

Find the prices p_1 and p_2 that maximize revenue.

Solution.

First partial derivatives:

$$R_{p_1} = 180 - 8p_1 - 2p_2, \qquad R_{p_2} = 120 - 2p_1 - 6p_2.$$

Solve the system:

$$180 - 8p_1 - 2p_2 = 0, \qquad 120 - 2p_1 - 6p_2 = 0.$$

Solving yields

$$(p_1, p_2) = \left(\frac{210}{11}, \frac{150}{11}\right).$$

Second derivatives:

$$R_{p_1p_1} = -8, \quad R_{p_2p_2} = -6, \quad R_{p_1p_2} = -2.$$

$$D = (-8)(-6) - (-2)^2 = 44 > 0 \quad \text{and} \quad R_{p_1p_1} < 0.$$

Conclusion. Revenue is maximized at

$$\left(\frac{210}{11}, \frac{150}{11}\right).$$

Practice Problems

1. Find and classify all critical points of

$$f(x, y) = x^2 + y^2 - 2x + 4y.$$

2. Find and classify all critical points of

$$f(x, y) = x^2 - y^2 + 4xy.$$

3. A firm's profit function is

$$P(x,y) = 50x + 40y - x^2 - 2y^2 - xy.$$

Find the production levels x and y that maximize profit.

Section Summary

- Optimization problems in two variables rely on partial derivatives.
- Critical points occur where both first partial derivatives are zero.
- The second derivative test classifies critical points.

- Saddle points behave like maxima in one direction and minima in another.
- These techniques are essential for business decision-making.

Appendix Selected Answers to Practice Problems

Review

Functions and Graphs

1. (a). $f(5) = 2(5) + 1 = 11$.

   (b). $2x + 1 = 9 \Rightarrow 2x = 8 \Rightarrow x = 4$.
2. The domain is all real numbers except $x = -3$.
3. (a). $R(0)$ is the revenue when no units are sold (typically $0).

   (b). A decreasing $R(q)$ means revenue goes down as the number of units sold increases.
4. The graph of $y = -3$ is a horizontal line 3 units below the x-axis.

Algebraic Operations and Simplification

1. $4x - 7 + 3x = 7x - 7$.
2. $x^2 - 16 = (x - 4)(x + 4)$.
3. $\dfrac{x^2 - 5x}{x} = \dfrac{x(x-5)}{x} = x - 5, \quad x \neq 0$.
4. $3x + 4 = 19 \Rightarrow 3x = 15 \Rightarrow x = 5$.
5. Simplification helps remove removable discontinuities (such as common factors that cancel), making limits easier to evaluate and reducing algebra errors.

Exponential and Logarithmic Functions

1. $2e^3$.
2. $\ln(4x) = \ln 4 + \ln x$.
3. $e^x = 10 \Rightarrow x = \ln 10$.
4. $\ln x = 3 \Rightarrow x = e^3$.
5. The growth rate 0.02 means the population grows continuously at about 2% per year.

The Derivative

Limits and Continuity

1. (a). 380

   (b). Revenue approaches $380 as output approaches 25 units.
2. (a). 500

(b). 500

(c). Continuous at $x = 40$.

3. A limit depends on values near the point, not the value at the point.

The Derivative

1. (a). $C'(x) = 0.02x + 4$.

   (b). $C'(80) = 0.02(80) + 4 = 5.6$.

   (c). At 80 units, cost is increasing at about \$5.60 per additional unit (marginal cost).

2. (a). $R'(x) = 150 - 0.5x$.

   (b). $R'(60) = 150 - 30 = 120$.

   (c). At 60 units, revenue is increasing at about \$120 per additional unit (marginal revenue).

3. (a). Left slope: $(27 - 22)/1 = 5$. Right slope: $(31 - 27)/1 = 4$. Average: $(5 + 4)/2 = 4.5$.

   (b). At week 4, profit is increasing at about \$4.5 thousand per week (about \$4,500 per week).

Power and Sum Rules for Derivatives

1. $f'(x) = 30x^5$.

2. $g'(x) = 12x^2 - 14x$.

3. (a). $C'(x) = 0.06x^2 + 12x + 150$.

   (b). $C'(40) = 726$.

4. (a). $R'(x) = 120 - 0.6x$.

   (b). At 50 units, revenue is increasing at about \$90 per additional unit.

Product and Quotient Rules

1. $f'(x) = 4(x^2 + 3) + 4x(2x) = 12x^2 + 12$.

2. $g(x) = \dfrac{6x^2 + 50x}{x} = 6x + 50$, so $g'(x) = 6$.

3. (a). $R'(x) = 120 - 6x$.

   (b). $R'(10) = 60$; at 10 units sold, revenue is increasing at about \$60 per additional unit.

4. (a). $A'(x) = 0.05 - \dfrac{2000}{x^2}$.

   (b). At $x = 50$, $A'(50) = 0.05 - \dfrac{2000}{2500} = -0.75$, so average cost is decreasing.

Chain Rule

1. $f'(x) = 5(x^2 + 3x + 1)^4(2x + 3)$.

2. $g'(x) = \dfrac{2x + 5}{\sqrt{2x^2 + 10x}}$.

3. (a). $R'(x) = -6(100 - 2x)^2$.

(b). $R'(20) = -6(60)^2 = -21{,}600$; revenue is decreasing at about \$21,600 per additional unit.

4. (a). $P'(x) = \dfrac{x+5}{\sqrt{x^2+10x+25}}$.

   (b). $P'(25) = \dfrac{30}{\sqrt{900}} = 1$.

Second Derivative and Concavity

1. $f'(x) = 4x^3 - 24x^2 + 36x, \quad f''(x) = 12x^2 - 48x + 36$.
2. (a). $R'(x) = 9x^2 - 60x + 90, \quad R''(x) = 18x - 60$.

   (b). $R''(x) = 0$ at $x = \dfrac{10}{3}$. Concave down for $x < \dfrac{10}{3}$ and concave up for $x > \dfrac{10}{3}$.
3. (a). $C''(x) = 1.5x - 12$.

   (b). $C''(x) = 0$ at $x = 8$, so there is a point of inflection at $x = 8$.

Optimization

1. (a). $P'(x) = -2x + 120 = 0 \Rightarrow x = 60$.

   (b). $P(60) = 2700$.
2. (a). $A'(x) = 1 - \dfrac{1000}{x^2} = 0 \Rightarrow x = \sqrt{1000} \approx 31.6$.

   (b). Average cost is minimized at about 32 units.
3. (a). $R'(x) = -6x + 180 = 0 \Rightarrow x = 30$.

   (b). $R(30) = 2700$.
4. (a). $P(x) = 80x - (x^2 + 40x + 600) = -x^2 + 40x - 600$.

   (b). $P'(x) = -2x + 40 = 0 \Rightarrow x = 20$.

Curve Sketching

1. (a). $f'(x) = 3x^2 - 18x + 24 = 3(x-2)(x-4)$, so the critical points are $x = 2$ and $x = 4$.

   (b). Increasing on $(2, 4)$; decreasing on $(-\infty, 2)$ and $(4, \infty)$.

   (c). $f''(x) = 6x - 18$, so $f''(x) = 0$ at $x = 3$ (inflection point at $x = 3$).
2. (a). $R'(x) = -3x^2 + 30x - 50 = 0 \Rightarrow x = 5 \pm \dfrac{5\sqrt{3}}{3}$.

   (b). Revenue is increasing on $\left(5 - \dfrac{5\sqrt{3}}{3}, 5 + \dfrac{5\sqrt{3}}{3}\right)$ and decreasing outside this interval.

   (c). $R''(x) = -6x + 30$. Concave up for $x < 5$ and concave down for $x > 5$.
3. (a). $C'(x) = 4x^3 - 12x^2 + 12x = 4x(x-1)(x-3), \quad C''(x) = 12x^2 - 24x + 12 = 12(x-1)^2$.

   (b). Since $C''(x) = 12(x-1)^2 \geq 0$ for all x, C is concave up for all x (no inflection point).

Applied Optimization

1. (a). Let x be units of Product X. Then $120 - x$ units of Product Y:

$$P(x) = 40x + 25(120 - x) = 15x + 3000.$$

(b). Since $P(x)$ is increasing, profit is maximized at $x = 120$. Produce 120 units of Product X and 0 units of Product Y.

2. (a). Let x be the width and y the length:

$$2x + y = 200 \Rightarrow y = 200 - 2x, \quad A(x) = xy = x(200 - 2x).$$

(b).

$$A'(x) = 200 - 4x = 0 \Rightarrow x = 50, \quad y = 100.$$

3. (a). Revenue as a function of price:

$$R(p) = pq = p(500 - 5p) = 500p - 5p^2.$$

(b).

$$R'(p) = 500 - 10p = 0 \Rightarrow p = 50.$$

Implicit Differentiation and Related Rates

1. Differentiate:

$$x^2 + y^2 + 4xy = 16$$

$$2x + 2y\frac{dy}{dx} + 4\left(x\frac{dy}{dx} + y\right) = 0$$

$$(2y + 4x)\frac{dy}{dx} = -(2x + 4y) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x + 2y}{y + 2x}.$$

2. $A = lw \Rightarrow \dfrac{dA}{dt} = l\dfrac{dw}{dt} + w\dfrac{dl}{dt}$. At $l = 20$, $w = 10$, $\dfrac{dl}{dt} = 3$, $\dfrac{dw}{dt} = -1$:

$$\frac{dA}{dt} = 20(-1) + 10(3) = 10 \text{ m}^2/\text{month}.$$

3. $V = pq \Rightarrow \dfrac{dV}{dt} = p\dfrac{dq}{dt} + q\dfrac{dp}{dt}$. At $p = 25$, $q = 400$, $\dfrac{dp}{dt} = 1$, $\dfrac{dq}{dt} = -30$:

$$\frac{dV}{dt} = 25(-30) + 400(1) = -350 \text{ dollars/month}.$$

The Integral

The Definite Integral

1. (a). The integral represents the total change in profit as production increases from 10 units to 40 units.

(b). A negative value indicates that overall profit decreases (a net loss) over that production range.

2. (a). The integral represents the total change in cost as production increases from $x = a$ to $x = b$.

   (b). It gives the accumulated cost incurred over that interval of production.

3. The integral represents the net change in inventory over the 8-week period.

The Fundamental Theorem of Calculus and Antidifferentiation

1. $\int_1^4 (2x^2 - x)\,dx = \left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]_1^4 = \left(\frac{128}{3} - 8\right) - \left(\frac{2}{3} - \frac{1}{2}\right) = 30.$
2. $\int_5^{20} (6x + 10)\,dx = \left[3x^2 + 10x\right]_5^{20} = (1200 + 200) - (75 + 50) = 1725.$
3. $\int_{10}^{30} (50 - 2x)\,dx = \left[50x - x^2\right]_{10}^{30} = (1500 - 900) - (500 - 100) = 600.$

Antiderivatives of Formulas

1. $\int (6x^2 - 4x + 9)\,dx = 2x^3 - 2x^2 + 9x + C.$
2. $\int (2e^x - 3 \cdot 4^x)\,dx = 2e^x - \dfrac{3 \cdot 4^x}{\ln 4} + C.$
3. $\int \dfrac{5}{x}\,dx = 5 \ln |x| + C.$
4. $\int (x^{-3} + 2x^{1/2})\,dx = -\dfrac{1}{2}x^{-2} + \dfrac{4}{3}x^{3/2} + C.$

Substitution

1. $\int x(2x^2 + 5)^3\,dx = \dfrac{(2x^2 + 5)^4}{8} + C.$
2. $\int e^{4x+1}\,dx = \dfrac{1}{4}e^{4x+1} + C.$
3. $\int \dfrac{6x}{x^2 + 4}\,dx = 3 \ln(x^2 + 4) + C.$

Additional Integration Techniques

1. $\int x \ln x\,dx = \dfrac{x^2}{2} \ln x - \dfrac{x^2}{4} + C.$
2. $\int_1^e \ln x\,dx = 1.$
3. $\int \dfrac{1}{x^2 - 16}\,dx = \dfrac{1}{8} \ln \left|\dfrac{x - 4}{x + 4}\right| + C.$

Area, Volume, and Average Value

1. $\int_0^3 4x\,dx = 18.$
2. $\int_0^2 (2x - x^2)\,dx = \dfrac{4}{3}.$
3. $V = \int_0^1 \pi (x^2)^2\,dx = \pi \int_0^1 x^4\,dx = \dfrac{\pi}{5}.$
4. $f_{\text{avg}} = \dfrac{1}{3 - 1} \int_1^3 (x^2 + 2)\,dx = \dfrac{13}{3}.$

Applications to Business

1. (a). $q^* = 25,\ p^* = 35$.

   (b). CS $= \int_0^{25} (60 - q)\, dq - 35(25) = \dfrac{625}{2}$.

   (c). PS $= 35(25) - \int_0^{25} (q + 10)\, dq = \dfrac{625}{2}$.

2. (a). PV $= \int_0^{5} 1000e^{-0.06t}\, dt = \dfrac{1000}{0.06}\left(1 - e^{-0.3}\right)$.

   (b). FV $=$ PV $e^{0.06\cdot 5} = \dfrac{1000}{0.06}\left(e^{0.3} - 1\right)$.

Differential Equations

1. $B'(t) = 0.03B(t) - 15$.
2. Not a solution.
3. $y^2 = 4x^2 + C$.
4. $\dfrac{dy}{dt} = 0.05y, \quad y = Ae^{0.05t}$.
5. $\dfrac{dy}{dt} = 0.02(10000 - y)$.

Functions of Two Variables

Functions of Two Variables

1. $R(q, p)$ represents the revenue generated by selling q units at price p.
2. (a). $C(4, 10) = 5(4) + 2(10) = 40$.

   (b). The total cost for $x = 4$ and $y = 10$ is \$40.
3. $C(2, 5) = 3(2) + 4(5) = 26$ and $C(5, 2) = 3(5) + 4(2) = 23$; they are not equal because the order of inputs changes the result.
4. The graph of $z = -1$ is a plane parallel to the xy-plane, 1 unit below it.
5. $\sqrt{(5-2)^2 + (4-0)^2 + (5-1)^2} = \sqrt{34}$.

Calculus of Functions of Two Variables

1. (a). $f_x = 2x, \quad f_y = 6y$.

   (b). $f_x(2, 1) = 4, \quad f_y(2, 1) = 6$.
2. $C_y(x, y)$ measures the rate at which cost changes as materials y change, holding labor x constant.
3. $g_x = \dfrac{1}{3}e^{x+y}, \quad g_y = \dfrac{1}{3}e^{x+y} + \dfrac{1}{y}$.
4. $f(2.1, 2.9) \approx f(2, 3) + 1.2(0.1) + (-0.5)(-0.1) = f(2, 3) + 0.07$.
5. Partial derivatives are useful because they approximate how the output changes for small changes in one input while the other input is held fixed, enabling quick estimation and sensitivity analysis.

Optimization

1. Critical point $(1, -2)$.

   $f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$ gives $D = 2 \cdot 2 - 0 = 4 > 0$ and $f_{xx} > 0$, so a local minimum.
2. Critical point $(0, 0)$.

   $f_{xx} = 2,\ f_{yy} = -2,\ f_{xy} = 4$ gives $D = 2(-2) - 4^2 = -16 < 0$, so a saddle point.
3. The profit function is maximized at $(x, y) = (12, 7)$.

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