PERMUTATIONS AND COMBINATIONS

Math 1001

Quantitative Skills and Reasoning

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PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- Up to this point we have been counting the number of permutations of *distinct* objects.
- We now look at the situation of arranging objects when some of them are identical.
- In the case of identical or indistinguishable objects, a modification of the permutation formula is necessary.
- The general idea for modifying the formula is to count the number of permutations as if all the objects were distinct, and then remove the permutations that are not different in appearance.

PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- Consider the permutations of the letters aaabb.
- We must first assume the letters are all different – for example:
  - $a_1 a_2 a_3 b_1 b_2$.
- Using the permutation formula, there are 5! = 120 permutations.
- We need to remove repeated permutations.

PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- Consider for a moment the letter *a* in the permutation aaab_1b_2. If the *a*'s are written as a_1, a_2, and a_3 (so that they are distinguishable), then

$$a_1a_2a_3b_1b_2 \quad a_1a_3a_2b_1b_2 \quad a_2a_1a_3b_1b_2$$
$$a_2a_3a_1b_1b_2 \quad a_3a_2a_1b_1b_2 \quad a_3a_1a_2b_1b_2$$

are all distinct permutations that end with b_1b_2.

- There are six of these permutations
- Note that 3! = 6, where 3 is the number of *a*'s.

PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- However, because the a's are not distinct, each of these permutations should have been counted only once.
- Thus, there are six times too may permutations of a's for each arrangement of b_1 and b_2.

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PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- A similar argument applies to the *b*'s. If the *b*'s are identical, then $b_1 b_2 aaa$ and $b_2 b_1 aaa$ are the same permutation.
- For each arrangement of *a*'s, there are two arrangements of *b*'s that yield identical permutations.

- Note that there are two *b*'s and that 2! = 2, the number of identical permutations.
- Thus, there are two times too many permutations of *b*'s for each arrangement of *a*'s.

PERMUTATIONS OF INDISTINGUISHABLE OBJECTS

- Combining the results above, the number of permutations of *aaabb* is

$$\frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{(3 \cdot 2 \cdot 1)} \cdot (2 \cdot 1)} = \frac{20}{2} = 10$$

- There are 10 distinct permutations of *aaabb*.

PERMUTATIONS OF OBJECTS, SOME IDENTICAL

The number of distinguishable permutations of *n* objects of *r* different types, where k_1 identical objects are of one type, k_2 of another, and so on, is given by

$$\frac{n!}{k_1! \cdot k_2! \cdot \cdots \cdot k_r!}$$

where $k_1 + k_2 + \cdots + k_r = n.$

PERMUTATIONS OF IDENTICAL OBJECTS

- A password requires 7 characters. If a person who lives at 177 Moon Road wants a password to be an arrangement of the characters 177MOON, how many different passwords are possible?

- We are looking for the number of permutations of the characters 177MOON, with *n* = 7, (the number of characters), $k_1 = 1$ (the number of 1s), $k_2 = 2$ (the number of 7s), $k_3 = 1$ (the number of Ms), k_4=2 (the number of Os) and $k_5 = 1$ (the number of Ns).

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PERMUTATIONS OF IDENTICAL OBJECTS

- We are looking for the number of permutations of the characters 177MOON, with *n* = 7, (the number of characters), $k_1 = 1$ (the number of 1s), $k_2 = 2$ (the number of 7s), $k_3 = 1$ (the number of Ms), k_4=2 (the number of Os) and $k_5 = 1$ (the number of Ns).

- We have

$$\frac{7!}{1! \cdot 2! \cdot 1! \cdot 2! \cdot 1!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2!}}{2 \cdot \cancel{2!}} = \frac{2520}{2} = 1260$$

- There are 1260 possible passwords.

COMBINATIONS

- For some arrangements of objects, the order of the arrangement is important. These are permutations.
  - For example, if a telephone extension is 1357, then the digits must be dialed in exactly that order.
- On the other hand, order does not matter in many situations.
  - For example, if you were to receive a $1 bill, a $5 bill, and a $20 bill, you would have $26 regardless of the order in which you received the bills.

COMBINATIONS

- A **combination** is a collection of objects for which the order is not important.
  - The three-letter sequences xyz and zyx are *different* permutations, but the *same* combination.

- If the particular order of the objects makes a difference, we are talking about permutations.
- Additionally, there are more permutations of objects than there are combinations.

COMBINATIONS

- From a group of 40 applicants, five identical scholarships will be awarded .
- Is the number of ways in which the scholarships can be awarded determined by permutations or combinations?
- Answer: Combinations because the order in which the scholarship winners are chosen is not important.

COMBINATIONS

- The formula for finding the number of combinations is derived in much the same manner as the formula for finding the number of permutations of identical objects.
- Consider the problem of finding the number of combinations possible when choosing three letters from the letters a, b, c, d, and e, without replacement.

- For each choice of three letters, there are 3! permutations.
- For example, choosing the letters a, d, and e gives the following six permutations:

ade aed dae dea eda ead

COMBINATIONS

- Because there are six permutations and each permutation is the *same* combination, the number of permutations is six times the number of combinations.
- This is true for each time three letters are selected.
- Therefore, to find the number of combinations of five objects chosen three at a time, divide the number of permutations by 3! = 6.
- The number of combinations of five objects chosen three at a time is

$$\frac{P(5,3)}{3!} = \frac{5!}{3! \cdot (5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!} = \frac{20}{2} = 10.$$

- There are 10 combinations of 5 objects chosen 3 at a time.

COMBINATION FORMULA

- The number of combinations of *n* objects chosen *k* at a time is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$\binom{n}{k}$ is read as "*n* choose *k*," but your book uses C(*n*, *k*) to denote the same concept.

COUNTING USING THE COMBINATION FORMULA

- An emergency room at a hospital has 10 nurses on staff. Each night, a team of 6 nurses is on duty.
- In how many different ways can the team of 6 nurses be chosen?

$$C(10,6) = \frac{10!}{6!\,(10-6)!} = \frac{10!}{6!\cdot 4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 4!}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$$

- There are 210 possible teams of 6 nurses.

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COUNTING USING THE COMBINATION FORMULA AND THE COUNTING PRINCIPLE

- A committee of 5 is chosen from 5 mathematicians and 6 economists. How many different committees are possible if the committee must include 3 mathematicians and 2 economists?

- Because a committee of professors A, B, C, D, and E is exactly the same as a committee of professors B, D, C, E, and A, choosing a committee is an example of choosing a combination.
- There are 5 mathematicians from whom 3 are chosen, which is equivalent to $C(5, 3)$ combinations.
- There are 6 economists from which 2 are chosen, which is equivalent to $C(6, 2)$ combinations.
- Therefore, by the counting principle, there are $C(5, 3) \cdot C(6, 2)$ ways to choose 3 mathematicians and 2 economists.

COUNTING USING THE COMBINATION FORMULA AND THE COUNTING PRINCIPLE

- A committee of 5 is chosen from 5 mathematicians and 6 economists. How many different committees are possible if the committee must include 3 mathematicians and 2 economists?

$$C(5,3)\cdot C(6,2)=\frac{5!}{3!\,(5-3)!}\cdot\frac{6!}{2!\,(6-2)!}=\frac{5!\cdot 6!}{3!\cdot 2!\cdot 2!\cdot 4!}=\frac{5\cdot \cancel{4!}\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot 2!\cdot 2!\cdot \cancel{4!}}$$

$$=\frac{5\cdot 6\cdot 5\cdot 4}{2\cdot 1\cdot 2\cdot 1}=150$$

- There are 150 possible committees consisting of 3 mathematicians and 2 economists.

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