PERMUTATIONS AND COMBINATIONS

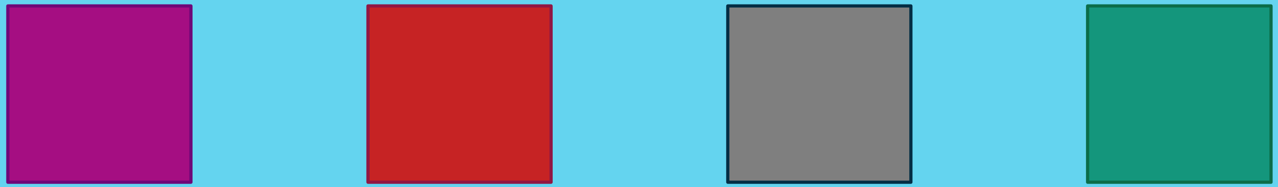
Math 1001

Quantitative Skills and Reasoning

Columbus State University

FACTORIAL

- Suppose four different colored squares are arranged in a row, as below:



- How many different ways are there to order the colors?

- There are four choices for the first square, three choices for the second square, two choices for the third square, and only one choice for the fourth square.
- By the counting principle, there are 4(3)(2)(1) = 24 different arrangements of the four squares.
- Note from this example that the number of arrangements equals the product of the natural numbers *n* through 1, where *n* is the number of objects.
- This product is called the *factorial*.

FACTORIAL

- ***n* factorial** is the product of the natural numbers *n* through 1 is symbolized by *n*!.
  - $n! = n \cdot (n-1) \cdot (n-2) \cdot \ \ldots \ \cdot 3 \cdot 2 \cdot 1$

- Some examples:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

$$1! = 1$$

FACTORIAL

- On some occasions it will be necessary to use 0! (zero factorial).
- Because it is impossible to define zero factorial in terms of a product of natural numbers, a standard definition is used.

$$0! = 1$$

FACTORIAL

- A factorial can be written in terms of smaller factorials.
- This is useful when calculating large factorials.
  - $100! = 100 \cdot 99!$
  - $100! = 100 \cdot 99 \cdot 98!$
  - $100! = 100 \cdot 99 \cdot 98 \cdot 97!$
  - etc…

SIMPLIFY FACTORIALS

- Evaluate:
  - 5! – 2!

$$5! - 2! = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - (2 \cdot 1) = 120 - 2$$
$$= 118$$

  - $\dfrac{10!}{6!}$

$$\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}} = 10 \cdot 9 \cdot 8 \cdot 7$$
$$= 5040$$

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PERMUTATIONS

- Determining the number of possible arrangements of distinct objects in a definite order, as we did with the squares earlier, is one application of the counting principle.

- Each arrangement of this type is called a *permutation*.

PERMUTATIONS

- For example, xyz and zyx are two different permutations of the letters x, y, and z.
- As a second example, 133 and 313 are two different permutations of one 1 and two 3s.

- The counting principle is used to count the number of different permutations of any set of objects.
- We will begin our discussion with *distinct* objects.
  - 1, 2, 3, 4 are distinct.
  - ● ● ♦ ⌘ are not all distinct.

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PERMUTATIONS

- Suppose that there are two songs, *Song-I* and *Song-II*, in a playlist, and you wish to play both on your music player.
- There are two ways to choose the first song, and only one way to choose the second.
- By the counting principle, there are 2! = 2 permutations (or orders) in which you can play the two songs.
- Each permutation gives a different playlist:

Permutation (playlist) 1	Permutation (playlist) 2
Song-I	***Song-II***
Song-II	***Song-I***

PERMUTATIONS

- With three songs in a playlist, *Song-I, Song-II* , and *Song-III* in a playlist, there are three choices for the first song, two choices for the second, and one choice for the third.
- By the counting principle, there are $3(2)(1) = 3! = 6$ permutations in which you can play three songs.

Permutation 1	Permutation 2	Permutation 3	Permutation 4	Permutation 5	Permutation 6
Song-I	*Song-I*	*Song-II*	*Song-II*	*Song-III*	*Song-III*
Song-II	*Song-III*	*Song-III*	*Song-I*	*Song-II*	*Song-I*
Song-III	*Song-II*	*Song-I*	*Song-III*	*Song-I*	*Song-II*

PERMUTATIONS

- With four songs in a playlist, there are 4(3)(2)(1) = 4! = 24 orders in which you could play the songs.
- In general, if there are *n* songs in a playlist, there are *n*! permutations, or orders, in which the songs could be played.

- Suppose now that you have a playlist that consists of six songs but you have time to listen to only three of the songs.
- You could choose any one of the six songs to play first, then any one of the five remaining songs to play second, and then any one of the remaining four songs to play third.
- By the counting principle, there are 6(5)(4) = 120 permutations in which the songs could be played.

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PERMUTATION FORMULA FOR DISTINCT OBJECTS

- The following formula can be used to determine the number of permutations of *n* distinct objects (in the example above, songs), of which *k* are selected.
- The number of permutations of *n* distinct objects selected *k* at a time is

$$P(n,k) = \frac{n!}{(n-k)!}$$

PERMUTATIONS FOR DISTINCT OBJECTS

- Applying this formula to the situation with 6 songs where we only listen to 3, we have n = 6 and k = 3.
- $P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 6 \cdot 5 \cdot 4 = 120$
- There are 120 permutations of playing the songs.
- This is the same answer we obtained using the counting principle.

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COUNTING PERMUTATIONS

- A university tennis team consists of six players who are ranked from 1 through 6.
- If a tennis coach has 12 players from which to choose for these six positions, how many different tennis teams can the coach select?

- Because the players on the tennis team are ranked from 1 through 6, a team with player A in position 1 is different from a team with player A in position 2.
- Therefore, the number of different teams is the number of permutations of 12 players selected six at a time.
- $P(12,6) = \frac{12!}{(12-6)!} = \frac{12!}{6!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}} = 12(11)(10)(9)(8)(7) = 665{,}280$
- There are 665,280 possible tennis teams.

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APPLYING SEVERAL COUNTING TECHNIQUES

- The permutation formula is derived from the counting principle.
- This formula is just a convenient way to express the number of ways the items in an ordered list can be arranged.
- Some counting problems require both the permutation formula and the counting principle.

COUNTING USING SEVERAL METHODS

- Six women and five men are to be seated in a row of eleven chairs.
- How many seating arrangements are possible if
  - **there are no restrictions on the seating arrangements?**
  - the women sit together and the men sit together?

- Because seating arrangements have a definite order, they are permutations.
- If there are no restrictions on the seating arrangements, then the number of seating arrangements is $P(11, 11)$.
- $P(11,11) = \frac{11!}{(11-11)!} = \frac{11!}{0!} = \frac{11!}{1} = 11! = 39916800$
- There are 39916800 seating arrangements.

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COUNTING USING SEVERAL METHODS

- Six women and five men are to be seated in a row of eleven chairs.
- How many seating arrangements are possible if
  - there are no restrictions on the seating arrangements?
  - **the women sit together and the men sit together?**

- This is a multi-stage experiment, so both the permutation formula and the counting principle will be used.
- There are 6! ways to arrange the women and 5! ways to arrange the men.
- We must also consider that either the women or the men could be seated at the beginning of the row – there are 2 ways to do this.
- By the counting principle, there are $2 \cdot 6! \cdot 5! = 2(6)(5)(4)(3)(2)(1)(5)(4)(3)(2)(1) = 172{,}800$
- There are 172,800 arrangements in which the women sit together and the men sit together.

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