QUANTITATIVE SKILLS & REASONING

For MATH 1001

at The University of West Georgia

First Edition 2021

Contents

Introduction

Welcome to Quantitative Skills and Reasoning! Just what are quantitative skills and reasoning? The simple answer is working with numbers, making sense of data, and using your brain to figure things out. This particular class at UWG will cover fundamental concepts from problem solving, statistics, probability, graphs, logic, sets, measurements, and finance.

Memorizing formulas and just being able to do problems are not enough to truly be successful in math classes or college. In order to succeed in math, students must fully understand the concepts, be able to apply them to real world problem solving, and be able to make sense of the results. Here we will focus on not just the what and when, but also the why, how, and where is it used. Most students who do not succeed in math or dislike math, probably were told: "Just do it the way I told you, get through it quickly and move on. Don't worry about understanding it". I have seen many students with a fear and dislike of math, who have succeeded and actually started to like it, once they are shown the "big picture" and how it all comes together.

The main excuse students give for not trying is "I won't ever need to know this or use this, so why should I learn this?" The benefit of learning mathematics is not necessarily to obtain particular knowledge about certain math topics. Retaining the knowledge is important, if you have to go on to the next level. However, even for those who just need to pass one core math course, there is still plenty to gain. Learning mathematics trains your brain to think logically and develops analytical skills, which can be used for the rest of your life. Some examples include doing your taxes, running a business, managing people or projects, building a tree-house for your kids, etc.

Think of it this way: a professional basketball player is mainly concerned with getting the ball into a basket, so why does he lift weights (the ball is not heavy), run laps (the court is only 94 feet long), and analyze film. Because these things help train him for being better at what he wants to do. The same can be said about learning math. It helps train your brain to be better at dealing with the real world, life, careers, etc. So even if you do not care to know the math itself, learning math can be a vehicle for increasing your brainpower and critical thinking skills.

Here are some tips on how to use this book and what else you can do to succeed in this course. These ideas are not new discoveries and can be applied to almost any course.

- Read each section of the book BEFORE you cover it in class. Let your brain mull it over so class time will seem like review or at least let you get a better grip on the material.
- TAKE GOOD NOTES!! I have noticed that many students (especially home-schoolers) do not take notes. Just because you have a textbook, does not mean that you should not write out your thoughts in your own words.
- Do ALL of the exercises, practice makes perfect (or at least closer to perfect).
- Work the problems and think them over BEFORE you look at the solutions.
- Start working on assignments and studying early, this way you avoid "something else came up, I couldn't finish".
- Many test problems will be similar to exercises, but some will combine topics and/or be longer. Prepare very thoroughly for tests.
- Don't be afraid to ask questions. Most teachers/tutors take them seriously. Questions help them to assess where you are BEFORE you're tested.
- Search the internet for one of the many FREE online math help sites.

Chapter 1

Problem Solving

Written by Robert Burnham

1.1 Inductive and Deductive Reasoning

1.1.1 Inductive Reasoning

Definition: Inductive Reasoning is the making of generalizations based on observed examples.

Example: The last four times I have driven downtown at 6pm there has been traffic. My conclusion is that there is always traffic downtown around 6pm.

Notice that I am making a generalization based off of my experiences. I am using examples to make a generalization. The four previous times I drove downtown around 6pm there was traffic. Since I have had the same experience all four times I come to the conclusion that there must always be traffic downtown at 6pm. This is inductive reasoning.

Let's consider another example.

Example: The past 3 times I have gone to an Atlanta Braves game they have lost. The Braves only lose games that I go to.

Again, I am making a generalization based off of my experiences. I am using examples to make this generalization. The three previous times I went to a Braves game, they lost. Since I have had the same experience every time I come to the conclusion that the Braves only lose when I attend the games. Once again, this is inductive reasoning.

Notice that with both of these examples I am making a generalization based off of a few examples. This does not necessarily mean that my generalizations are true, however, it gives evidence to support my hypotheses. For instance, I can check the Atlanta Braves website and see that the Braves have lost other games that I have not attended. This shows that my generalization is incorrect.

Now I want to encourage caution. Be aware that we can use inductive reasoning and make a generalization and be incorrect. Therefore we call generalizations made from using inductive reasoning **hypotheses**, because there is no guarantee that they are true or false. If we believe that a hypothesis is true and we want to show it, then we must prove that it holds for every possible case. If we believe that a hypothesis is false, then we can prove it by providing a **counterexample**. A counterexample is a case where the hypothesis does not hold.

1.1.2 Counterexamples

Example: Consider the following hypothesis,

The sum of two three-digit numbers is a four-digit number.

Identify a counterexample to show that it is false.

Solution: Consider the two examples of the sum of two three-digit numbers

$$\begin{array}{r} 472 \\ +731 \\ \hline 1203 \end{array} \qquad \begin{array}{r} 825 \\ +634 \\ \hline 1459 \end{array}$$

For each of these examples we have that the sum gives a four digit number. Is this always the case? The counterexample below shows that the statement is false.

$$\begin{array}{r} 340 \\ +121 \\ \hline 461 \end{array}$$

This is a three digit number. This shows that the hypothesis is not true for the sum of two three-digit numbers in general.

Example: Consider the following hypothesis,

All prime numbers are odd.

Identify a counterexample to show that it is false. ***Note*** A prime number is a number that has exactly two factors: 1 and the number itself.

Solution: To solve this problem we will need to know the set of primes. The primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

We can see right away that there is a prime that is not odd and it is 2. This gives a counterexample to the general statement and shows that the hypothesis is not in general true.

Numerical Sequences

When completing numerical sequences we also use inductive reasoning. We will look at earlier terms in the sequence, identify a pattern, and use it to find the other members of the sequence.

Example: Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8
Value	1	3	9	27	81			

Solution: With this problem we see that the pattern to get the next number in the sequence is to multiply the previous term in the sequence by 3. So to find the 6th, 7th, and 8th terms in the sequence we will use this pattern. The 5th term is 81. The 6th term is $3 \times 81 = 243$, the 7th term is $3 \times 243 = 729$, and the 8th term is $3 \times 729 = 2187$.

Example: Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8
Value	58	46	34	22	10			

Solution: With this problem we see that the pattern to get the next number in the sequence is to subtract 12 from the previous term in the sequence. To find the 6th, 7th, and 8th terms in the sequence we will use this pattern. The 5th term is 10. The 6th term is $10-12=-2$, the 7th term is $-2-12=-14$, and the 8th term is $-14-12=-26$.

Example: Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8	9	10
Value	5	10	30	120	240	720	2880			

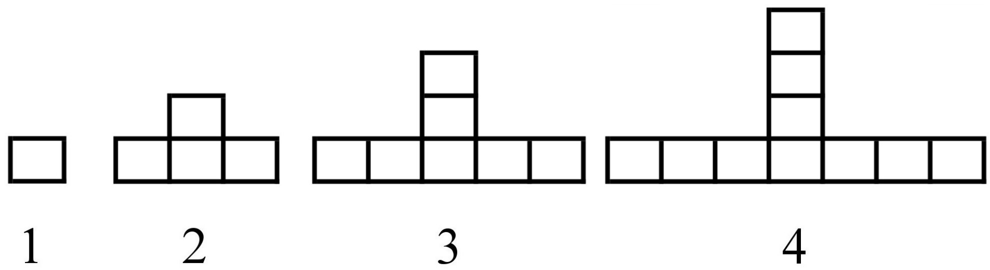
Solution: With this sequence we see to go from 5 to 10 we multiply by 2. To go from 10 to 30 we multiply by 3. To go form 30 to 120 we multiply by 4. Then we see that this pattern repeats to get the next three terms in the sequence. $2 \times 120 = 240$, $3 \times 240 = 720$, and $4 \times 720 = 2880$. So we will use this same pattern to get the 8th, 9th, and 10th terms. The 8th term is $2 \times 2880 = 5760$, the 9th term is $3 \times 5760 = 17280$, and the 10th term is $4 \times 17280 = 69120$.

****Try this on your own**: Describe the pattern found in the following sequence of numbers and then find in the next two values: 80, -40, 20, -10.

Visual Sequences

Our next examples deal with recognizing visual sequences.

Example: In the figure below, the first four terms of a visual sequence are given. If this pattern continues, how many boxes will be needed to make the 13th term of the sequence?



Solution: For term 1 we have 1 box. For term 2 we have 4 boxes. For term 3 we have 7 boxes. For term 4 we have 10 boxes. We can generalize this into a linear equation because to get the next term in the sequence we add the same number of boxes. In this case we add 3 boxes to the previous term to get the next term. We can use these relationships between term number and number of boxes to make ordered pairs where x is the term number and y is number of boxes. Two ordered pairs we can use are $(1, 1)$ and $(2, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

Then we can use the point-slope formula for the equation of a line

$$y - y_1 = m(x - x_1)$$

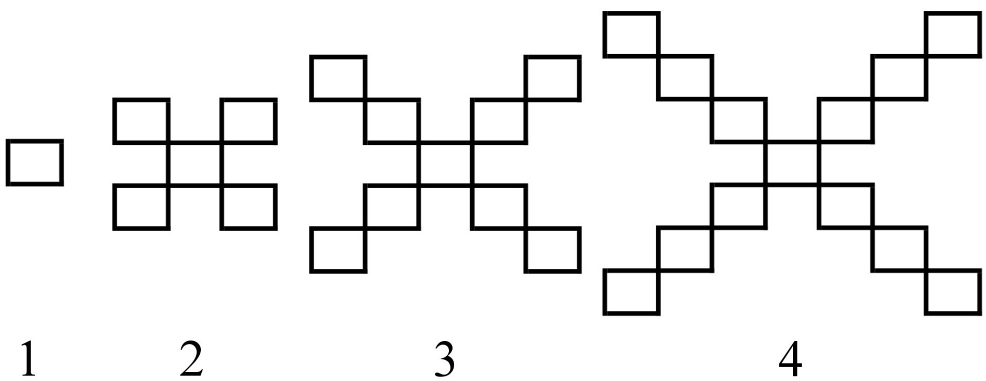
where the slope is $m = 3$ and we can use the point $(1, 1)$. This gives

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 3(x - 1) \\ y - 1 &= 3x - 3 \\ y &= 3x - 2 \end{aligned}$$

We can use this formula to find the number of boxes that will be needed for the 13th term. To do this plug in 13 for x and compute. The output will be the number of boxes needed for the 13th term.

$$y = 3(13) - 3 = 39 - 3 = 36$$

Example: In the figure below, the first four terms of a visual sequence are given.



If this pattern continues, how many boxes will be needed to make the 20th term of the sequence?

Solution: For term 1 we have 1 box. For term 2 we have 5 boxes. For term 3 we have 9 boxes. For term 4 we have 13 boxes. Just as in the previous example, we can generalize this into a linear equation because to get the next term in the sequence we add the same number of boxes. In this case it is 4 boxes. We can use the relationship between term number and number of boxes to make ordered pairs where x is the term number and y is number of boxes. Two ordered pairs we can use are $(1, 1)$ and $(2, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{2 - 1} = \frac{4}{1} = 4$$

Then we can use the point-slope formula for the equation of a line

$$y - y_1 = m(x - x_1)$$

where the slope is $m = 4$ and we can use the point $(1, 1)$. This gives

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 4(x - 1) \\ y - 1 &= 4x - 4 \\ y &= 4x - 3 \end{aligned}$$

We can use this formula to find the number of boxes that will be needed for the 20th term. To do this plug in 20 for x and compute. The output will be the number of boxes needed for the 20th term.

$$y = 4(20) - 3 = 80 - 3 = 77$$

1.1.3 Deductive Reasoning

Definition: Deductive Reasoning is the process of applying a given generalization (or rule) to make conclusions about specific examples.

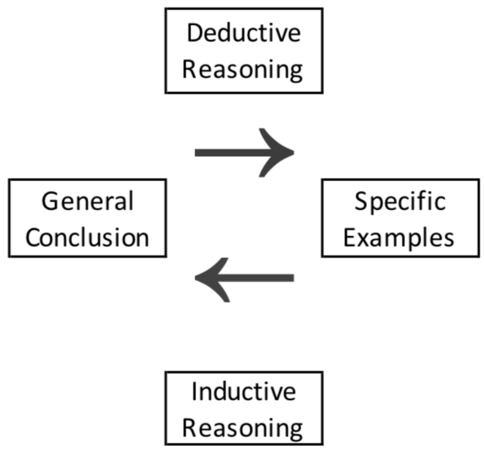
Example: Consider the statement and determine if it is inductive or deductive:

" Every month has 30 days in it. July is a month. Therefore it has 30 days in it. "

Solution: This statement starts with a generalization and it's then applied to a specific case. This follows the pattern of deductive reasoning. The statements are not necessarily true, but if every month had 30 days in it, then it would be true.

Inductive Reasoning vs Deductive Reasoning

There are two types of reasonings that we need to be able to recognize and use. They are Inductive Reasoning and Deductive Reasoning. A good question is what identifies these types of reasonings?



- For Inductive Reasoning we start with examples or cases, and then draw general conclusions.

- For Deductive Reasoning we start with a general statement and apply it to examples or cases.

Example: Consider the statement and determine if it is inductive or deductive:

" Pizza Hut has a lunch buffet. Stevi B's has a lunch buffet. Therefore all pizza restaurants have a lunch buffet."

Solution: This statement starts with two examples about pizza restaurants having lunch buffets. Based on these examples a generalization is made. This follows the pattern of inductive reasoning.

Example: Consider the statement and determine if it is inductive or deductive:

" All pro wrestlers have a catch phrase. Macho Man Randy Savage was a pro wrestler. Therefore he had a catch phrase. ”

Solution: This statement starts with a generalization about pro wrestlers having catch phrases. It's then applied to the specific case of Macho Man Randy Savage. This follows the pattern of deductive reasoning.

1.1.4 Exercises: Reasoning

Solutions appear at the end of this textbook.

1. Is the following statement inductive or deductive reasoning? Explain why. " All Noble prize winners get a monetary award. Jennifer Doudna won a Noble Prize, so she must have received money."
2. Is the following statement inductive or deductive reasoning? Explain why. " My friend and my brother graduated from Harvard and immediately got great jobs. Therefore, everyone who graduates from Harvard will immediately get a great job. "
3. Find a counter example to disprove the hypothesis: If two even numbers are divided, the quotient is a whole number.
4. Find a counter example to disprove the hypothesis: If a number is added to itself, the sum is greater than the original number.
5. Describe the pattern found in the following sequence of numbers and then find in the next two values: 1, 2, 4, 7, 11, 16.
6. Describe the pattern found in the following sequence of days and then find in the next two values: Monday, Thursday, Sunday, Wednesday, Saturday.

1.2 Estimation

Definition: Estimation is the process of approximating a rough answer to a question using the best available information.

1.2.1 Rounding Whole Numbers

The set of **Whole Numbers** is

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots.$$

We call the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9 **digits**. Numbers are comprised of digits. Where a digit is in a number, indicates its place value. For instance, the number 243 is made up of the digits 2, 3, and 4. Their placement in the number tells us that we have 2 hundreds, 4 tens, and 3 ones. For us to round correctly we will need to be more familiar with the names of the place values.

Example: Here is a whole number with its place values labelled.

Billions	Hundred-Millions	Ten-Millions	Millions	Hundred-Thousands	Ten-Thousands	Thousands	Hundreds	Tens	Ones
3,	7	4	5,	2	9	8,	1	6	4

How to Round Whole Numbers

1. Locate the digit where rounding is to occur then observe the digit to its right.

2. **a.** If the digit to the right is less than 5, then round down. Leave the digit to be rounded as it is and change all the digits to its right to zeros.

   b. If the digit to the right is greater than or equal to 5, then round up. Increase the digit to be rounded by 1 and change all the digits to its right to zeros.

Example: Round $3,745,298,164$

a. to the nearest hundred-million

Solution: Begin by underlining the hundred-millions digit and look at the digit to its right.

$$3,\underline{7}\underset{\Uparrow}{4}5,298,164$$

The digit to the right of the hundred-millions place is 4. This is less than 5 so we round down. This means we leave all the digits at the hundred-millions place and higher the same and the lower digits will be changed to 0's. Our answer is $3,700,000,000$.

b. to the nearest hundred

Solution: Begin by underlining the hundreds digit and look at the digit to its right.

$$3,745,298,\underline{1}\underset{\Uparrow}{6}4$$

The digit to the right of the hundreds place is 6. This is greater than 5, so we round up. This means we will add 1 to the digit in the hundreds place and all the digits to the left of the hundreds place will remain the same and the lower digits will be changed to 0's. Our answer is $3,745,298,200$

c. to the nearest ten-thousand

Solution: Begin by underlining the ten-thousands digit and look at the digit to its right.

$$3,745,2\underline{9}\underset{\Uparrow}{8},164$$

The digit to the right of the ten-thousands place is 8. This is greater than 5, so we round up. This means we add 1 to the digit in the ten-thousands place. This makes the 9 in this position 10 so we add 1 to the hundred-thousands place, and all the digits to the left of the hundred-thousands place will remain the same and the lower digits will be changed to 0's. Our answer is $3,745,300,000$

1.2.2 Rounding Numbers with Decimals

We will also round numbers with decimals. This will require us to be familiar with the names of the decimal place value positions.

Example: Here is a number with decimals and its place values labelled.

Ones	Decimal Point	Tenths	Hundredths	Thousandths	Ten-Thousandths	Hundred-Thousandths	Millionths	Ten-Millionths	Hundred-Millionths
1	.	4	1	4	2	1	3	5	6

How to Round Numbers with Decimals

To round numbers with decimals, follow the same procedure as with rounding whole numbers, except, instead of changing all the digits to zeros below the position that is to be rounded, just simply drop those digits.

Example: Round 1.41421356 as follows:

a. to the nearest hundredth

Solution: Begin by underlining the hundredths digit and look at the digit immediately to its right.

$$1.4\underline{1}\underset{\Uparrow}{4}21356$$

The digit to the right of the hundredths place is 4. This is less than 5, so we round down. This means we leave the digit at the hundredths place and the digits to its left as they are and the digits to the right will be dropped. Our answer is 1.41.

b. to the nearest millionth.

Solution: Begin by underlining the millionths digit and look at the digit immediately to its right.

$$1.41421\underline{3}\underset{\Uparrow}{5}6$$

The digit to the right of the millionths place is 5. This is equal to 5, so we round up. This means we will add 1 to the millionths digit and leave all the digits to its left as they are and the digits to its right will be dropped. Our answer is 1.414214.

1.2.3 Estimation by Rounding

Example: Estimate $768 + 9831 + 1421$ by first rounding each number to the nearest thousand and then carrying out the operation(s).

Solution: Begin by rounding the 3 numbers to the nearest thousand. 768 rounds to 1000, 9831 rounds to 10000, and 1421 rounds to 1000. So we have that

$$768 + 9831 + 1421 \approx 1000 + 10000 + 1000 = 12000.$$

Example: Estimate $7203 - 4925$ by first rounding each number to the nearest hundred and then carrying out the operation(s).

Solution: Begin by rounding the 2 numbers to the nearest hundred. 7203 rounds to 7200 and 4925 rounds to 4900. So we have that

$$7203 - 4925 \approx 7200 - 4900 = 2300.$$

Example: Estimate $43.773+52.345+22.544$ by first rounding each number to the nearest whole number and then carrying out the operation(s).

Solution: Begin by rounding the 3 numbers to the nearest whole number. 43.773 rounds to 44, 52.345 rounds to 52, and 22.544 rounds to 23. So we have that

$$43.773 + 52.345 + 22.544 \approx 44 + 52 + 23 = 119.$$

Applications

Here we look at some application problems that involve estimation to determine their solutions.

Example: Use your judgement to answer the following problems. Be aware that within the context of each part, traditional rounding may not be the best strategy.

a. Jon is buying paint to update a classroom at UWG. He estimates that he will need 2.37 gallons of paint. The hardware store sells paint by the gallon. How many gallons should he buy?

Solution: Since the store sells paint by the gallon then Jon will need to purchase more than he needs because 2.37 gallons is 0.37 gallons over the nearest gallon. So he will need to purchase 3 gallons of paint.

b. Katherine is a caterer and she is getting ready to bake an order of cupcakes for a child's birthday party. She needs 12.76 pounds of flour. At the grocery store they only have five pound bags of flour. How many five pound bags does Katherine need to buy?

Solution: Since at this grocery store, flour is only available in five pound bags, Katherine will need to buy more than what she needs. She needs 12.76 pounds of flour. So she will have to buy 3 five pound bags of flour.

c. Alejandra is shopping online and has computed that she has enough money in her bank account to buy 3.74 video games. How many video games can she buy?

Solution: Alejandra does not have enough money for 4 video games so she will have to settle for 3 video games in this case.

Example: Joe is building a birdhouse for his grandmother. He's going to the hardware store to purchase the tools and materials necessary to complete the project. The tools and materials along with their prices are given below:

Tool/Materials	Price
Wood: 1 in x 6 in x 5 ft	\$9.37
1 Box of Screws	\$8.58
Power Drill	\$50.47
Handsaw	\$9.83
Drill Bit Set	\$11.77

Use this information to complete the following questions.

a. Without using a calculator, estimate the total cost of the five items by rounding each price to the nearest dollar and then adding them together.

Solution: Begin by rounding the cost of the five items to the nearest dollar. 9.37 rounds to 9, 8.58 rounds to 9, 50.47 rounds to 50, 9.83 rounds to 10 and 11.77 rounds to 12. Now sum the values up.

$$9.37 + 8.58 + 50.47 + 9.83 + 11.77 \approx 9 + 9 + 50 + 10 + 12 = 90$$

b. What is the exact total cost of the items? You may use a calculator for this part.

Solution:

$$9.37 + 8.58 + 50.47 + 9.83 + 11.77 = 90.02$$

c. How close is your estimate to the exact total cost?

Solution: The rounded estimation was \$90 and the actual amount was \$90.02. The estimation is very close to the actual cost. The estimation was off by only 2 cents.

****Try this on your own**: Round 136.295 to the nearest hundred, one, and tenth.

1.2.4 Exercises: Estimation

Solutions appear at the end of this textbook.

1. Round $1,842,508$ to nearest million, ten-thousand, thousand, and then ten.

2. Round 6.20819 to nearest one, hundredth, and then ten-thousandth.

3. Rina spent \$86.37 on gas, \$49.99 on an oil change, \$5.75 on a car wash. Use rounding to estimate how many dollars she spent on car expenses.

4. Mr. Walters will be driving to several cities for his job. He will be driving 312 miles to Atlanta, 555 miles to St.Louis, then 821 miles to Baltimore. To get a quick estimate of the minimum amount of gas he will need, he decides to round each mileage up to the next hundred. How many miles will he travel by this estimation?

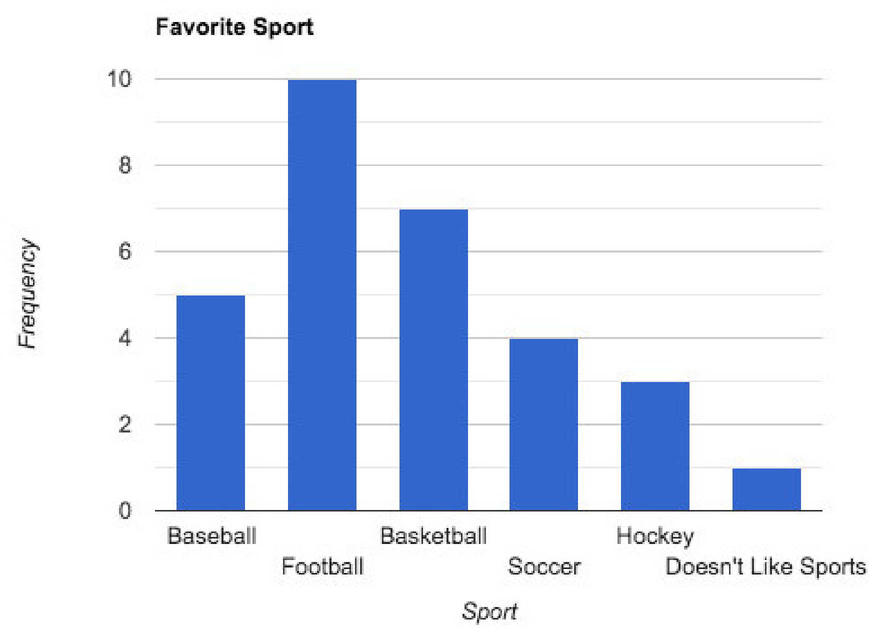
1.3 Interpreting and Estimating Graphs

In this section we will consider three types of graphs: Bar graphs, Circle graphs (or pie charts), and Line graphs.

1.3.1 Bar Graphs

A **Bar Graph** is a graph that consists of bars for each category with the length/height of the bars specifying the frequency for each category. One axis will indicate the categories and the other axis will indicate the frequency. Bar graphs are often used for comparing different characteristics of items. Bar graphs can have horizontal or vertical bars.

Example: A group of students surveyed their class about what sport was their favorite. The results are given below.



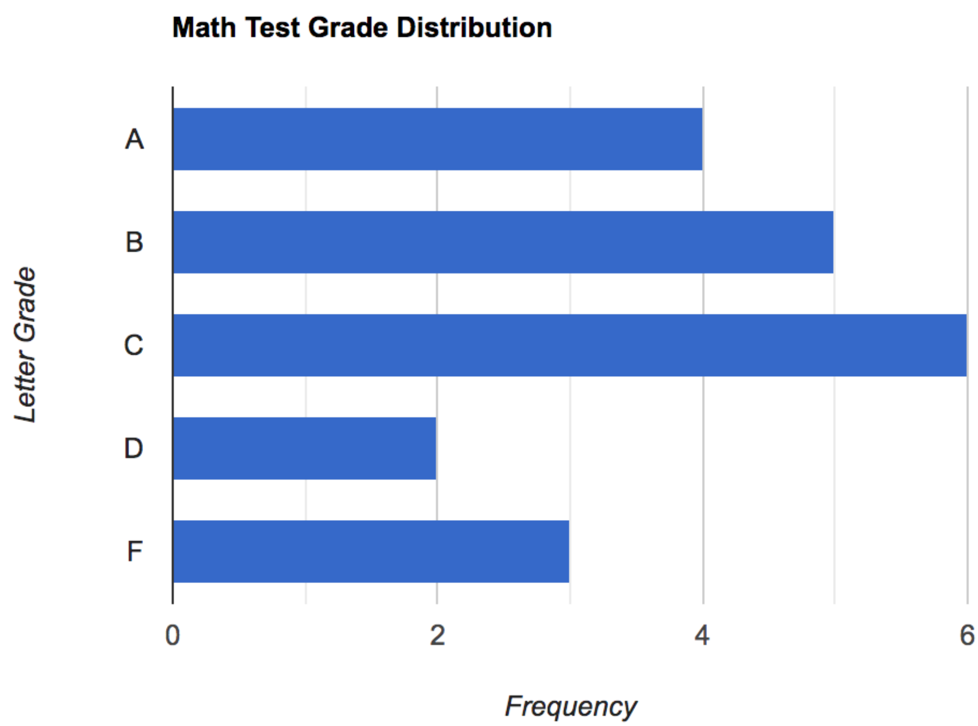
a. How many students said basketball was their favorite?

Solution: Find the bar above that is labelled "Basketball" on the horizontal axis. The number of students whose favorite is basketball is equivalent to the height of the bar. Read the height of the bar from the vertical axis. The number of students who said basketball was their favorite sport was 7.

b. How many more students liked Football than Soccer?

Solution: Begin by observing the number of students who liked Football and Soccer from the bar graph. The number of students who liked Football is 10 and the number of students who liked Soccer is 4. There are $10 - 4 = 6$ more students that liked Football than Soccer.

Example: The bar graph below shows scores on a Math test.



a. How many B's were there?

Solution: This is a horizontal bar graph. The grade categories are on the vertical axis. Begin by looking for "B". Then look at the horizontal axis for the length of the bar for B's. This gives 5 B's.

b. How many test grades are there?

Solution: Add all the lengths of each bar for each letter grade together to find the number of test grades. This gives a total of $4 + 5 + 6 + 2 + 3 = 20$

c. How many students scored a C or higher?

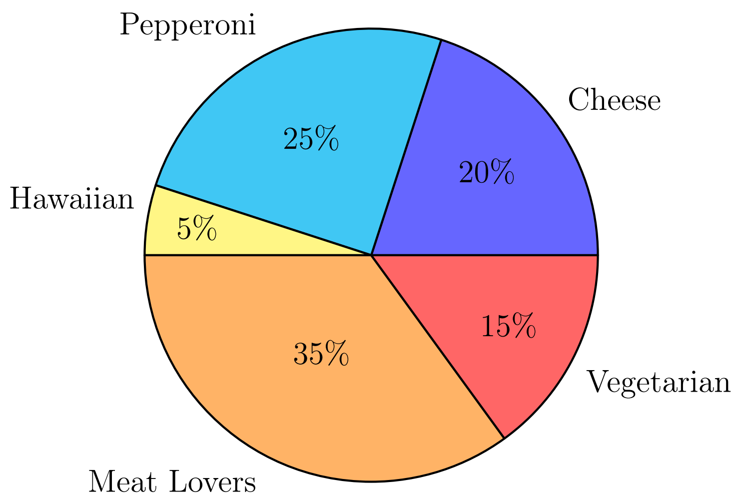
Solution: Begin by finding the number of test grades for A's, B's, and C's, which are 4, 5, and 6 respectively. Number of grades of C or better are $4 + 5 + 6 = 15$.

1.3.2 Circle Graphs (or Pie Charts)

A **Circle Graph** (or **Pie Chart**) is a circle cut into sections with varying sizes shaped like slices of a pizza. The sizes of the sections are based on the relative frequencies of the categories. The percent or frequency for each category can be specified on the sections of the pie chart.

Example: A pizza chef at Mario's Pizza makes a list of all the types of pizza he made on a particularly busy Tuesday. He then used this list to create a pie chart for the pizza types.

Pizza Type Preference



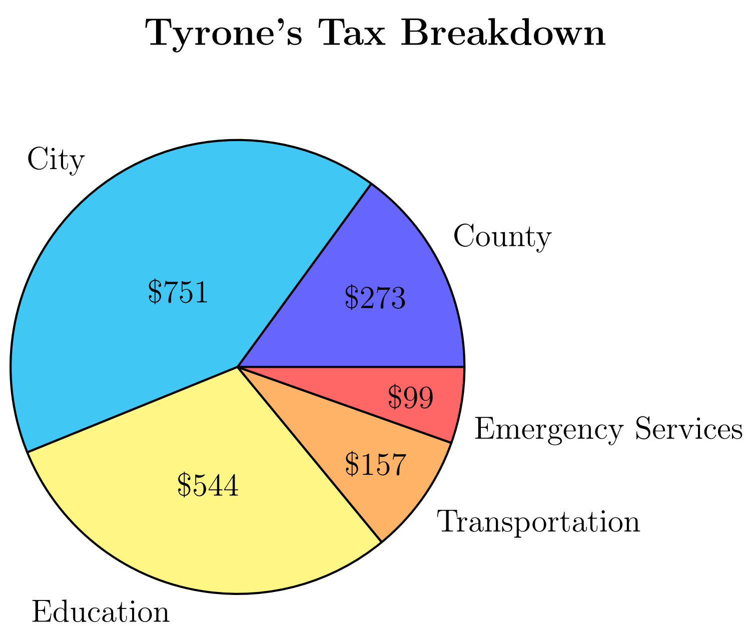
a. What was the least ordered type of pizza?

Solution: Start by identifying the smallest piece on the pie chart. Looking at this graph shows that this category is that of Hawaiian at 5%.

b. What was the most ordered type of pizza?

Solution: Identifying the largest piece on the pie chart. Looking at this graph shows that this category is that of Meat Lovers at 35%.

Example: Using the Circle Graph answer the following questions.



a. How much in taxes did Tyrone pay in total?

Solution: Sum of the money amount for each category represented on the chart. The total amount is $\$99 + \$157 + \$273 + \$544 + \$751 = \$1824$

b. How much of Tyrone's Taxes went to the City?

Solution: Look on the chart for the piece labeled City and read the value there which is $751

c. Which of the categories on the pie chart received the smallest amount of money?

Solution: Look at the chart and look for the category with the lowest amount of money. In this case the category is Emergency Services at $99.

Before we consider any more examples, let's review how to convert a percent to a decimal.

How to Convert a Percent to a Decimal

1. Drop the percent sign.
2. Move the decimal point two places to the left.

Example: Convert 47% to a decimal.

Solution: To convert 47% to a decimal first drop the percent sign. Then move the decimal over two places to the left. This gives the following.

$$47\% = .\underset{\curvearrowleft\curvearrowleft}{47}.0 = 0.47$$

Example: Convert 125% to a decimal.

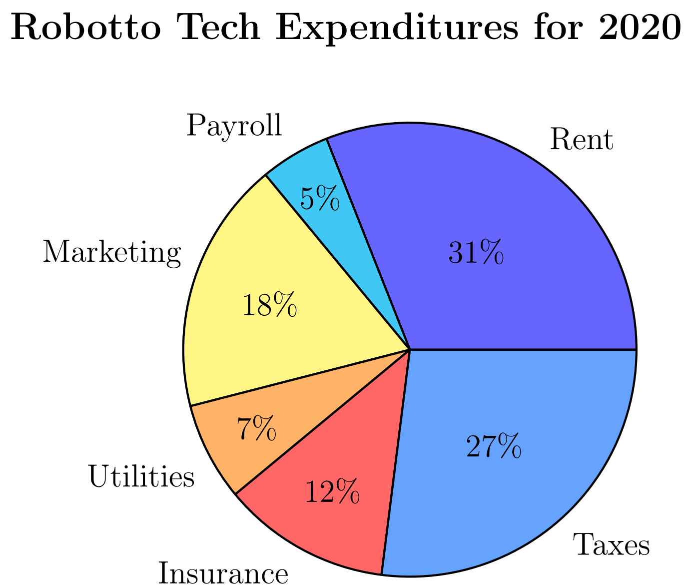
Solution: Just as with the previous example, to convert 125% to a decimal first drop the percent sign. Then move the decimal over two places to the left. This gives the following.

$$125\% = 1.\underset{\curvearrowleft\curvearrowleft}{25}.0 = 1.25$$

How to Find the Percent of a Total

To find the percent of the total, multiply the total by the percent in decimal form.

Example: Mary is the CEO of Robotto Tech. The percents of the expenditures of her company are presented in the given graph.



If Robotto Tech spent a total of \$435000 in 2020, then how much was spent on the following categories?

a. Payroll

Solution: To answer this question first identify the percentage associated with Payroll. This is 5% and convert it to a decimal which is 0.05. Then multiply 0.05 by the total amount for 2020, \$435000. The amount spent on Payroll for 2020 was $0.05 \times 435000 = \$21750$.

b. Insurance

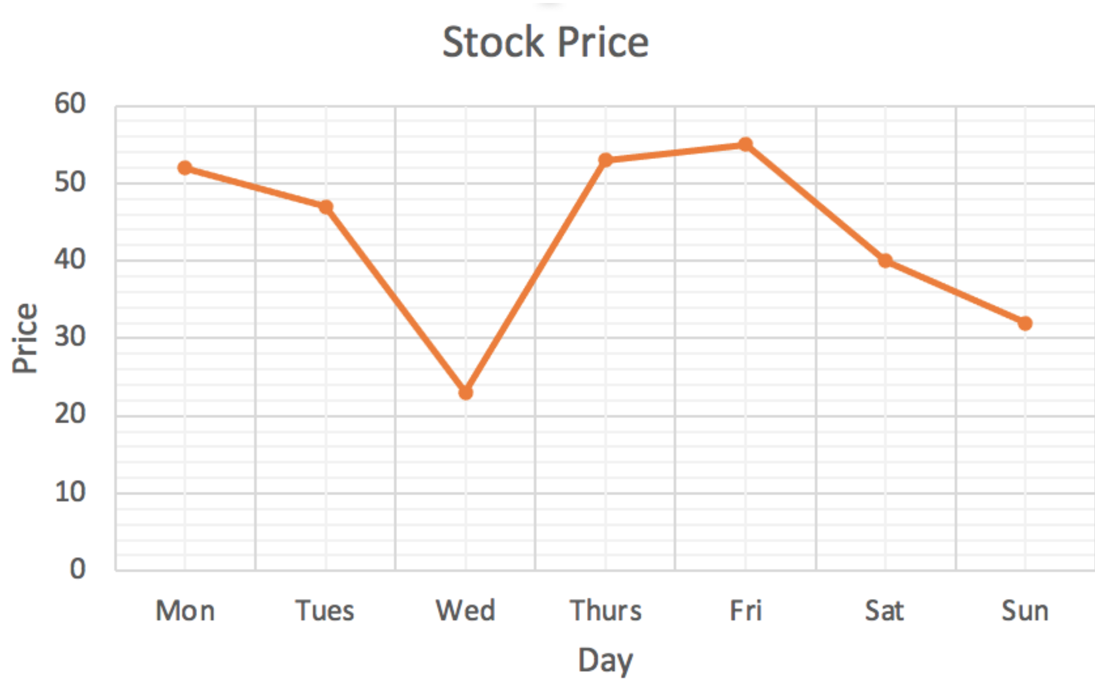
Solution: To answer this question first identify the percentage associated with Insurance. This is 12% and convert it to a decimal which is 0.12. Then multiply 0.12

by the total amount 2020, \$435000. The amount spent on Insurance for 2020 was $0.12 \times 435000 = \$52200$.

1.3.3 Line Graphs

Line Graphs show trends over a period of time. Time is located on the horizontal axis and amounts will be located on the vertical axis. Information will be organized into ordered pairs. To draw the graph, you plot these points and then connect them with straight lines.

Example: Veronica is a stock trader. She followed the value of a stock and recorded the following graph for this past week. Use it to answer the following questions.



a. What was the value of the stock on Tuesday?

Solution: When reading the graph the horizontal axis is the day and the vertical axis tells us the value of the stock that day. For Tuesday the value of the stock was \$47.

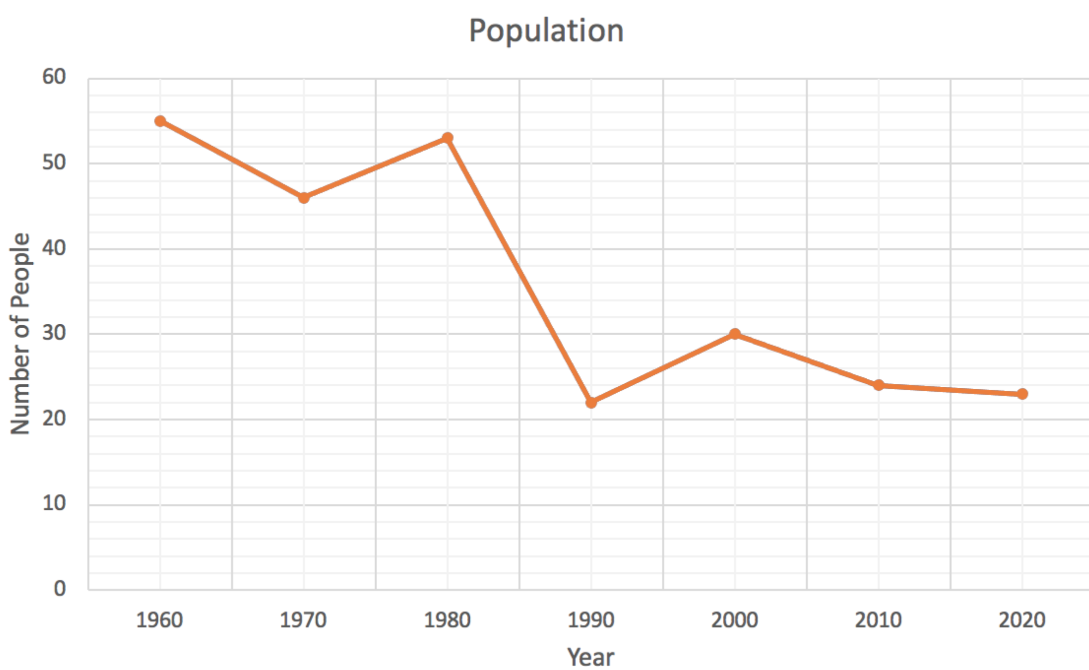
b. What day of the week was the stock price equal to $40?

Solution: Start by reading the vertical axis and find where the stock price is $40 and then find the corresponding day on the horizontal axis which is Saturday.

c. Between which two consecutive days was the biggest change in stock prices?

Solution: We need to look at the graph and find two consecutive points who have the biggest difference between stock prices. Looking at the graph this occurs between Wednesday and Thursday.

Example: Edge Hill is the smallest incorporated city in the state of Georgia. The line graph below shows its population every ten years starting in 1960. Use this graph to answer the following questions.



a. What was the population in 2000?

Solution: When reading this graph the horizontal axis represents the year and the vertical axis represents the population size. To answer this problem look for the year

2000 on the horizontal axis and then read the population size from the vertical axis which is 30 people.

b. What year was the population equal to 46 people?

Solution: For this problem look for the point on the line graph where the population is 46 people. This happens for the year 1970.

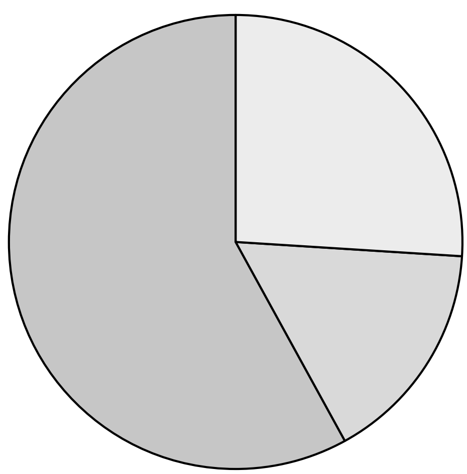
c. At approximately what rate did the population decrease from the year 1980 to 1990?

Solution: To answer this question we need to compute a rate. Start by identifying two points from the graph at the years 1980 and 1990: (1980,53) and (1990, 22). Now compute the rate. To do this we will use the slope formula.

$$\text{Rate} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{22 - 53}{1990 - 1980} = \frac{-31}{10} = -3.1$$

What this tells us is that on average the population decreased by approximately 3 people each year between 1980 and 1990.

****Try this on your own**: Estimate the percent size of each slice.



1.3.4 Exercises: Interpreting and Estimating Graphs

Solutions appear at the end of this textbook.

1. Based on the graph below, give an estimate for the year at which the cost of gas heat is equal to the cost of solar heat. Then estimate the cost they both equal to.

2. Based on the graph below, estimate child growth between the ages 3 and 6.

3. Estimate how many trees are there in the data for the bar graph below.

1.4 Problem Solving

Polya's Four Steps in Problem Solving

Step 1: Understand the Problem.

Step 2: Devise a Plan.

Step 3: Carrying out the plan to solve the problem.

Step 4: Look back and check answer.

In this section we will practice using these steps, however, we will spend a great amount of time focusing on devising a plan.

1.4.1 Using a Formula

For these examples we rely on some known formulas to find the solutions.

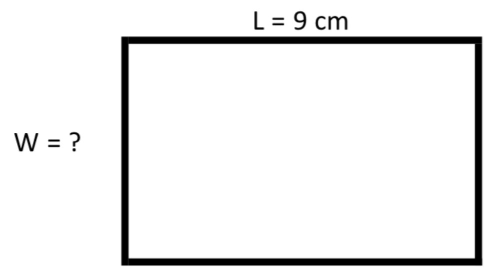
Example: Charlie is the owner of a Minor League baseball team. He is hoping that the average attendance for a 4 game series will be at least 20000 people. On Friday, Saturday, and Sunday the attendance was 23000, 19000, and 17000, respectively. How many people must attend Monday's game to reach the owner's goal?

Solution: To solve this question we need to recall how to take the average of a set of values. The average in general is the sum of all the values in a set divided by the number of values in that set. Let's call the attendance of the fourth game x. This gives us

$$\begin{aligned}
\frac{23000+19000+17000+x}{4} &= 20000 \\
4\times\frac{59000+x}{4} &= 4\times 20000 \\
59000+x &= 80000 \\
x &= 21000
\end{aligned}$$

Therefore the attendance for Monday's game will need to be at least 21000 people to have an average of at least 20000 for the 4 game series.

Example: Consider the rectangle below.



The perimeter for this rectangle is 32 cm and its length is 9 cm. Find the value of the width. Note that the perimeter formula for a rectangle is $P = 2L + 2W$, where P is perimeter, L is length, and W is width.

Solution: To answer this problem we will plug in the known information which is $P = 32$ and $L = 9$ in the formula for perimeter and then solve for W.

$$\begin{aligned} P &= 2L + 2W \\ 32 &= 2(9) + 2W \\ 32 &= 18 + 2W \\ 14 &= 2W \\ 7 &= W \end{aligned}$$

Therefore the width of this rectangle is $W = 7$ cm.

1.4.2 Using a Table or List

For these examples we will use tables or lists to determine the solutions.

Example: At a poorly stocked country store, batteries are sold in packages of 1, 2 and 4. Your tv remote control needs 7 batteries to work. In how many ways can you purchase 7 batteries?

Solution: Use a table to help count the number of ways to buy 7 batteries from the purchasing options of buying a package of 1 battery, 2 batteries, or 4 batteries.

Count	1 Battery	2 Batteries	4 Batteries	Computation
1	1	1	1	$1 \times 1 + 1 \times 2 + 1 \times 4 = 7$
2	3	0	1	$3 \times 1 + 1 \times 4 = 7$
3	1	3	0	$1 \times 1 + 3 \times 2 = 7$
4	3	2	0	$3 \times 1 + 2 \times 2 = 7$
5	5	1	0	$5 \times 1 + 1 \times 2 = 7$
6	7	0	0	$7 \times 1 = 7$

This shows that there are 6 ways to purchase 7 batteries with the given packaging options.

Example: Brenda is a bank teller at Bank of Coweta. Her customer is there to cash a check for \$40 dollars. He requests that she give him his money in \$5, \$10, and \$20 dollar bills. In how many ways can you make \$40 using \$5, \$10, and \$20 bills?

Solution: Just like in the previous example, we will use a table to help us count the number of different ways that we make \$40 dollars using fives, tens, and twenty dollar bills.

Count	Number of Five Dollar Bills	Number of Ten Dollar Bills	Number of Twenty Dollar Bills	Computation
1	0	0	2	$2 \times 20 = 40$
2	0	2	1	$2 \times 10 + 1 \times 20 = 40$
3	2	1	1	$2 \times 5 + 1 \times 10 + 1 \times 20 = 40$
4	4	0	1	$4 \times 5 + 1 \times 20 = 40$
5	0	4	0	$4 \times 10 = 40$
6	2	3	0	$2 \times 5 + 3 \times 10 = 40$
7	4	2	0	$4 \times 5 + 2 \times 10 = 40$
8	6	1	0	$6 \times 5 + 1 \times 10 = 40$
9	8	0	0	$8 \times 5 = 40$

From this table we can see that we counted a total of 9 ways to make 40 dollars using combinations of five, ten, and twenty dollar bills.

1.4.3 Using Diagrams or Sketches

With these problems we will use diagrams to help us determine the problem solutions.

Example: Brothers Matthew, Nathan, Andrew, and Will are playing Mario Kart on their family's Nintendo Switch. During a particularly competitive race the following results occurred: Andrew finished last, Nathan finished ahead of Matthew but behind Will. List who came in first, second, third, and fourth place.

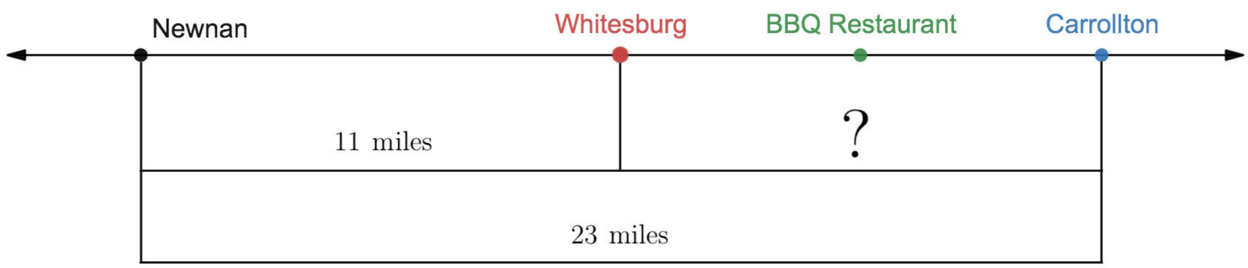
Solution: To get started with this problem let's use some variables to label the brother's names. Let M = Matthew, N = Nathan, A = Andrew, and W = Will. We will use these in our diagram. We're told that Andrew finished last. If we are using a number line as our diagram this means that Andrew is the slowest so we have

Next we're told that Nathan finished before Matthew but after Will. For this to be true it implies that Will finished before Nathan and Matthew and that Nathan finished before Matthew. So the race finishing order must be.

This diagram gives us the answer to the problem. The farther right I am on the line the sooner I finish the race. Therefore, the results of the race are Will was first, Nathan was second, Matthew was third and Andrew was fourth.

Example: You have just finished your morning classes at UWG Newnan and are driving to UWG Carrollton for your afternoon classes. As you travel down the highway you pass a sign that says you are leaving Newnan and that Whitesburg is 11 miles away and Carrollton is 23 miles away. You plan on meeting up with some friends for lunch at a BBQ restaurant that is exactly half way between Whitesburg and Carrollton. How far is Newnan to the BBQ restaurant? Assume that all three towns are in the same direction.

Solution: This is another problem where drawing a diagram will help with our reasoning. Draw a line and mark where the towns are and indicate any known information from the description above.



Our goal is to find the distance from Newnan to the BBQ restaurant half way between Whitesburg and Carrollton. We know the distance between Newnan and Whitesburg, however we do not know the distance between Whitesburg and Carrollton. If we knew this distance then we could take half of it and add it to the distance between Newnan and Whitesburg and this would be the answer to this question. To find the distance between Whitesburg and Carrollton, find the difference between the distance from Newnan to Carrollton and the distance from Newnan to Whitesburg. This gives $23 - 11 = 12$. Then take half of this and we get $12/2 = 6$. Therefore the distance between Newnan and the BBQ Restaurant is $11 + 6 = 17$ mi.

1.4.4 Building Equations

With these examples we will build equations to find the solutions.

Example: Nancy is a pharmaceutical sales representative for the Big Pharma company. She earns a monthly salary plus an end of year bonus of \$5800 dollars. In 2020, her total income was \$103600 dollars. What was her monthly salary in dollars during 2020?

Solution: Notice that we are interested in determining Nancy's monthly salary. She gets paid a monthly salary plus an end of year bonus of \$5800 dollars. She earns a total of \$103600 dollars for the whole year. The bonus is a one time payment and the monthly salary is a periodic payment. This follows the pattern of a linear equation $y = mx + b$. Where y is the amount of money made in a year, x is the number of months, m is the monthly salary, and b is the end of year bonus. This gives us an equation to work with.

$$\begin{aligned} 103600 &= 12m + 5800 \\ 97800 &= 12m \\ \frac{97800}{12} &= \frac{12m}{12} \\ 8150 &= m \end{aligned}$$

Therefore Nancy's monthly salary is \$8150.

****Try this on your own**: Edera works as a salesperson and gets paid \$500 per month plus \$100 for each car she sells. If she makes \$1,600 in a month, how many cars did she sell? Setup a formula and solve.

Example: Joe has 3 sisters, Courtney, Brooke, and Elizabeth, and one brother Zack. The ages of the five children have the following relationships: Courtney is twice as old as Zack, Brooke is twice as old as Elizabeth, Joe is 6 years older than Courtney, Elizabeth is 11 years older than Zack, and the average age of Zack and Courtney is 3. What is everyone's age?

Solution: Begin by naming some variables. Let

$$\begin{aligned} Z &= \text{Age of Zack} \\ C &= \text{Age of Courtney} \\ J &= \text{Age of Joe} \\ E &= \text{Age of Elizabeth} \\ B &= \text{Age of Brooke} \end{aligned}$$

Now we're told that Courtney is twice as old as Zack so this gives us $C = 2Z$. We're also told that the average age of Courtney and Zack is 3. This gives us

$$\begin{aligned} \frac{Z+C}{2} &= 3 \\ \frac{Z+2Z}{2} &= 3 \\ 2 \times \frac{3Z}{2} &= 3 \times 2 \\ 3Z &= 6 \\ \frac{3Z}{3} &= \frac{6}{3} \\ Z &= 2 \end{aligned}$$

Therefore Zack's age is 2. This means that Courtney's age is $C = 2Z = 2(2) = 4$. We also know that Joe's age is 6 years more than Courtney's age. This gives the relationship $J = C + 6$. Therefore Joe's age is $J = 4 + 6 = 10$. We're told that Elizabeth is 11 years older than Zack. This gives the relationship $E = Z + 11$. Therefore Elizabeth's age is $E = Z + 11 = 2 + 11 = 13$. Then finally Brooke's age is twice Elizabeth's age. This will give us that $B = 2E = 2(13) = 26$. Therefore Brooke is 26 and we have identified everyone's age.

1.4.5 Exercises: Problem Solving

Solutions appear at the end of this textbook.

1. A car rents for \$210 per week plus \$0.28 per mile. Find the rental cost for a two-week trip of 400 miles for a group of four people, if they make five stops for gas and restroom as they drive.

2. Jessie needs to get food for 24 people. Which is a better deal? Assume each person will eat one bagel and 1 ounce of cream cheese. Store A, bagels are \$0.50 each and 12oz. tub of cream cheese is \$3 or Store B, bagels are \$5.50 per dozen and 24oz. tub of cream cheese is \$8.

3. A student wants to get on the principal's honor roll (average 90). They got grades for four classes 95, 91, 88, and 82. The fifth class grade has not been released yet. What grade do they need in the fifth class in order to get a 90 average overall?

Chapter 2

Descriptive Statistics

Written by James Bellon

2.1 Collecting Data

2.1.1 Key Terms

If we wish to analyze data, we need to understand what data is. People and things have characteristics which can be observed or measured. A **Variable** is a general type of characteristic (or type of status), which can be different for each person or thing. Some variables are: name, height, color, texture, mood, wingspan, density, anxiety level, etc., **Data** is the collection of all observations for a particular variable or variables, from one or more people or things.

The branch of mathematics that covers the methods and procedures in analyzing data is called **Statistics**. Statistics includes methods for planning studies and experiments, obtaining data, and then organizing, summarizing, presenting, analyzing, interpreting, and drawing conclusions. This course and textbook will focus on **Descriptive Statistics**, which is collecting, organizing, summarizing, and presenting data. Other courses may also touch on **Inferential Statistics**, which involves analyzing, interpreting, and drawing conclusions based on data.

Below are some important terminology we will be using in statistical data analysis.

A **Population** is the collection of all individuals or items under consideration in a study.

A **Census** is information (data) obtained from the entire population.

In reality, most large censuses (such as the US national census) are an attempt to collect from the entire population, but being so large, some data is never collected. There are just some people who do not wish to be found and others who are too busy to report their data. The results are often adjusted to be a good approximation of the population information. The US census is so large and takes so much time, money, and staff, that it is done only every ten years.

In many cases, it is usually easier and sufficient to collect data from only some of the elements in the population. A **Sample** is the part of a population from which information is actually collected.

A **Parameter** is a numerical measurement describing some characteristic of a population. Examples: The average starting salary of elementary school teachers in Georgia is $33,673. The average for the whole United States is $35,763.

A **Statistic** is a numerical measurement describing some characteristic of a sample. **Example:** A survey of ten job postings for elementary school teachers in the Atlanta area, had an average starting salary of $38,541.

****Try this on your own**: For the following scenario, describe the population of interest, describe the sample, state the parameter of interest, and the statistic that was calculated.

A farm wants to track the weight gain of their chickens after they switched to a new feed. The farm has over 10,000 chickens. They isolated 200 chickens and weighed them before the switch, then every week for the next 10 weeks. At the end of 10 weeks, the 200 isolated chickens gained an average of 1.2 pounds.

2.1.2 Sampling

There are two main types of variables. In this book, we will focus mainly on data from numerical variables, since you can perform calculations with numbers more easily.

Qualitative Variables are variables which have values that are words, symbols, or categories. They can also be numbers that have no absolute measure, order or units. Examples: gender, job title, letter grade, phone number, numbers on football jerseys, etc. These are also referred to as **Categorical Variables**.

Quantitative Variables are variables which have values that are numerical values with a specific order and units. They represent counts or measures. Examples: height, weight, temperature, number of siblings, hours of sleep, etc.

There are several ways to pick the subjects that will be in a sample. **Random Sampling** is using a random method to select the sample from population. This is better than human judgment, but no guarantee of getting a perfect sample. It is important to make sure you have selected a **Representative Sample**, which means that the sample has characteristics similar to the population being studied, in order to avoid having bias.

For example, if you want to know what students at a college think about proposed changes to graduation requirements, you should get a sample that has students from all class levels, both male and female students, and not all from the same major.

Here is an example of a sample that is not representative. If you need 10 people for a survey, then ask the first ten people you come in contact with. This can often lead to extreme bias. The ten people might be related or friends, and have similar opinions.

****Try this on your own**: Would the following sample be representative of the population? A teacher would like to know how students feel about the new math curriculum. They selected a sample of students from the ones who are failing the class and come for extra help.

Whether conducting statistical analysis of data that we have collected, or analyzing a statistical analysis done by someone else, we should not rely on blind acceptance of mathematical calculation. We should consider these factors:

1. Context of the data: What do the values represent? Why were they collected? An understanding of the context will directly affect the statistical procedure used.

2. Source of the data: Is the source objective or biased? Is there something to gain or lose by distorting results? Be vigilant and skeptical of studies from sources that may be biased, such as a nutrition study done by a fast food company.

3. Sampling method: Is the method chosen appropriate and help eliminate bias? Voluntary response(self-chosen) samples often have bias (those with strong opinions are more likely to participate). These sample results are not necessarily valid. Other methods are more likely to produce good results.

4. Conclusions: Make statements that are clear to those without an understanding of statistics and its terminology. Avoid making statements not justified by the statistical analysis.

5. Practical implications: State practical implications of the results. The results may be valid and significant yet there may be NO practical significance. Does anyone even care about it? Common sense might suggest that the finding does not make enough of a difference to justify its use or to be practical.

6. Consider the likelihood of getting the results by chance. If results could easily occur by chance, then they are not statistically significant (you did not justify anything). If the likelihood of getting the results by chance is so small, then the results are statistically significant (you found strong evidence).

2.1.3 Exercises: Collecting Data

Solutions appear at the end of this textbook.

1. Identify the population, sample, parameters, and statistics for the following situation. A textbook company wants to know the average price of homeschool science textbooks in the United States. They obtain a list of 15 science books and compute the average price of the 15 books is $52.

2. A teenager put the following information on his myspace page. What are the variables, what are their values, which variables are qualitative, which are quantitative? Name: Sean Higgins, Ht: 5ft.10in., Wt: 185 lbs., Eyes: Green, Hair: Red, Page-hits: 142

3. What is a census? The US government does a census every ten years. Why don't they do one every year? Is the US census an actual census? Why or why not?

4. Which samples are representative of their populations, which are not? Explain why.

   (a) A marketing firm wants to know how much time teenagers spend on youtube. They post ads to take their survey on the top ten youtube videos.

   (b) A large company wants to know how far their employees drive to work. They pick employees from several cities, several different job levels, and a mix of young and older employees.

2.2 Summarizing Data

2.2.1 Distributions

After data is collected, it is a good idea to summarize and display the data in ways that show the important characteristics. One of the best ways is to create a **Distribution** of each variable, which is information that tells us what values the variable takes on and how often. Large sets of data are often grouped according to **Classes**, which are categories or groupings (quantitative data is in consecutive intervals). The three guidelines for grouping data into classes are:

1. Small number of classes to be effective, but enough to show differences.

2. Each observation must belong to only one class.

3. Whenever feasible, classes should have same width.

Then compute the **Frequency** of each class, which is the number of observations that fall into a class (count). A listing of all classes of the data and their frequencies is called a **Frequency distribution**.

When data sets are different sizes, it is hard to compare them. A good way to compare is to compute **Relative Frequency**, which is the ratio of the frequency of a class to the total number of observations. A listing of all classes and their relative frequencies is called a **Relative Frequency distribution**. Most distributions show frequencies as well as relative frequencies.

Example: Bradley worked a summer job to earn money for college. His weekly hours over a 12 week period were 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36. We can group the hours into 3 classes in various ways, but one simple choice would be 10-19, 20-29, and 30-39. The Distribution would be as follows:

Hours	Frequency	Relative Frequency
10-19	3	$\frac{3}{12} = 0.25 = 25\%$
20-29	2	$\frac{2}{12} = 0.167 = 17\%$
30-39	7	$\frac{7}{12} = 0.583 = 58\%$
Total	12	100%

Notice that the frequencies add up to 12, and there are 12 weeks of data. Also the relative frequencies add up to 100%. These must always happen, or the distribution was done incorrectly. If the relative frequencies are rounded, then total percentage may be slightly off, between 99 to 101%. Any larger difference is not valid.

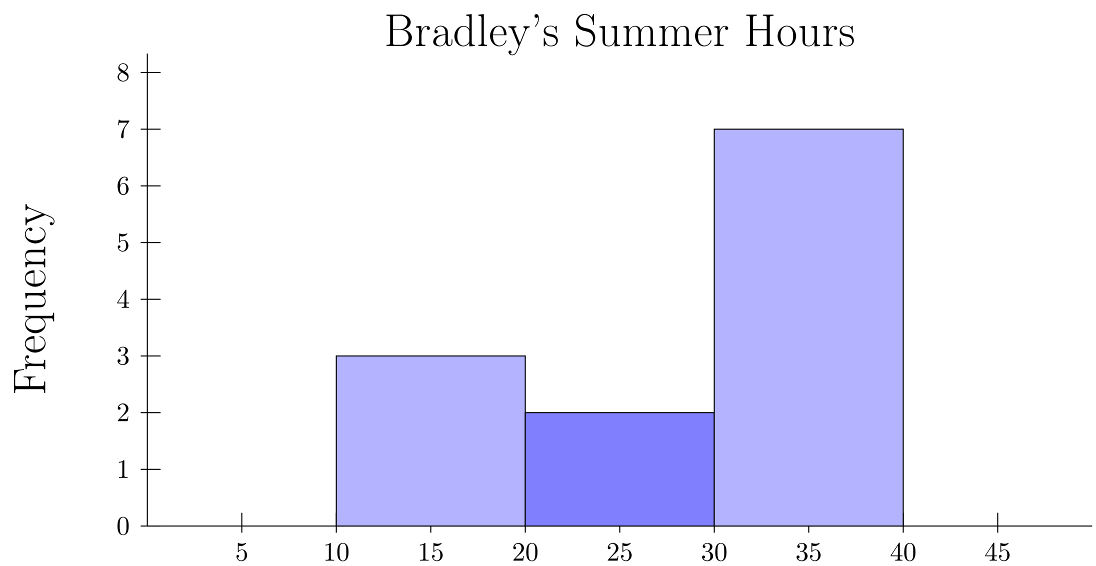
2.2.2 Graphs of Data

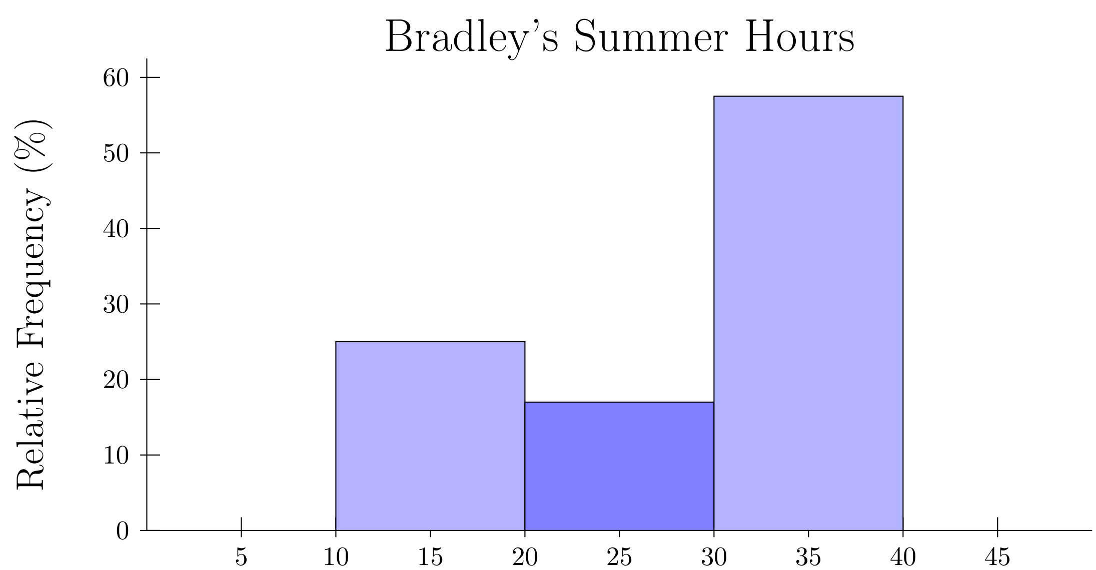
As the old saying goes, a picture is worth a thousand words. Data summaries can come in pictures or graphs. Here are some of the typical types of graphs to display distributions. They can give us a quick overview of the big picture and the characteristics of the data.

A **Frequency Histogram** is a graph that displays the classes on the horizontal axis and the frequencies on the vertical axis. It consists of vertical bars, whose height is equal to the frequency of the class(interval). The bars are drawn next to each other (without gaps), since they encompass the range of the data in numerical order. The left side of each bar starts at the lower limit of the class interval. The right side goes up to the lower limit of the next interval. A Histogram is only for quantitative data, not qualitative.

A **Relative Frequency Histogram** is the same as a frequency histogram, except it uses relative frequencies for the vertical axis and the bar heights.

Example: The frequency and relative frequency histograms for Bradley's summer job data are shown below.

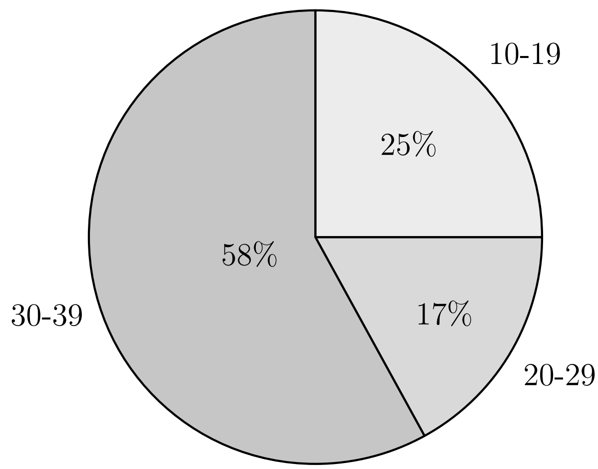




Notice that these graphs are the same shape, this is because the relative frequencies are based on the frequencies, so the same relationships between the classes are maintained.

A **Pie Chart** is a disk (circle) divided into pie-shaped pieces proportional to the relative frequencies. A pie chart should be labeled well, with class and the relative frequency for each slice. If a slice is very small, then the labels can go outside with an arrow pointing to the corresponding slice. The preferred way to sketch a pie chart is to start slices at 12 o'clock and rotate clockwise.

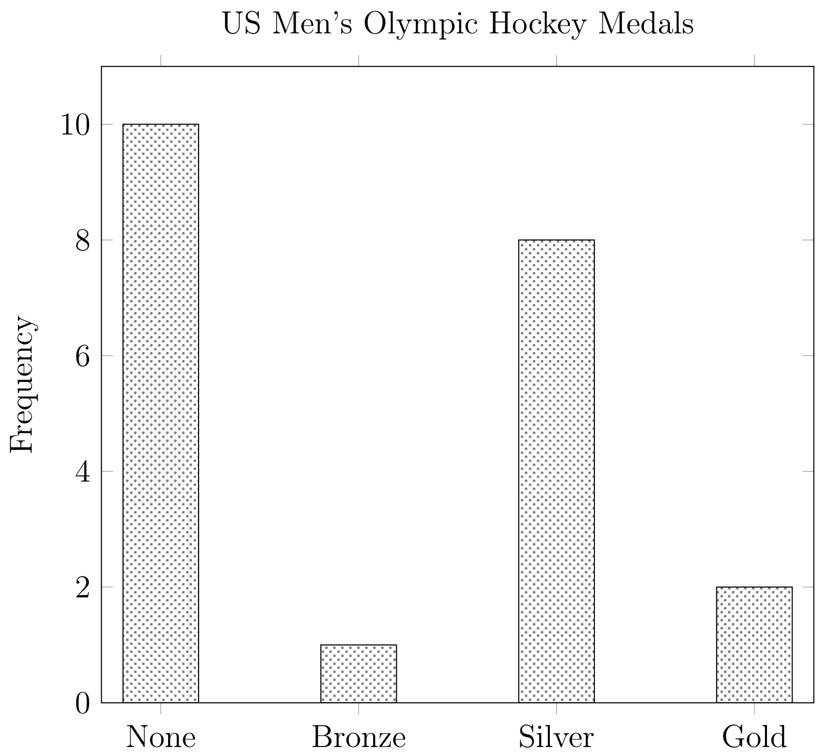
Example: The pie chart for Bradley's summer job data is shown below.

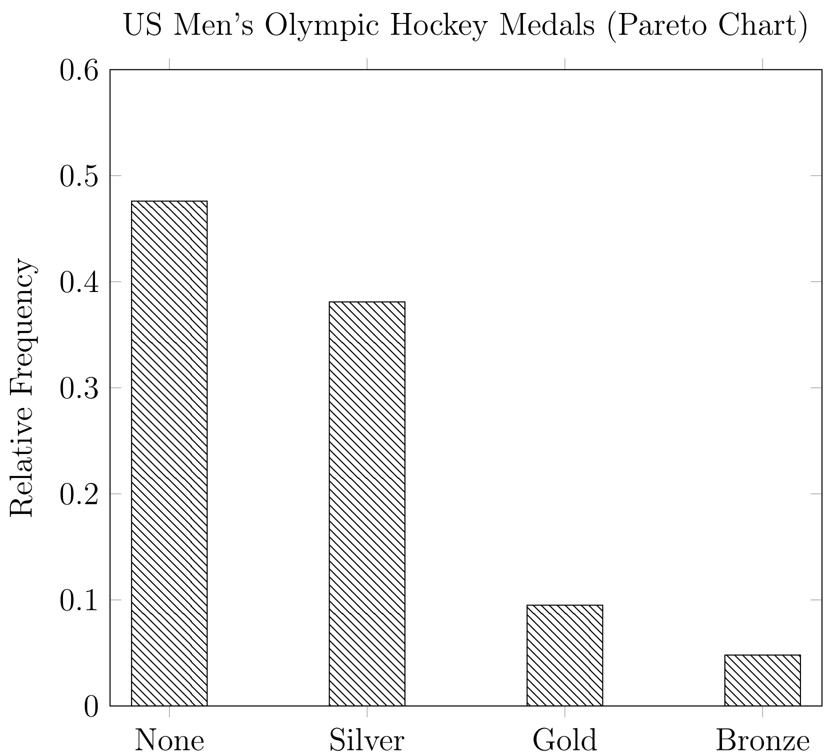


****Try this on your own**: The grades on a science final exam were 75, 83, 96, 82, 90, 78, 60, 76, 82, 71, 92, 86, 83, 88. Create a table with frequencies and relative frequencies using the intervals 60-69, 70-79, 80-89, 90-99. Then sketch a frequency histogram and a relative frequency pie chart.

Histograms are for quantitative data. There is a similar graph for qualitative data (categories), called a **Bar Graph**. In a bar graph, the width of the bars is arbitrary and the bars are not connected. Can show frequency or relative frequency. A **Pareto chart** is a specific type of bar graph where the classes are reordered so that the bars are in size order.

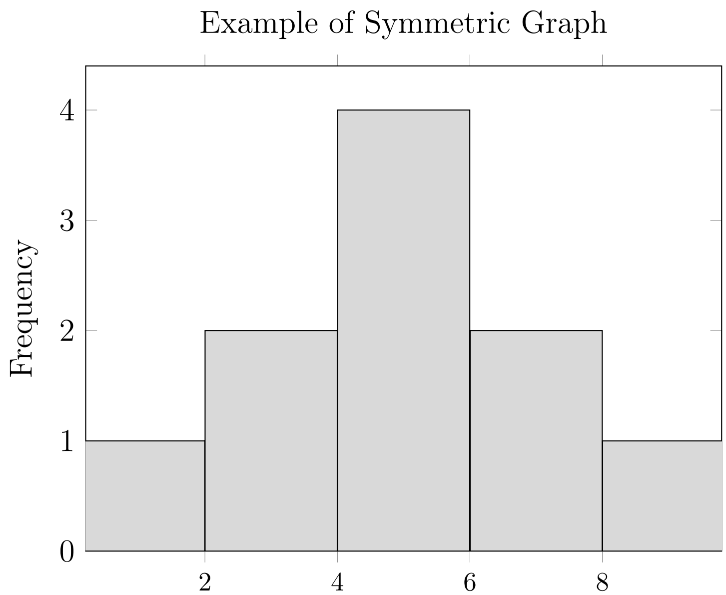
Example: The US Men's olympic hockey teams have played in 21 olympic games, winning 11 medals (2 gold, 8 silver, 1 bronze). Below are a frequency bar graph (ordered from worst to best finishes) and a relative frequency Pareto chart (ordered from highest to lowest).

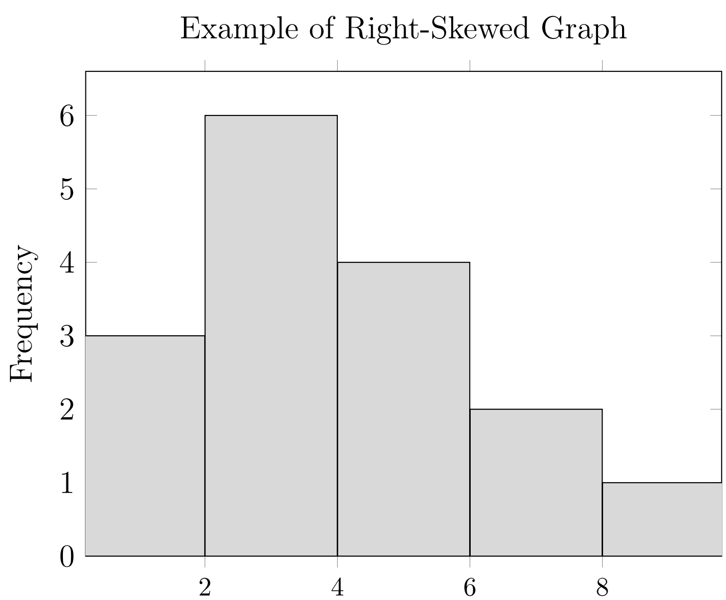


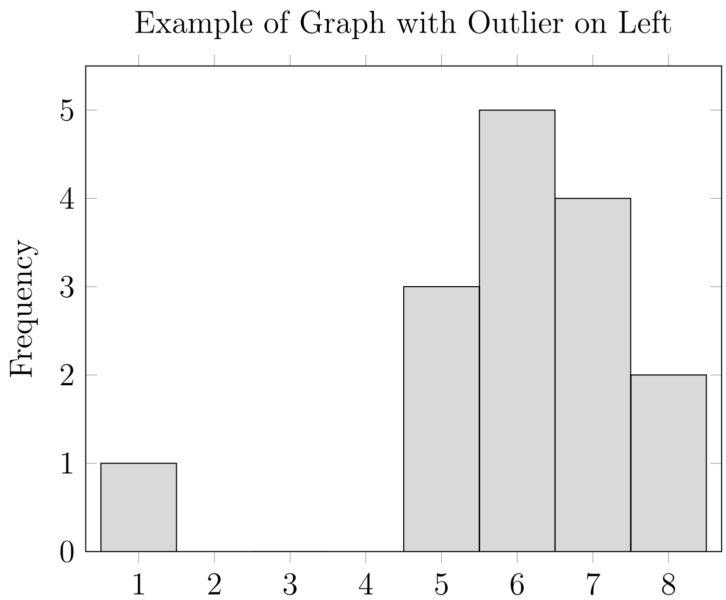


Graphs can help us get an overall view of the data set. When looking at a graph, pay attention to the following:

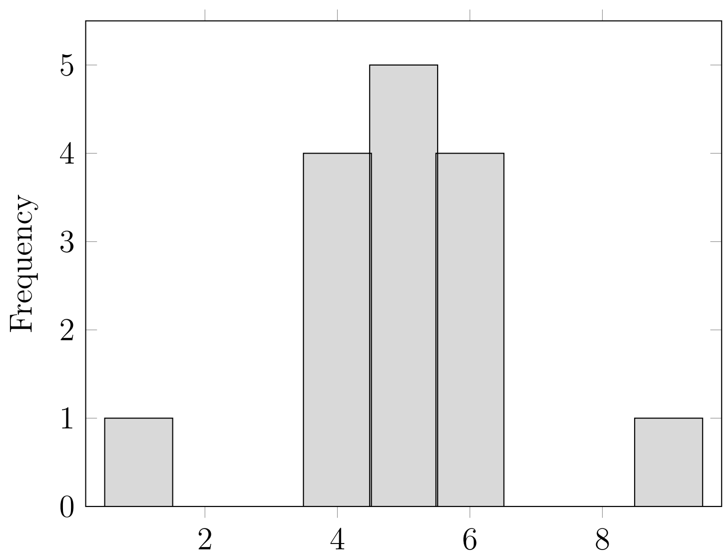
- Center: where is the middle of the graph, and the highest point.
- Spread: how are the parts of the graph spread out from each other?
- Shape: what shape does the graph have? Bell shape, straight across, repeatedly up and down, random?
- Symmetry: graph can be split in half with two mirror image parts, almost equal amount on both sides. Graph that extends more out to left is **Left-skewed**. Graph that extends more out to right is **Right-skewed**.
- Outliers: are data values (small parts of the graph) that are far from the other data (parts).


Example of Symmetric Graph
Frequency
0
1
2
3
4
2
4
6
8


Example of Right-Skewed Graph
Frequency
0
1
2
3
4
5
6
2
4
6
8


Example of Graph with Outlier on Left
Frequency
0
1
2
3
4
5
1
2
3
4
5
6
7
8

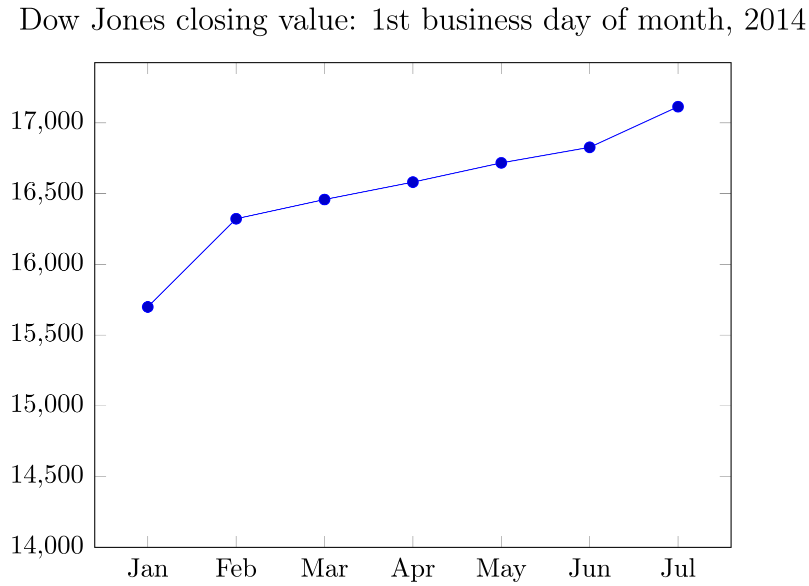
Example: what are the characteristics of the following graph? What does is suggest about the data?



Solution: The center of the graph is at data value 5, also the highest point is at 5. Most of the data in concentrated near the middle (from 4 to 6) with outliers on both extremes (only one value at 1 and one value at 9). It is not spread out very much. The graph is symmetric, mirror images on either side of 5. It is somewhat bell-shaped.

Another type of simple graph that plots the values of a single variable over time, is called a **Time Series**. The horizontal axis is in time units and the vertical axis is scaled for the values of the variable being graphed. It has values plotted as points and connected by lines, although the lines themselves do not represent data, they are just to show the pattern of the points. The vertical scale should start at zero, unless all of the data is very large, then it is better to start higher.

Example: Here is a time series graph for the Dow Jones stock market closing average for the first business day of each month of 2014.

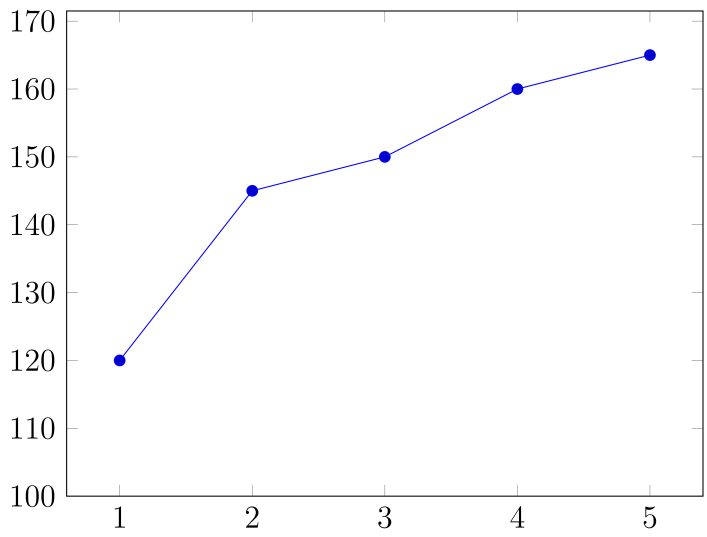


From this graph, we can see a general upward trend (increasing over time), with the sharpest increase from January to February. Notice the vertical axis for the price, does not start at zero. This is because the values are all very large, and starting at zero would collapse it into a small area at the top. It would be hard to see any details.

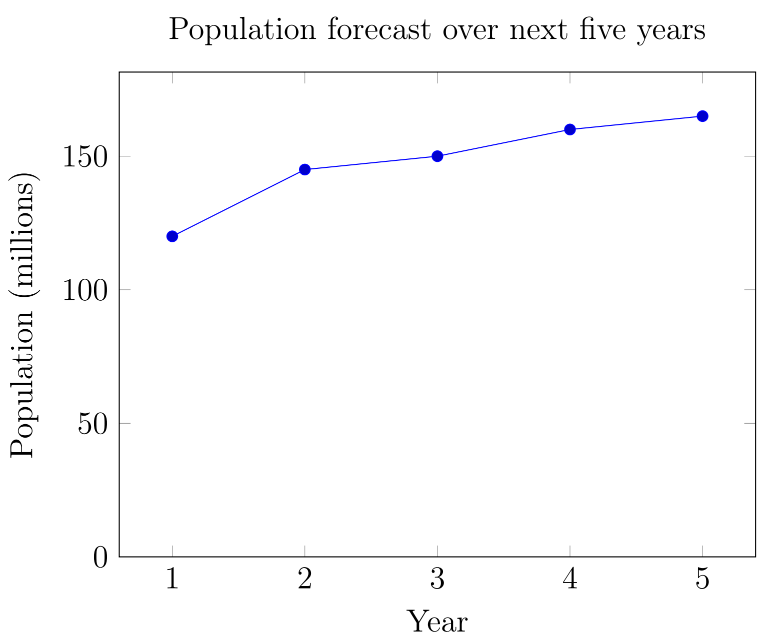
Beware of misleading or bad graphs!! Any data can be shown accurately but with different graphs and seem like it is showing very different results. There are several common ways that graphs can be misleading.

- Starting the vertical axis (values or frequencies) above zero. This chops the bars down and exaggerates the differences. The only exception is for line graphs (time series) with very large values for all of the data, to avoid having the graph squished into a small area.
- Using uneven scales on the axes.
- Using multiple dimensions when the data is just one dimension (using area or volume, when data is only the height for the bars).
- Unclear labelling.
- Too many cosmetic enhancements. This makes the graph hard to read, it is too busy!
- Poor choice of grouping.

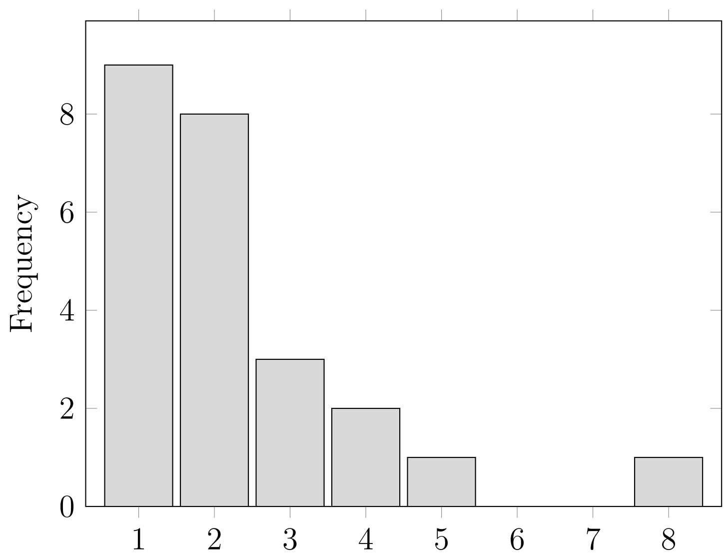
Here is an example of how to create a good or bad graph with the same data. The first graph is bad because it starts the vertical axis at 100, which makes the increase look very steep. It is also not labelled. The second graph shows a better view of the data starting at zero and labelled.



Notice the increase from 1 to 2, looks very steep above, due to the vertical axis being chopped down. Below, the axis starts at zero and shows a less drastic increase. Also the graph below is easier to read with labels.



****Try this on your own**: What are the characteristics of the following graph? Examine the spread, symmetry, and outliers.



2.2.3 Exercises: Summarizing Data

Solutions appear at the end of this textbook.

1. Why should we use only a small number of groups? If a data set has 1000 values, why not use 100 groups?

2. When computing relative frequencies, how do we know that we have calculated them correctly?

3. What is the difference between a bar graph and a Pareto chart? Can a Pareto chart be done from quantitative data? Why or why not?

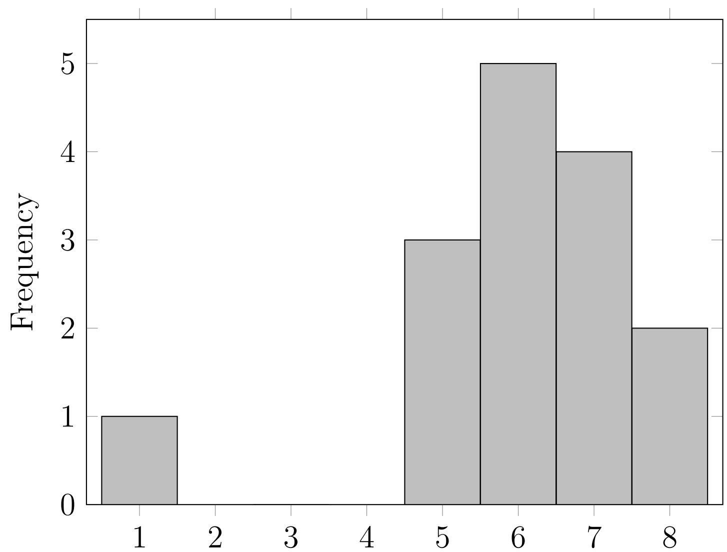
4. Group the following data into classes, calculate the frequencies and relative frequencies. Convert relative frequencies to nearest whole percent. A Farmer kept a log for one month of which days it rained. Here is the order. Tuesday, Saturday, Sunday, Tuesday, Friday, Sunday, Wednesday, Sunday, Friday, Tuesday

5. Make a pie-chart and a bar graph for the distribution you created from the farmer's observations in the previous exercise.

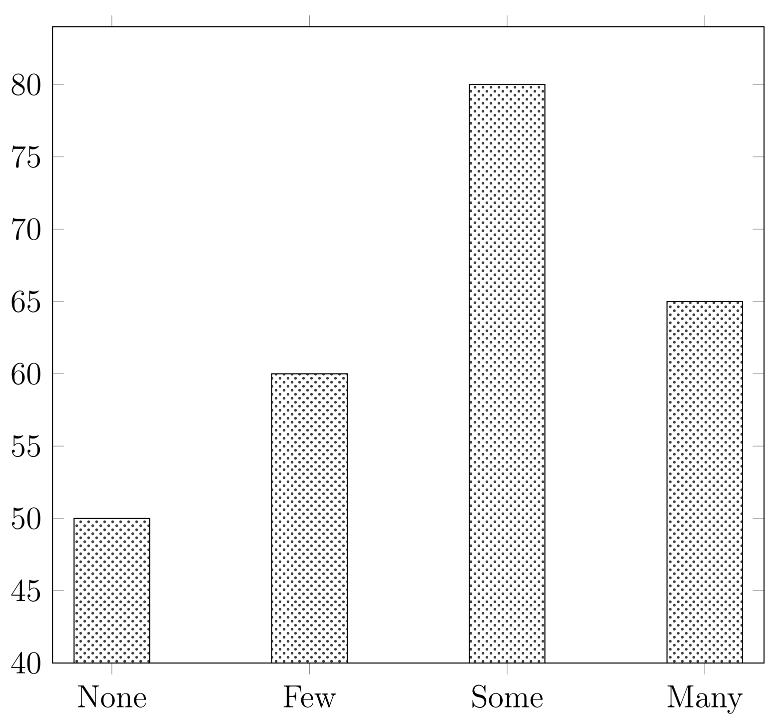
6. Group the following data into classes, calculate the frequencies and relative frequencies. Convert relative frequencies to nearest tenth of a percent. The grades on a history test were as follows: 67, 72, 99, 100, 82, 83, 94, 90, 80, 85, 85, 77, 48, 88, 75, 50, 75, 82, and 95. Use Classes of F, D, C, B, and A.

7. Make a pie-chart and a histogram for the distribution you created from the history grades in the previous exercise.

8. Describe the characteristics of the following graph (center, spread, shape, symmetry, outliers).



9. Describe two issues that are bad or misleading about the following graph.



2.3 Measures of Center

2.3.1 Mean-Median-Mode

In order to use mathematics to measure data sets, there will be several formulas that we will use. We will typically use letters to represent variables (typically X or Y or a letter that has some relevant meaning like S to represent the variable salary).

Particular values (observations) of a variable X can be denoted by subscripts x_1, x_2, etc. For summation of values, we use the Greek capital letter Sigma Σ, with a variable next to it to show which one is being summed. For **Example:** $\sum x$ stands for the sum of the values of the variable X.

The **Round-Off Rule**: Round all calculations to one more decimal place than is present in the data. Round only the final answer, not the steps along the way. For example, if the data set has values that are go out to 2 decimal places, then all calculated statistics should be reported showing 3 decimal places.

Measures of Center (or measures of central tendency) are descriptive measures that indicate where the center or most typical value of a data set lies. Some of the specific measures of center are shown below.

The **Mean** is sum of all the values of the observations, divided by the number of observations. The mean is also more commonly just called average .

The Sample mean is represented by the symbol $\overline{x}$, which is called 'x-bar': $\overline{x} = \dfrac{\sum x}{n}$, where n is the number of values in the sample.

The Population mean is represented by the symbol μ, which is called 'mu': $\mu = \dfrac{\sum x}{N}$, Where N is the number of values in the population.

Both of these are the same procedure and give the average of the values, the only difference is where the values come from, a sample, or the whole population.

Example: In the previous section, we looked at graphs for Bradley's weekly hours at a summer job: 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36. Find the mean (average) hours he worked in a week.

Solution: We add up the values and divide by 12 (the count of how many values). Since this is the entire set of his summer job hours, it is a population mean. $\mu = \frac{\sum x}{N} = \frac{336}{12} = 28$, which we will report, according to the round off rule, as 28.0 hours worked in an average week.

The **Median** is the value that divides the bottom 50% of data from top 50%. It is the middle value when the values are placed in size order. To find the median, first arrange the data in increasing order. If there are an odd number of observations, the median is the middle value in order. If there are an even number of observations, the median is the average of two middle values in order .

Example: Find the median of Bradley's summer weekly hours.

Solution: First we must put the values in order from lowest to highest: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. There are two values in the middle, 30 and 32, with five values below and above. The average of the two middle values is 31, which is the median. Notice that the median is not one of the original data values here. According to the round-off rule, we will report the median as 31.0 hours worked in a week. Half of the data is below this and other half above this.

The **Mode** is the value that has the most number of observations (frequency), but must occur more than once. There can be multiple modes (a tie for the most often).

Example: Find the mode of Bradley's summer weekly hours.

Solution: Having the values in order makes this easier: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. The values that occurs the most often is 36, which is the only mode here.

Here are some cautions about measures of center. The mean is sensitive to extreme values. If a company has 20 workers making \$15,000 each and the owner makes \$500,000, then the mean would be \$38,095. This mean does not give a complete picture of the company salaries. Nobody makes close to that value, everyone except the owner is way below average.

When dealing with salaries or prices, the median is often a better measure of the data set. The median for the company would be \$15,000 and a better representation of what prospective employees could expect to be paid.

Most of the time, data that is left-skewed, will have a mean that is less than the median. For right-skewed, the mean is greater than the median. This is because the skewed data out to the extreme, pulls the mean closer to that extreme, but the middle values are still in place so the median is not affected.

2.3.2 Mean of Frequency Table

If a large data set has already been grouped and you have only the frequency distribution (but not the actual data), the average can be estimated using the midpoint of each class (group) as the estimate of the typical value in each class and multiplying that by the frequency of that class. The formula is: $\frac{\sum \widehat{x} f}{\sum f}$, where $\widehat{x} f$ is the product of each class midpoint $\widehat{x}$, times the class frequency, f.

Example: Estimate the mean of the following frequency distribution.

Class	1-8	9-16	17-24	25-32	33-40
Frequency	3	5	7	2	1

Solution: The midpoints of the classes ($\widehat{x}$) are: $4.5, 12.5, 20.5, 28.5, 36.5$.
Then the formula would be $\frac{\sum \widehat{x} f}{\sum f} = \frac{4.5(3)+12.5(5)+20.5(7)+28.5(2)+36.5(1)}{3+5+7+2+1} = \frac{313}{18} = 17.3889$

According to the round-off rule, we should report this as 17.4. Let's think about the estimate of 17.4, does it make sense? Notice in the frequency distribution, that the class 17-24 occurs the most, with more in the lower classes than in the upper classes. Then the estimate of 17.4, in the beginning of the class 17-24, makes sense.

****Try this on your own**: The grades for a sample of a science final exam were 75, 83, 96, 82, 90, 78, 60, 76, 82, 71, 92, 86, 83, 88. Calculate the mean, median, and mode.

Another special case of finding average is when we know the values, but they are not all equally weighted. This idea is know as a **Weighted Mean**. The formula is $\overline{x} = \frac{\sum xw}{\sum w}$, where xw is the product of each data value x multiplied by its weight w. The weight could be a dollar amount or a credit amount (as in college classes), or other amounts.

Example: If an investment earns 6% interest on \$1,000 and 4% interest on \$250, what is the average interest rate for the entire investment.

Solution: here we are looking for the average interest rate, so the data values are the interest rates. The dollar amounts are the weights. More money, means more weight for its corresponding interest rate. Therefore, the average rate will be closer to 6%, since it corresponds to the biggest dollar amount invested. $\overline{x} = \frac{6(1000)+4(250)}{1000+250} = \frac{7000}{1250} = 5.6$, so the average interest rate on the entire investment is 5.6%.

Example: Rachel's class is graded in the following manner: Test average counts 30%, homework average counts 25%, participation counts 10%, project counts 15%, and final exam counts 20%. What is her overall grade average if she has the following grades:
Test avg 86, HW avg 94, part. 100, project 80, final 82.

Solution: here we are looking for the average grade, so the data values are grades. The percentages are the weights. Test average has the most weight (30%). Class Avg $= \frac{86(30)+94(25)+100(10)+80(15)+82(20)}{30+25+10+15+20} = \frac{8770}{100} = 87.7$, so her overall grade in the class is about 88, a high B. Almost an A, but not quite.

Academic Grade Point Average (or GPA) is a very common concept, but one that most people don't understand how to compute. Letter grades earned in classes are assigned a numerical value called quality points, ranging from 0 to 4. Typical grade scale is $A = 4.0$, $B = 3.0$, $C = 2.0$, $D = 1.0$, and $F = 0$. Some schools have grades in between with a $\pm$, with fractional values. The weights of the grade points are the credit hours for the classes. A longer class for 4 credit hours has more weight than a short elective for 2 credits.

Example: Compute the overall semester GPA for a college student who earned the following grades. Use the standard letter points shown above, assume there are no $\pm$ grades in between.

Course	Grade	Credits
Physics	B=3.0	4
English	C=2.0	3
Math	C=2.0	3
Study Skills	B=3.0	1
History	A=4.0	3

Solution: here we are looking for the average grade points, so the data values are grade (quality) points. The credit amounts are the weights.

$$GPA = \frac{3.0(4) + 2.0(3) + 2.0(3) + 3.0(1) + 4.0(3)}{4 + 3 + 3 + 1 + 3} = \frac{39}{14} = 2.79,$$

which is a high C, close to an overall B. Not the greatest, but it is passing.

2.3.3 Exercises: Measures of Center

Solutions appear at the end of this textbook.

1. For the following data, compute the mean, median, and mode. Use the round-off rule.

   The ages of students in a college class were: 17, 19, 20, 18, 18, 19, 19, 18, 18, 25, and 19.

2. How are the mean, median and mode affected by extreme values?

3. Does every data set have a mode?

4. Estimate the mean of the following frequency distribution.

Class	10-14	15-19	20-24	25-29	30-34	35-39
Frequency	12	5	7	2	6	3

5. Compute the GPA for a college student who earned the following grades.

Course	Grade	Credits
Biology	B=3.0	3
English	C=2.0	3
Math	A=4.0	3
Bio Lab	A=4.0	1
Spanish	A=4.0	3

6. Find the median of the data set: $14, 7, 23, 5, 14, 22, 19, 20, 27, 9$

2.4 Measures of Variation

2.4.1 Range

Measures of Variation (or measures of spread) are descriptive measures that indicate how much variation is in the data or how spread out the data values are from each other.

The **Minimum** (Min) is the lowest value in the data set.

The **Maximum** (Max) is the highest value in the data set.

The **Range** is the difference between Min and Max: ($Range = Max - Min$). To give a more detailed range, some studies will just say the range goes from MIN to MAX.

Example: Find the min, max, and range of Bradley's summer weekly hours.

Solution: Having the values in order makes this easier as well: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. Here $min = 12$, $max = 36$, and $Range = 36 - 12 = 24$ hours. We could also say the range is from 12 to 36 hours.

2.4.2 Standard Deviation

Standard Deviation is the measure of how far, on average, the data is from the mean. Another related measure, is the **Variance** which is standard deviation squared. It will not be used much here, except as a step in the computation of standard deviation.

The standard deviation and variance for a SAMPLE are calculated by the following symbols and formulas:

Variance: $s^2 = \frac{\sum (x - \overline{x})^2}{n-1}$ Standard deviation: $s = \sqrt{\frac{\sum (x - \overline{x})^2}{n-1}}$

The variance and standard deviation for a POPULATION are calculated by the following symbols and formulas:

Variance: $\sigma^2 = \dfrac{\sum (x-\mu)^2}{N}$ Standard deviation: $\sigma = \sqrt{\dfrac{\sum (x-\mu)^2}{N}}$

Example: Find the standard deviation of the sample data set $10, 50, 10, 30, 30$.

Solution: To work out standard deviation yourself, it is very important to work out each step carefully and organized. The data set is a sample, so we will use the formulas with s. First we find the mean $\overline{x} = 26$ and the count $n = 5$ from the data. Then it helps to make a table like the one below.

x (hours)	$x - \overline{x}$	$(x - \overline{x})^2$
10	$10 - 26 = -16$	$(-16)^2 = 256$
10	$10 - 26 = -16$	$(-16)^2 = 256$
30	$30 - 24 = 4$	$4^2 = 16$
30	$30 - 26 = 4$	$4^2 = 16$
50	$50 - 26 = 24$	$24^2 = 576$
Sum		1120

The standard deviation, $s = \sqrt{\dfrac{\sum (x-\overline{x})^2}{n-1}} = \sqrt{\dfrac{1120}{4}} = \sqrt{280} = 16.7332$

Using the round-off rule, $s = 16.7$ hours. Computing the standard deviation can be done very quickly and easily using special calculator functions or computers, but interpreting the meaning of it requires human understanding. In this example, it means that the data values are somewhat spread out, on average the data values are 16.7 away from the mean of 26 Sometimes they were close to 26 (the 30's are only 4 away), and sometimes farther away (50 is 24 units away).

2.4.3 Coefficient of Variation

If we wish to compare two data sets, to figure out which is spread out more, there are two cases to consider. First, if the data sets are from similar variables with similar sizes, then we can directly compare the standard deviations, since they are the same units. As an example, we already discussed Bradley's hours versus his fellow employee.

The other case is when comparing two very different data sets. For that we will use a special measure called the **Coefficient of Variation** (or CV). It is equal to the standard deviation divided by the mean, converted into a percent. It has no units, it is only a ratio as percent. The formulas are slightly different, depending upon the data set being from a sample or a population. The CV states how big the standard deviation is, relative to the average size of the data.

For a sample: $CV = \frac{s}{\overline{x}} \times 100\%$ For a population: $CV = \frac{\sigma}{\mu} \times 100\%$

Example: Which data set is more spread out, the weight of elephants in a herd: $s = 1,175$ pounds and $\overline{x} = 12,342$ pounds, or the price of regular unleaded gasoline in a US: $s = \$0.26$ and $\overline{x} = \$3.73$?

Solution: For the elephant weights, $CV = \frac{1,175}{12,342} \times 100\% = 9.5\%$. For the gas prices, $CV = \frac{0.26}{3.73} \times 100\% = 7.0\%$. The weights of elephants are a more spread out set of data than US gas prices. This is NOT because the values are larger. Another set of large values could have a lower CV than gas prices.

****Try this on your own**: A runner kept a record of their miles ran for the past 6 training sessions. The miles were 9, 3, 6, 8, 6, 10. Calculate the mean, range, and standard deviation of this sample.

2.4.4 Exercises: Measures of Variation

Solutions appear at the end of this textbook.

1. For the following data, compute the min, max, and range. The ages of students in a college class were: 17, 19, 20, 18, 18, 19, 19, 18, 18, 25, and 19.

2. The range of a data set is 83. The maximum value is 124, what is the minimum value?

3. The Bronson family are having a reunion celebration. The ages of the five members at one table are 10, 25, 40, 50, 75. Compute the variance and standard deviation, using the sample formulas for s^2 and s.

4. Which data set is more spread out, the shot put throws for a high school track team: $s = 5.5$ feet and $\overline{x} = 38$ feet, or the gymnastics scores for a college team: $s = 1.4$ points and $\overline{x} = 8.45$ points?

5. All of the tigers at the zoo were weighed. Their weights in pounds are 350, 420, 400, 520, 570, and 650. Compute the variance and standard deviation, using the population formulas.

2.5 The Normal Distribution

2.5.1 Z-scores

Many variables in the real world fit a special pattern called a **Normal Distribution**. When the data is grouped and shown on a histogram, the graph is symmetric and bell shaped. Most of the values fall in the middle, close to the mean. The frequencies get smaller as the values go further out, with only a few values on the extremes.

There are many real world phenomena that follow a normal distribution, such as heights of adult males in any given area, heights of adult females, weights of babies born full term (not premature), IQ scores, and many others. Since all of these variables have very different values and units, it is hard to analyze them directly.

Luckily, all variables can be converted to a standard variable called the z-score. This allows all of the different normal variables to be converted into one universal bell shape distribution called the **Standard Normal Distribution**.

A **z-score** (or standardized score) is the number of standard deviations that a given value is above or below its mean. Whenever a value is below the mean, its corresponding z-score will be negative. Usual values are z-scores from -2 to $+2$. Unusual values are z-scores outside this range. More than 3 standard deviations away from the mean is very unusual for most data sets. Z-scores have no units.

Z-scores are found by the following formulas and are generally rounded to two decimal places.

$$\text{For a sample: } z = \frac{x - \overline{x}}{s} \qquad \text{For a population: } z = \frac{x - \mu}{\sigma}$$

Example: A scientist took a sample of tree heights in a forest. His results were $s = 1.8$ meters and $\overline{x} = 9.1$ meters. Calculate the z-score for a tree that is 14 meters tall, and explain what the value means.

Solution: $z = \dfrac{14 - 9.1}{1.8} = 2.72$. This means that the tree is 2.72 standard deviations above average, it falls into the category of unusually tall (above 2), but not extreme (more than 3).

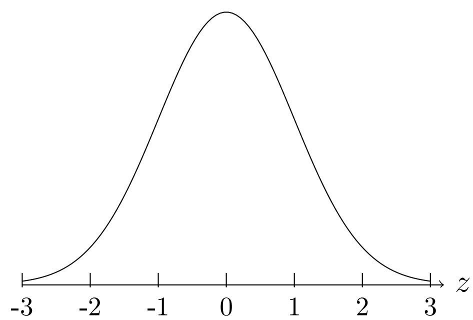
Example: Standardized IQ test scores, for the overall population, have $\sigma = 15$ points and $\mu = 100$ points. Andy Warhol was a famous artist and leader of the 70's pop art movement. It is reported that his IQ was only 86. Calculate his z-score and explain what the value means.

Solution: $z = \dfrac{86 - 100}{15} = -0.93$. This means that his IQ is about 1 standard deviation below average, it falls just slightly below average into the category of usual intelligence (between -2 and 2).

****Try this on your own**: Calculate the z-score of a woman who is 5 feet tall if the mean height is 65 inches and standard deviation is 3 inches. Is she unusually short or not? Round Z to two decimal places.

The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$, which makes it very easy to work with.

What makes the standard normal distribution so nice to work with, is that probabilities directly correspond to area under the curve. The total area under the curve is equal to 1 or 100%. The standard normal distribution graph is shown below. Notice that a large part of the graph in concentrated between $z = -1$ and $z = +1$, with most of the graph between $z = -2$ and $z = +2$ (usual values), and just about all of the graph between $z = -3$ and $z = +3$. The graph technically goes out forever, but it gets so low that effectively there is not much outside of ± 3.



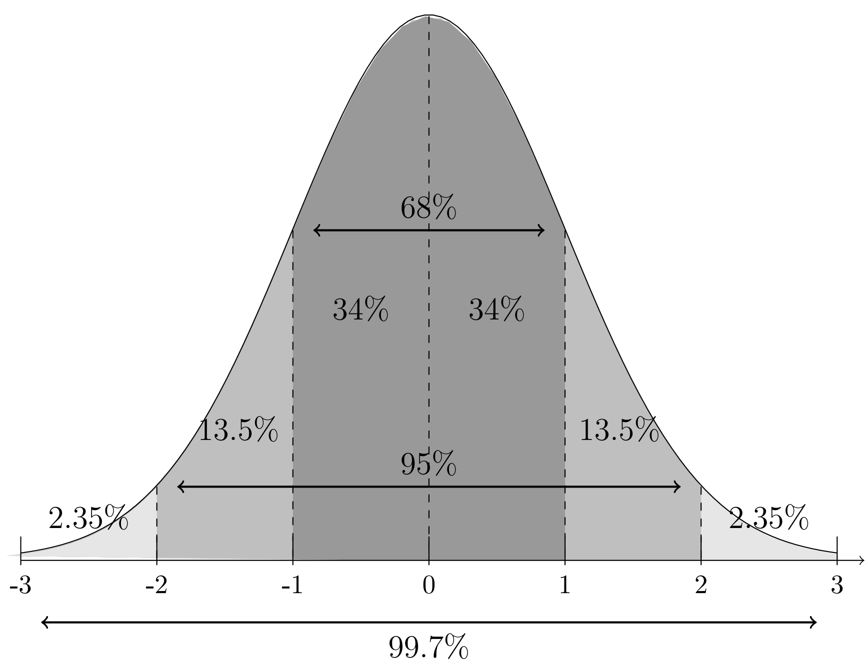
There are two methods to calculating probabilities for areas under the bell curve. One is very quick and easy, but limited to intervals between the whole values of z-scores and only requires the use of a general rule. The other is more detailed but requires formulas, tables, and/or technology. This textbook and course will focus on the general method only. The more detailed method is used in other courses such as Statistics.

2.5.2 The 68-95-99.7% Rule

The quick and easy method is to use the **The Empirical Rule** which is stated below.

- 68% of the data will be located within one standard deviation to either side of the mean
- 95% of the data will be located within two standard deviations to either side of the mean
- 99.7% of the data will be located within three standard deviations to either side of the mean

The Empirical Rule is also known as the **69-95-99.7 Rule**. This scope of this rule can be best shown with the following diagram.



Since the bell curve is symmetric, each side is a mirror image of equal size. The percentage for one side, from 0 on up or up to 0, is exactly half of the graph, so 50%. Each of the intervals mentioned above between $\pm$ whole units, are split down the middle. That is how we know that the two sections in the middle (darkest) are 34% each, half of 68.

The next tier on either side is 13.5% each. This is found by taking difference between $95 - 68 = 27\%$ and dividing by two, since there is a section on either side. The two small slices on the ends (between 2 and 3) are 2.35% each. This is found by taking difference between $99.7 - 95 = 4.7\%$ and dividing by two.

To solve problems about area/probability of certain intervals, just add up or subtract the appropriate sections.

Example: What percent of the bell curve lies between $z = -1$ and $z = +2$?

Solution: Between -1 and 1 is 68%, with additional area of 13.5% between 1 and 2. This results in $68 + 13.5 = 81.5\%$

Example: What percent of the bell curve lies between $z = -2$ and $z = 0$?

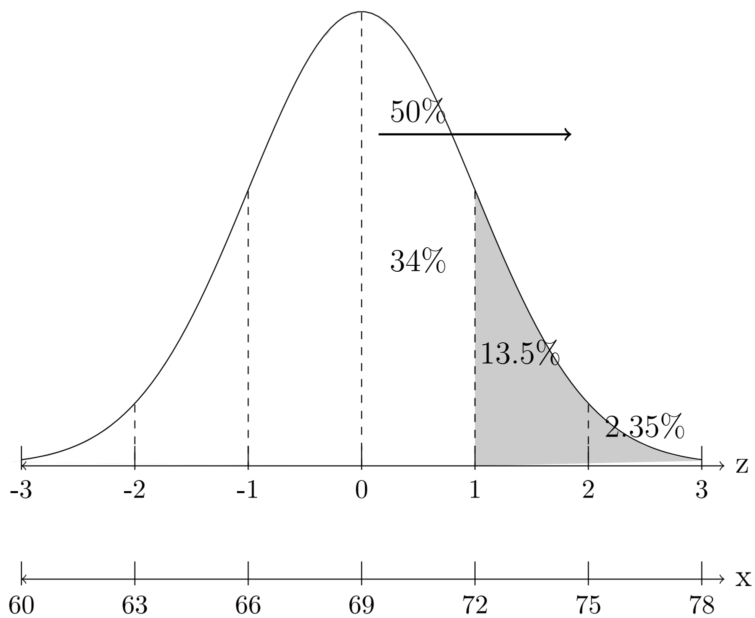
Solution: Between -2 and -1 is 13.5%, with additional area of 34% between -1 and 0. This results in $34 + 13.5 = 47.5\%$

This graph and the **69-95-99.7 Rule** can be applied to real world data that follow a bell curve. When working problems applied to specific data (such as heights, IQ scores, etc.), we can match the data values with z-scores on the graph. Since the z-scores are actually the number of standard deviations from the mean, then we can lineup the data mean below $z = 0$, then add/subtract the data standard deviation to put data values in line with the z-score units from -3 to 3.

Example: Adult male heights are normally distributed (follow bell curve), with a mean of $\mu = 69$ inches and a standard deviation of $\sigma = 3$ inches. What percent of men are taller than 6 feet?

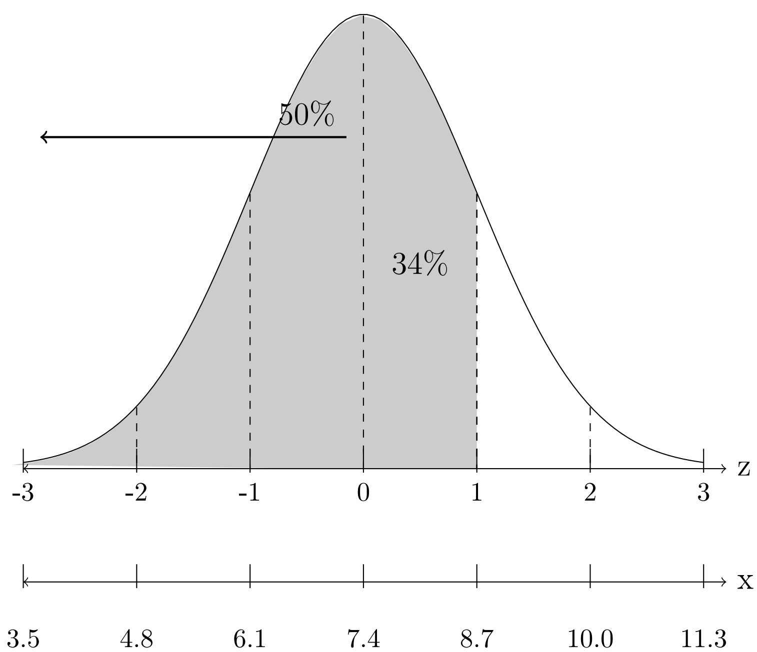
Solution: Draw a bell curve with standard z-axis from -3 to 3 and below that, an x-axis with heights that correspond to the z marks. The mean height of 69 will go below $z = 0$, one standard deviation higher ($69+3 = 72$ inches) will go below $z = 1$, two deviations higher, 75, goes below $z = 2$, and 78 below $z = 3$. Do similar process on left side, subtracting standard deviation to go under the negative z-values.

Change 6 feet into 72 inches. So we are looking for the slice of the graph above 72, which is above $z = 1$. The area is the upper half (50%) minus the section from $z = 0$ to 1 of 34%. The answer is $50 - 34 = 16\%$. Notice that this is also the same as adding the pieces to the right $13.5 + 2.35 = 15.85\%$, where the extra bit to make the full 16% is the tiny slice after $z = 3$. The graph is shown below.



Example: The birth weights of babies in the USA are normally distributed, with a mean of $\mu = 7.4$ pounds and a standard deviation of $\sigma = 1.3$ pounds. What percentage of babies are born with a weight less than 8.7 pounds?

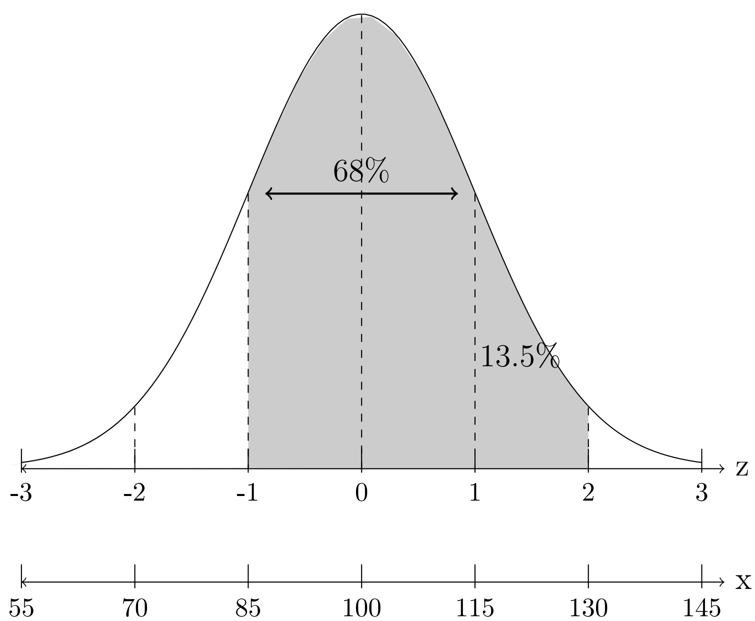
Solution: Draw a bell curve with standard z-axis from -3 to 3 and below that, an x-axis with weights that correspond to the z marks. The mean weight of 7.4 will go below $z = 0$, one standard deviation higher (8.7 pounds) will go below $z = 1$, etc. So we are looking for the slice of the graph below 8.7, which is above $z = 1$. The area is the lower half (50%) plus the section from $z = 0$ to 1 of 34%. The answer is $50 + 34 = 84\%$. The graph is shown below.



Example: IQ test scores are normally distributed, with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$. What percent of the population have IQ scores between 85 and 130?

Solution: Draw a bell curve with standard z-axis from -3 to 3 and below that, an x-axis with IQ scores that correspond to the z marks. The mean of 100 will go below $z = 0$, one standard deviation higher $(100 + 15 = 115$ inches) will go below $z = 1$, etc. Do similar process on left side, subtracting standard deviation to go under the negative z-values.

We are looking for the slice of the graph from 85 to 130, which are z-scores from -1 to $+2$. The area is the middle (68%) plus the section from $z = 1$ to 2 of 13.5%. The answer is $68 + 13.5 = 81.5\%$. The graph is shown below.



****Try this on your own** The birth weights of babies in South America are normally distributed, with a mean of $\mu = 3,100$ grams and a standard deviation of $\sigma = 400$ grams. Find the percentage of babies born with a weight more than $2,300$ grams.

2.5.3 Exercises: The Normal Distribution

Solutions appear at the end of this textbook.

1. Using the 68-95-99.7 rule, compute the area of a bell curve between $z = -1$ and $z = 3$.
2. Using the 68-95-99.7 rule, compute the area of a bell curve between $z = -3$ and $z = -1$.
3. SAT test scores for English are normally distributed, with a mean of $\mu = 500$ and a standard deviation of $\sigma = 100$. Match the whole z-scores on the bell curve with SAT test scores.
4. Adult female heights are normally distributed, with a mean of $\mu = 64$ inches and a standard deviation of $\sigma = 2.5$ inches. What percent of women are shorter than 5 feet 1-1/2 inches?
5. The birth weights of hospital born babies in Pakistan are normally distributed, with a mean of $\mu = 2.9$ kg and a standard deviation of $\sigma = 0.5$ kg. Find the percentage of babies being born with a weight greater than 3.9 kg.
6. SAT test scores for Math are normally distributed, with a mean of $\mu = 480$ and a standard deviation of $\sigma = 110$. What percent of scores are between 260 and 480?

Chapter 3

Probability

Written by James Bellon

3.1 Probability Basics

3.1.1 Key Terms

In the first chapter, we learned about descriptive statistical methods for summarizing and displaying data, and descriptive measures (mean, variance, percentiles, etc.). In most studies, information about the population is the goal, but data is usually only collected from a sample, due to a census being too difficult to do. Even for single observations, we would like to know if and when something will happen. This leads us to the mathematics behind uncertainty. The science of uncertainty is called **Probability**. There are some important terms we need to know.

An **Experiment** is an action or procedure whose outcome cannot be predicted with certainty.

An **Event** is some specified result that may or may not occur when an action or procedure is performed. For example, a roll of a 6-sided die is an action. Getting a result of an even number is an event. An event is a collection of results or outcomes. In the die rolling example, the event of "even" is the collection of outcomes of the numbers 2, 4, or 6. An outcome that cannot be further broken down into components is called a **Simple Event**.

The **Sample Space** is the set of all possible outcomes of an experiment. The number of possible outcomes for a sample space is denoted by the capital letter N. All events are subsets of the sample space.

Generally we assign capital letters to represent events (A, B, C, etc.). The letter can be something meaningful such as E for even. We might assign letters in the following way. Let M = the event that someone chosen from the class is male. Let F = the event that someone chosen from the class is female. Now we can refer to M and F instead of writing out the events.

The probability value (or just probability) of an event, is the chance that the event will happen, relative to all of the possible outcomes. There are three basic properties for probability values:

1. The probability of an event is always between 0 and 1, inclusive (or 0% to 100%).
2. The probability of an event that cannot occur is equal to 0 (the event is said to be impossible).
3. The probability of an event that must occur is equal to 1 (the event is said to be certain).

When referring to the probability of an event, the capital letter P is used with the associated event (or its assigned letter) inside parentheses and placed next to the capital P. For example the probability of event male would be shown as $P(M)$.

There are three commonly used ways to calculate a probability value. The particular situation and the level of detail desired, determines which way to use.

3.1.2 Calculating Probability

Rule #1 for calculating a probability is the **Empirical Probability**. The probability is based on the actual results observed for some number of trials of an experiment. This is similar to relative frequency. The formula for the empirical probability of event A is $P(A) = \frac{f}{n}$, where f is the number of times the event occurred (like frequency) and n is the total number of trials of the experiment.

Example: If a women makes 14 out of 20 free-throws in her WNBA basketball tryout, what is her empirical probability of making a free-throw?

Solution: Here $n = 20$ and the number of shots she made was $f = 14$. Let F be the event of making a free-throw. Then $P(F) = \frac{14}{20} = 0.7 = 70\%$. Assuming this is what she normally does, then for the near future, her probability of making a free-throw can be estimated as 70%. If she makes the team and practices, the probability will hopefully go up.

Rule #2 for calculating a probability is the **Theoretical Probability**. This is a logical approach, but only applies to equally likely outcomes. The formula for the theoretical probability of event A is $P(A) = \frac{e}{N}$, where e is the number of possible outcomes that fall under the event and N is the total number of possible outcomes in the sample space.

Example: A father buys two raffle tickets for his son's baseball team raffle. There were 80 tickets sold, and one will be picked out of a hat as the winner. What is the theoretical probability of the father winning the raffle?

Solution: Here the number of possible tickets that could be picked is $N = 80$ and the number of tickets the father has is $e = 2$. Let W be the event of the father winning. Then

$P(W) = \frac{2}{80} = 0.025 = 2.5\%$. Notice this probability was calculated BEFORE the ticket was picked. Theoretical can be reasoned out based on logic, but only when outcomes are equally likely, such as picking a ticket out of a hat.

The empirical probability of the free-throws in the previous example could not be reasoned out before we had the data of the woman's attempts. Also, making or missing the free-throw are not equally likely for most people, so empirical probability was done for that and not theoretical.

Rule #3 for calculating a probability is the **Subjective Probability**. Subjective probability is estimated by using personal knowledge or experience. It is not scientific nor mathematical and rarely logical. For example, if a young teen sneaks out of his house twice, late at night, without getting caught, he might assume his chances of getting caught are very low. Another example of this is betting on your favorite team (or against the opposing team). Most people bet with emotion and not logic.

****Try this on your own**: In each situation below, calculate the probability, deciding whether to use empirical or theoretical probability.

1. 200 people are at a banquet and 8 people are at your table including you. What is the probability that someone at your table is chosen at random to win a prize out of the entire banquet?

2. Danny has played 20 tennis matches this season and has won 17 of them. What is the probability that he wins his next match?

Sometimes rule #1 and #2 give about the same value, especially in the long run. This is known as the **Law of Large Numbers**, which states that as an experiment is repeated again and again, the empirical (relative frequency) probability of an event TENDS to approach the theoretical probability. For example, the theoretical probability of flipping a coin and getting the result 'Tails' is $\frac{1}{2}$ or 50% (it is one out of two equal outcomes). If you flip a coin once, the outcome will either be 100% heads or 100% tails, never 50%, but as you flip a coin many times, most likely it will be close to even for number of heads and tails.

Here is an experiment for you to try, that will demonstrate the Law of Large Numbers. Find any coin that you can distinguish one side as heads and the other as tails. Make a table on paper with 5 columns: Flips, heads, P(H), tails, P(T). For each step, fill in the values in each column, along a row and then go to the next step/row.

In step one, flip the coin five times and in the first row, put 5 under flips. Under heads, write down however many heads came up for you and same for tails. Calculate $P(H) = \frac{heads}{5}$ and $P(T) = \frac{tails}{5}$, multiplying them by 100 and rounding to nearest whole percent. For example, if you get 3 heads and 2 tails, then $P(H) = \frac{3}{5} \times 100 = 60\%$, etc. With five flips, it will be impossible to get 50%, and you could easily get values far from it.

In step two flip the coin 25 times and in the second row, put 25 under flips. Under heads, write down however many heads came up for you and same for tails. Calculate $P(H) = \frac{heads}{25}$ and $P(T) = \frac{tails}{25}$, multiplying them by 100 and rounding to nearest whole percent. For example, if you get 11 heads and 14 tails, then $P(H) = \frac{11}{25} \times 100 = 44\%$, etc. With twenty-five flips, it will be impossible to get exactly 50%, but the values are most likely somewhat close. Since this is random, it could be farther away from 50% that in the first step, but unlikely.

In step three flip the coin 150 times and in the second row, put 150 under flips. Under heads, write down however many heads came up for you and same for tails. Calculate $P(H) = \frac{heads}{150}$ and $P(T) = \frac{tails}{150}$, multiplying them by 100 and rounding to nearest whole percent. For example, if you get 68 heads and 82 tails, then $P(H) = \frac{68}{150} \times 100 = 45\%$, etc. With 150 flips, it is possible to get exactly 50%, but the values are more likely to be somewhat close and probably closer that in the previous steps.

In rare cases, your values may have gotten further away instead of approaching 50%. That is why the Law states the values TEND to approach, but anything could happen. If many people did this experiment, most of them would see the values get closer to 50%. You may or may not have seen this happen. If not, try it again and I'm pretty sure it will work this time.

3.1.3 Complements

The **Complement** of an event A consists of all outcomes in which event A does NOT occur. There are several ways to show the complement in symbolic form, A', $\overline{A}$, A^c and A. In this book and course we will use A^c. Since every outcome must either belong to set A or its complement, there is a special relationship for the probabilities. The **Complement Rule** is stated as a formula: $P(A) + P(A^c) = 1$ or 100%. So once you know the probability value of one, the other is automatically determined.

Example: A bag contains 29 marbles: 3 blue, 4 red, 6 yellow, 7 orange, 4 brown, and 5 green marbles. Calculate the theoretical probability of picking a marble that is not blue. Then use the complement rule to find the same probability and compare these.

Solution: $P(\text{not blue}) = \dfrac{\#\text{not blue}}{total} = \dfrac{4+6+7+4+5}{29} = 0.897 = 90\%$
Using complement rule $P(B^c) = 1 - P(B) = 1 - \frac{3}{29} = \frac{26}{29} = 0.897 = 90\%$, which is the same value as it should be, since this is the same event.

****Try this on your own**: If a team has a $\frac{1}{3}$ chance of making the playoffs, find the probability they do not make the playoffs.

3.1.4 Exercises: Probability Basics

Solutions appear at the end of this textbook.

1. List the sample spaces for the following experiments:

   (a) Flipping a coin once

   (b) Flipping a coin 3 times

   (c) Rolling a 6-sided die once

   (d) Rolling two 6-sided dice and computing the sum of the dice

   (e) Randomly picking a color of the rainbow and a season of the year

2. Which of the following are valid values for a probability?

$$0.35,\ 0.004,\ 1.23,\ 213\%,\ -0.25,\ \tfrac{3}{8},\ \tfrac{8}{3}$$

3. A student is about to roll a 6-sided die. Find the following theoretical probabilities. P(even), P(3), P(>2).

4. John is trying out for the basketball team. He shoots from the foul line 12 times. His results are: make, miss, miss, make, miss, make, miss, miss, miss, make, make, miss. What is the empirical probability of John making a shot from the foul line? What is your subjective probability of John making it onto the team?

5. If the probabilities for the types of precipitation are as follows, what is the probability of no precipitation? P(rain) = 0.20, P(snow) = 0.10, P(sleet) = 0.15.

6. Find a 6-sided die. Roll it 15 times and compute the empirical probabilities of rolling each number (1 to 6). Then compute the theoretical probabilities of rolling each number. Compare the empirical to the theoretical. How different are they? Explain why or not.

3.2 Counting Rules

3.2.1 The Fundamental Counting Principle

You have been counting since you were a toddler. It is pretty easy to do, or so you thought! Counting can get very complicated, especially if you had to count how many different combinations of clothing you could put together from 12 shirts, 8 pants, 3 belts, and 6 pair of shoes. Believe it or not, there are actually $1,728$ different combinations of clothing (choosing one of each type). The long way to reach that value is to count each combination one at a time. The short way is to use some really cool math rules for counting. There are some advanced rules, but in this course we will only focus on the most basic rule.

The **Fundamental Counting Principle** states that when picking one item each, from several groups of items, the total number of combinations of those items is equal to the product of how many items are in each type. In simple terms, multiply the number of items of the first type by the number of items of the second type, etc.

For the clothing example, the answer was calculated using the Fundamental Counting Principle, $12 \times 8 \times 3 \times 6 = 1,728$.

Example: A student is considering taking four classes out of the following subjects this semester: Math, Physics, Literature, Economics, Psychology, and Music. How many different sets of four classes are possible?

Solution: There are 6 courses to choose from for their first class, then 5 courses remaining to pick a second class. After that there are 4 and 3 courses to pick from. $6 \times 5 \times 4 \times 3 = 360$ possible schedules.

Example: If a furniture shop is going to paint three chairs and they can select from 8 different colors, How many different ways can the chairs be painted assuming that colors can be used more than once?

Solution: There are 8 colors to choose from for their first chair, but also 8 colors for the second and third. $8 \times 8 \times 8 = 512$ possible combinations of chair colors. A few examples are red-blue-white, red-red-red, and white-yellow-white.

Example: A popular type of lottery game is where players choose a sequence of three digits from. To win the main prize, players must match the exact order of the winning sequence of digits, for example 048 or 993. How many different sequences of numbers can be played? What is the probability of winning the main prize?

Solution: For each digit, there are ten possible choices, 0 through 9. The number of different sequences of numbers that can be played are $10 \times 10 \times 10 = 1,000$. Only one exact sequence is the winning set, so the probability is $\frac{1}{1,000}$

****Try this on your own**: If a room is being decorated by choosing one of five colors for the walls, one of three choices of carpet, and one of four furniture sets, how many ways are there to create a look for the room?

3.2.2 Combinations and Permutations

For any number of items selected from a total, the number of possible **Combinations** for a selection of r elements drawn from a population of N elements, is found using the formula ${}_N C_r = \dfrac{N!}{r!(N-r)!}$, where the exclamation symbol, "!", is the factorial symbol. To make sure you understand the factorial notation, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. Factorial is a whole number multiplied by all whole numbers going down to 1.

Example: Compute ${}_{14}C_6$

Solution: ${}_{14}C_6 = \frac{14!}{6!(14-6)!} = \frac{14!}{6!(8!)} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$
Now notice that we can reduce and simplify this. The numbers 8 down to 1 on the top, will cancel off the 8 to 1 on the bottom. So the combinations $= \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
To even further simplify, we could cancel 10 and 12 on top, using the 4,3,5, and 2 on bottom. This leaves $\frac{14 \cdot 13 \cdot 11 \cdot 9}{6 \cdot 1} = \frac{18018}{6} = 3003$. So there are 3,003 different combinations of 6 elements chosen from just 14. I don't know about you, but this seems crazy even though I know it is the correct value. It is amazing how large combinations can get, just picking from a small amount like 14.

Example: A student is considering taking the following subjects this semester: Math, Physics, Literature, Economics, Psychology, and Music. How many different 4 course combinations can this student possibly take?

Solution: The student must pick 4 out of 6 courses.

$${}_6C_4 = \frac{6!}{4!(6-4)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1(2 \cdot 1)} = 15 \text{ possible schedules.}$$

A **Permutation** is an ordering of r elements selected from a set of N distinct elements. The elements selected will be in r positions. For example selecting first, second, third in a talent contest. The number of permutations of r objects chosen from a possible N is found using the formula ${}_NP_r = \frac{N!}{(N-r)!}$. Here, a different ordering of the same picks, is considered a different permutation.

Example: How many different permutations can be selected from a group of 10 paintings, if the judges must select a first, second, and third place award?

Solution: There are three awards, so ${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = 720$ possible ordering of three paintings for the awards.

There are some special cases of the counting rules.

1. It is necessary for mathematicians to define $0! = 1$. This makes sense in the context of the next special case, or else the formula would lead to no answer.

2. There is only one way to select no items from a group of items, that is to not select at all. As formulas: ${}_NP_0 = 1$ and ${}_NC_0 = 1$. Since nothing is selected, there is no difference between combination or permutation (no ordering possible).

3. There are as many ways to select one item, as there are items, ${}_NP_1 = N$ and ${}_NC_1 = N$.

4. The number of ways to select all items from a group of items is just to take them all, so only one combination is possible ${}_NC_N = 1$.

5. There are many ways to create an ordering of all items, ${}_NP_N = N!$.

6. Selecting some items, is the same as leaving behind the others, so ${}_NC_r = {}_NC_{N-r}$. For example, selecting 5 out of 12 items is the same as leaving behind the other 7. Either way you have separated the items into two groups, one with 5 and the other 7. Notice that ${}_{12}C_5 = {}_{12}C_7 = 792$

Sometimes when working on word problems, it can be difficult to know if it involves combinations or permutations. The way to tell is if there is any wording that denotes some specific order of the elements. Such as award place, rank of officers (president, VP, secretary), or order based on timing (eat a donut first, then a muffin next, etc.).

If you forget which formula to use, simply calculate them both. There are always more permutations than combinations, so the larger answer comes from permutation formula.

This is easy to see for a simple case of picking 3 letters out of A, B, C, D. There are only 4 combinations (leave any one of the letters out). However, there are 24 permutations, since ACB and ABC are the same combination, but different ordering.

3.2.3 Probability using Counting Rules

Counting rules are useful for finding probabilities when there are a large number of outcomes. A common situation is a lottery game. **Example**: There is a popular lottery ticket game called Lotto. In each play, you choose 6 different numbers from 1 to 59. To win the big jackpot, all 6 of your numbers must match the winning combination. How many different combinations of 6 numbers can be played? What is the probability of winning the jackpot with each play? What are the odds against winning?

Solution: The number of different plays of 6 numbers out of 59, is ${}_{59}C_6 = 45,057,474$ (over 45 million!!). Only one combination is a winner. The probability of winning $P(Win) = \frac{1}{45057474} = 0.0000000222$ (extremely small!). By the complement rule, probability of not winning the jackpot is 0.9999998778 (very large!). The odds against winning are the ratio $\frac{\frac{45057473}{45057474}}{\frac{1}{45057474}} =$ 45,057,473 to 1.

NOTE: most lottery rules will incorrectly use the word odds when they are stating the probability and vice-versa. Since the probabilities are so small, there is not much difference, and most people don't understand the difference, so they don't bother to be mathematically correct.

****Try this on your own**: For a lottery in which you pick five numbers from 1 to 50, how many different sets can you pick if they can be in any order, and if they must be in a specific order?

3.2.4 Exercises: Counting Rules

Solutions appear at the end of this textbook

1. How many different outfits can you put together, if you get to pick one of five shirts, one of four pair of pants, and one of three belts?

2. How many different movie sequences can three friends watch, if they have 10 movies and only enough time to watch 3 of them?

3. A restaurant offers 10 entree choices and 5 side dishes, with a choice of dessert. If there 200 possible meal options, how many desserts are there to choose from?

4. What is the probability of winning a lottery by matching five digits in exact order?

5. Four friends Mike, Jose, Kenji, and Hakim, are running a sprint race against each other. What is the probability that Kenji wins?

6. Compute the following: ${}_8C_3,\ {}_{11}C_9,\ {}_7P_4,\ {}_8P_8,\ {}_5C_1$

7. The Fantasy 5 Lottery game consists of picking 5 different numbers from 1 to 39. How many sets of 5 numbers can be played?

3.3 More Probability

3.3.1 Compound Events

What chance did you think you had of being done with probability? Well sorry, there is plenty more. Some people actually get a PhD doctoral degree in probability theory. We won't go that far here, but we will look at the next level of concepts.

Events that cannot occur at the same time for one outcome of an experiment, are called **Disjoint Events** or **Mutually Exclusive**. For example, when rolling a 6-sided die, the events are rolling a 3 and rolling a 4 are mutually exclusive, because you cannot roll two numbers on one roll. You could roll a 3, then roll a 4 on the next roll, but they cannot both occur for the same roll.

Since an event and its complement never have any outcomes in common, it should be clear that complementary events are mutually exclusive. When you take a test, you either pass or fail, you can't do both at the same time. An example of events that can happen at the same time are passing a test and getting an A on the test. They are not the same, but they do share outcomes (scores of 90+). Actually, getting an A is a subset of passing.

Here is where the concepts from sets and Venn diagrams will come into play. Since events are sets of outcomes, we can combine events to get compound events with intersections and unions.

The Intersection of two events, is the set of outcomes that are part of the first event AND part of the second event at the same time. It is the set of outcomes they share in common. The symbol for intersection is $\bigcap$.

The Union of two events, is the set of outcomes that are part of the first event OR part of the second event (or both). It is the set of outcomes from both combined into one larger set. The symbol for union is $\bigcup$.

First let's look at probabilities of compound events, from a logical or reasoning perspective. We can find $P(A \textit{ or } B)$ if we know the individual outcomes in each event (not just the probability values). We find the sum of the number of outcomes from A, and the number of outcomes from B, in such a way that every outcome is counted only once. Then divide this sum by the total number of outcomes in the sample space.

In a similar way, we can find $P(A \textit{ and } B)$ by finding the number of outcomes that A and B both share in common. Then divide this sum by the total number of outcomes in the sample space.

Example: A class consists of 14 boys (8 are juniors, 6 are seniors) and 12 girls (8 are juniors, 4 are seniors). If one student is to be selected at random to come up to the board, find the following probabilities: P(boy $\bigcup$ junior) and P(girl $\bigcap$ senior).

Solution: The event boy $\bigcup$ junior is the same as boy OR junior. There are 26 students total. There are 14 boys and 16 juniors which equals 30??? How can that be? Remember that 8 of the juniors are also boys, so when we count the students for our compound event, we should only count the 14 boys (which include 8 juniors) and then the other 8 juniors (girls) to get 22 students who are boys or juniors. Now probability is $\frac{22}{26} = \frac{11}{13} = 0.846 = 84.6\%$.

The event girl $\bigcap$ senior is the same as girl AND senior. There are 26 students total. There are 4 girls who are seniors. The probability P(girl $\bigcap$ senior)$= \frac{4}{26} = \frac{2}{13} = 0.154 = 15.4\%$.

If we have the probabilities of each event, then we can find the probabilities of compound events using formulas.

The **Addition Rule** states $P(A \textit{ or } B) = P(A) + P(B) - P(A \textit{ and } B)$ or $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$, where $P(A \textit{ or } B)$ is the probability at least one of the events occur in an outcome, and $P(A \textit{ and } B)$ is the probability that both A and B occur at the same time in an outcome.

Example: If the probability of rain today is 0.7, the probability you forget your umbrella is 0.4, and the probability they both happen together is 0.3, what is the probability that it rains or you forget your umbrella?

Solution: P(rain or forget) $= P(rain) + P(forget) - P(both) = 0.7 + 0.4 - 0.3 = 0.8$ or 80%.

****Try this on your own**: A football team has 42 players. There are 18 players who play offense, 20 players who play defense, and 10 players who play on special teams. Six of the offensive players play both offense and special teams. Find the probability that a player is on the offense or special teams.

3.3.2 Conditional Probability

The addition rule requires that we know the intersection probability at the end of the formula. There is a rule for calculating the intersection probability directly, but before we work with that formula, we need to define **Conditional Probability**. The conditional probability of an event is the probability that results after another event has already happened and could affect the new event.

The logical way to compute conditional probability is to take into account what has already happened and adjust the possible outcomes accordingly. For example, your probability of passing a test depends upon certain conditions. If you studied well, the probability will likely increase. If you didn't realize there was a test and didn't study at all, then the probability will likely decrease. Another example could be, the probability of rolling a 5 on a 6-sided die is $\frac{1}{6}$, but the probability of rolling a 5 on a 6-sided die, after you know the roll is an even number, is 0, since 5 is odd.

If two events affect the occurrence of each other, they are said to be **Dependent**. If two events do not affect the occurrence of each other, they are said to be **Independent**.

When two events are dependent, their probabilities must be calculated using conditional probability. As a formula, the probability of the intersection of events A and B is given by $P(A \bigcap B) = P(A) \cdot P(B|A)$. This is known as the **Multiplication Rule**. The notation $B|A$ is read as "B given A". The formula states that the probability of both A and B, is equal to the probability of event A (considered to happen first) times the probability of event B, given that event A has already happened. P(B|A) is the conditional probability of event B, given A.

When two events are independent, then the condition of one happening does not matter, and the multiplication rule simplifies to $P(A \bigcap B) = P(A) \cdot P(B)$. Remember, this only happens for independent events.

Many probability problems deal with picking cards from a standard deck of playing cards. Here is description. A standard deck has 52 cards split into four symbols (called suits). The two red symbols are hearts and diamonds. The two black symbols are clubs and spades. Each suit has 13 cards with a rank (or value). The ranks are $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$. 'J' stands for jack, 'Q' for queen, 'K' for king, and 'A' for ace. The jack, queen and king are called face cards, because they usually have faces of people on them. There are 4 of each rank card, one from each suit (symbol).

Example: If two cards selected at random from a standard deck of playing cards, what is the probability of picking two aces?

Solution: The probability of picking the first ace is $P(ace) = \frac{4}{52} = \frac{1}{13}$. The probability of the second ace being picked, depends upon the condition of an ace already being picked. P(2nd ace|1st ace) $= \frac{3}{51}$, since there would be 3 aces out 51 remaining cards. Therefore, $P(ace \bigcap ace) = \frac{1}{13} \cdot \frac{3}{51} = \frac{3}{663} = 0.005 = 0.5\%$.

Example: If two 6-sided dice are rolled, what is the probability of rolling two ones?

Solution: Here the two dice have no effect on each other, they are independent rolls. $P(one \bigcap one) = P(one) \cdot P(one) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = 0.028 = 2.8\%$.

In some situations, the conditional probability may be unknown and you wish to compute it. If the intersection probability is known, then we can rearrange the multiplication rule to find the probability of event B under the condition that event A had already happened, by $P(B|A) = \frac{P(A \bigcap B)}{P(A)}$. The formula states that the probability of event B, given that event A has already happened, is equal to the probability both events happen divided by the probability of event A.

Example: If one card is selected at random from a standard deck of playing cards, what is the probability of picking a Jack, given that the card picked is a face card?

Solution: Without any conditions being known, the probability of picking a jack would simply be $\frac{4}{52}$. Under these conditions, $P(jack|face) = \frac{P(jack \bigcap face)}{P(face)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{4}{12} = \frac{1}{3}$. Once you know it is a face card, it is more likely to be a jack, that just getting a jack out of all cards.

3.3.3 Probability Distributions

It is helpful to know all of the outcomes in a sample space for an experiment and their corresponding probabilities. A table which lists all outcomes and the probabilities is know as a **Probability Distribution**. There are two requirements for a valid probability distribution. Each probability must be between 0 and 1 (0% and 100%). The sum of all the probabilities must equal 1 or 100%. That way you know every outcome has been included properly.

Example: Which of the two tables are valid probability distributions (if any)? Why or why not?

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Probability	0.22	0.13	0.34	0.24	0.07

Value	1	2	3	4	5	6
Probability	-0.3	0.3	0.4	0.4	0.1	0.1

Solution: The first table is a valid probability distribution, since each probability value is between 0 and 1, and they add up to 1. The second table is not. The probabilities do add up to 1, but one of them is negative.

3.3.4 Expected Value

For a quantitative variable, we can use the probability distribution to find the **Expected Value**. The expected value is like an average, weighted by the probabilities. It gives the typical value of the variable over many observations. It is used everyday in many ways in the real world: to set prices for insurance, to choose investments, and in most of the sciences. The formula for the expected value of a variable x is $E(x) = \sum x \cdot p(x)$, where $p(x)$ is the probability of a value x. basically you just multiply each value times its probability and then add up all of the products.

Example: A particular insurance policy has a claim distribution shown below. Find the expected value of a claim for a one year period. Then compute the price of the monthly insurance premium, if the company will charge 10% profit margin and the premium is paid over 12 months.

Claim Amount	$0	$500	$1,500	$5,000
Probability	0.75	0.15	0.07	0.03

Solution: $E(x) = 0(.75) + 500(.15) + 1500(.07) + 5000(.03) = \$330$, so the insurance company expects to pay out $330 on average each year, for every policy it sells. Some policies will pay out more (large claim of $5,000), most less ($0). In order to stay in business, the company must charge more than $330. The annual profit margin they charge will be $10\%(330) = \$33$, so the annual premium is $363. Then the monthly premium is $\frac{\$363}{12} = \$30.25$. This is a simplified example of insurance, but the concept is the same as what insurance companies use everyday.

Example: A casino game costs $10 to play and pays $150 if you win. The probability of winning is 3%. Find the expected value of net winnings for a person playing one game.

Solution: If the person wins, they net $150 minus the amount to play, $150 - 10 = \$140$. If the person loses, their net is a loss $-\$10$. The expected net winnings are $E(x) = 140(.03) - 10(.97) = -\$5.50$, so the person playing expects to lose $5.50 on average each time they play. Some people will get lucky and win more than they lose, others could be very unlucky and never win. Casinos and lotteries always want to have a negative expected value for the players, which is a positive expected value for them. This is how they stay in business, by winning most of the time and having a net gain.

****Try this on your own**: A particular game has the prize distribution shown below. Find the expected value of a prize.

Prize Amount	$0	$25	$100	$500
Probability	0.7	0.2	0.09	0.01

3.3.5 Exercises: More Probability

Solutions appear at the end of this textbook.

1. Give an example of two events that are mutually exclusive.

2. At Mega University there are 32 physics majors, 49 math majors, and 112 engineering majors. Out of these, 8 are double majors in physics and math, and 14 are double majors in physics and engineering. Find the probability that one of these students selected at random is a physics or engineering major.

3. Given $P(A) = 0.5$, $P(B) = 0.7$, and $P(A\ and\ B) = 0.3$. Find $P(A\ or\ B)$.

4. Given $P(A) = 0.65$, $P(A\ or\ B) = 0.85$, and $P(A\ and\ B) = 0.25$. Find $P(B)$.

5. If two cards are picked at random from a deck of cards, what is the probability of picking two red sixes?

6. If one card is picked at random from a deck of cards, what is the probability of picking the ten of hearts, given that you know the card is red?

7. Give an example of two events that are independent.

8. Can two events be both mutually exclusive and independent? Explain.

9. Is this a valid probability distribution? Why or why not?

Meal	Pizza	Chicken	Steak	Pasta	Fish
Probability	0.48	0.12	0.12	0.20	0.05

10. Find the expected value of the cash prize for a lottery game. How much should they charge to play, if they want to make some profit?

Prize	\$0	\$5	\$100	\$2,500	\$20,000	\$100,000
Probability	$\frac{752{,}944}{800{,}000}$	$\frac{45{,}000}{800{,}000}$	$\frac{2{,}000}{800{,}000}$	$\frac{50}{800{,}000}$	$\frac{5}{800{,}000}$	$\frac{1}{800{,}000}$

Chapter 4

Financial Math

Written by Kyle Carter

4.1 Percent, Fraction and Decimal

4.1.1 Converting Numbers

Before we drift into the more interesting and fun aspects of financial math, we need to be sure we have a strong understanding of three particular ways to represent numbers: fraction, percent and decimal. When working through the problems and formulas in this chapter, we may need to change a number from one of these three forms, into another form. One example of this that will happen frequently is being able to change an interest rate (represented by the letter r in the formulas we will discuss later) from a percent to a decimal before performing calculations.

Converting Numbers Between Decimal and Percent Form: From Percent to Decimal: As mentioned above, the conversion from a percent to a decimal is going to be used frequently in later sections. Fortunately, it is also a very elementary process. When given a number in percent form, you can convert it to a decimal by following the steps below:

1. Divide the percent by 100 (this is the same as moving the decimal point two places to the left)
2. Remove the percent sign

Example: Convert the following numbers from percent form to decimal form:

- 25%

  Step 1: 25% divided by 100 is 0.25%

  Step 2: Removing the percent sign yields the answer 0.25.

- 237%

  Step 1: 237% divided by 100 is 2.37%

  Step 2: Removing the percent sign yields the answer 2.37.

From Decimal to Percent: Converting a number from a percent to a decimal can be achieved by doing the opposite from the steps outlined above. When given a number in decimal form, you can convert it to a percent form by following the steps below:

1. Multiply the decimal by 100 (this is the same as moving the decimal point two places to the right)
2. Add a percent sign

Example: Covert the following numbers from decimal form to percent form:

- 7.343

  Step 1: 7.343 multiplied by 100 is 734.3

  Step 2: Adding a percent sign yields the answer 734.3%.

- 0.49

  Step 1: 0.37 multiplied by 100 is 49

  Step 2: Adding a percent sign yields the answer 49%

Converting Numbers Between Fraction and Percent Form: From Fraction to Percent: When we are given a number in fraction from, we can follow the steps below to turn the fraction first into a decimal and then from a decimal to a percent, so this is two conversions in one. To go from fraction to decimal, you can stop after Step 1 below:

1. Divide the numerator (top of the fraction) by the denominator (bottom of the fraction). If you have a whole number and a fraction, you can first turn this number into an improper fraction before dividing. This step yields a decimal, so if you are looking to convert a fraction to a decimal, you can stop here. If you want to turn the fraction into a percent, then keep going.

2. Multiply the decimal by 100

3. Add a percent sign

Example: Convert the following numbers from fraction form to percent form:

- $\frac{4}{5}$ Step 1: 4 divided by 5 is equal to 0.8 Step 2: 0.8 multiplied by 100 is 80 Step 3: Adding a percent sign yields the answer 80%.

- $3\frac{1}{8}$ Step 1: We can convert $3\frac{1}{8}$ into the improper fraction $\frac{25}{8}$. 25 divided by 8 is equal to 3.125 Step 2: 3.125 multiplied by 100 is 312.5 Step 3: Adding a percent sign yields the answer 312.5%.

4.1.2 Find a Percentage of a Number

Now that we have a working understanding of fraction, decimal and percent forms of numbers and converting between the three forms, we can move on to the first application of percentages we will discuss in this class: Finding a percentage of a number.

The formula for finding the percentage of a number is $A = P * B$ In this formula: P is the percent converted to decimal form B is the original number A is the answer or amount

Example: Using the percentage of a number formula, solve the following problems:

1. What number is 40% of 65?

   Solution: Using the formula for finding the percentage of a number, and remembering to first convert 40% into decimal form, we have $A = 0.40 * 65 = 26$.

2. 15 is 25% of what number?

   Solution: Notice that this question is worded slightly differently than example 1. This time we have A and are instead trying to find B so we will plug 0.25 in the place of P and 15 in the place of A: $15 = 0.25 * B$ In order to solve for B, we need to divide both sides of the equation by 0.25, so $B = \frac{15}{0.25} = 60$.

Sales Tax: One application of the percent of a number formula is found in calculating sales tax. Whenever you buy anything from a store, you pay more than what your items totaled to. For instance, if you purchase $50 worth of groceries, you end up paying closer to $54 for the items. The reason for this increase in price is sales tax. These are taxes that are calculated as a percentage of your purchase total and go to funding states, counties and/or municipalities. Sales tax is known as a consumption tax because it is collected when we purchase goods and services. Typical sales tax rates in the Carroll County area are 7-8%, and this number can vary between different counties and states. Calculating sales tax is fairly elementary and is really just a specialized version of the percent of a number formula. You can see the symmetry below:

The Sales Tax formula is as follows:

Sales Tax Amount = Sales Tax Rate * Total Cost of Items Purchased

This calculation yields the tax amount, so in order to find the total cost after tax, you will need to add your answer back to the Total Cost of Items Purchased. Keep this in mind and read questions carefully to be sure you are providing the right answer to the problem. Finally, the sales tax rate will be provided in percent form, so be sure to change it to its equivalent decimal form before plugging it into the formula.

Example: Using the Sales Tax formula, solve the following problems:

1. Suppose you purchase the following items from the grocery store:

   Coffee: \$7.99 Creamer: \$2.99 Doughnuts: \$4.50

   What is the sales tax owed for this purchase if the sales tax rate is 8%?

   Solution: Before using The Sales Tax formula we need to convert 8% to decimal form ($8\% = 0.08$) and find the total cost of the three items. $7.99 + 2.99 + 4.50 = 15.48$. Plugging these values into the formula, we have:

   Sales Tax Amount $= 0.08 * 15.48 = \$1.24$, rounding the final answer to two decimal places since we are working with money. So the tax owed on this purchase is \$1.24.

2. Suppose you purchase the following items from the grocery store:

   Coffee: \$7.99 Creamer: \$2.99 Doughnuts: \$4.50

   What is the total cost owed for this purchase after tax if the sales tax rate is 8%?

   Solution: To find the total cost after sales tax, we need to add the total before tax to the sales tax amount. Using our work from the previous problem where we found the total before tax to be \$15.48 and the sales tax amount to be \$1.24, we find the answer to be $\$15.48 + \$1.24 = \$16.72$.

Discounts:

A second common application of the percent of a number formula is in calculating discounts. When you purchase items at a discount, you end up paying less than the original or normal price. Amazon Prime Day is a great opportunity to get items 25% or more off their original price. This is known as a discount. Once again, the discount formula is just a specialized version of the percent of a number formula and it closely mirrors the sales tax formula. The only difference is that the calculated amount is subtracted from what we owe this time instead of added to it as is the case with sales tax.

The Discount Formula is

discount amount = discount rate * total cost of items purchased

This calculation yields the discount amount, so in order to find the total cost after the discount, you will need to subtract your answer from total cost of items purchased. Keep this in mind and read questions carefully to be sure you are providing the right answer to the problem. Finally, the discount rate will be provided in percent form, so be sure to change it to its equivalent decimal form before plugging it into the formula.

Example: Using the discount formula, solve the following problems:

1. Suppose you purchase the following items from a going out of business sale:

   Canoe: \$399.99 Tent: \$89.50 Sleeping Bag: \$112.59

   What is the discount amount if everything in the store in on sale 35% off?

   Solution: Before using the discount formula we need to convert 35% to decimal form ($35\% = 0.35$) and find the total cost of the three items. $399.99+89.50+112.59 = 602.08$. Plugging these values into the formula, we have:

Discount Amount $= 0.35 * 602.08 = \$210.73$, rounding the final answer to two decimal places since we are working with money. So the discount amount on this purchase is \$210.73.

2. Suppose you purchase the following items from a going out of business sale:

   Canoe: \$399.99 Tent: \$89.50 Sleeping Bag: \$112.59

   What is the total cost owed for this purchase after the 35% discount?

   Solution: To find the total cost after discount, we need to subtract the discount amount from the total price of the three items. Using our work from the previous problem where we found the total price of the three items to be \$602.08 and the discount amount to be \$210.73, we find the answer to be $\$602.08 - \$210.73 = \$391.35$.

4.1.3 Percent Increase and Decrease

Things are always in flux in the world of finance, and even in personal finance we see increases and decreases in our savings accounts, investment accounts and monthly budget surpluses and deficits. The final concept we are going to discuss in this lesson is that of measuring the amount of change in terms of a percent. We will look at two formulas below that are really just two versions of the same formula dependent on whether the change is an increase or decrease:

$$\text{Percent Increase} = \frac{Amount of Increase}{Original Amount} \qquad \text{Percent Decrease} = \frac{Amount of decrease}{Original Amount}$$

In each case, after plugging values into the formula, we will get a fraction that we need to convert to a percent using the steps at the beginning of this lesson.

Example: Use percent increase/decrease formulas to solve the following problems:

1. Suppose you get a raise at work that increases your salary from \$52,000 a year to \$55,000 a year. Calculate the percent increase rounded to two decimal places of your raise.

   Solution: We need two things before we use the percent increase formula. First, we need the amount of increase, and this can be found by subtracting the two salary amounts we are given. $\$55,000 - \$52,000 = \$3,000$ (this will be the number that goes in the numerator of the fraction). Secondly, we need to know the original amount. This is \$52,000 (this will be the number that goes in the denominator of the fraction) for this problem because that was the original salary before the raise. Plugging these values into formula yields the answer: Percent Increase $= \frac{3000}{52000} = 0.05769 = 5.77\%$.

2. Suppose your house has lost value over the last year. Last year the house was worth $225,000 whilethisyearitisworth$205,500. Calculate the percent decrease of your home's value rounded to two decimal places.

   Solution: We need two things before we use the percent decrease formula. First, we need the amount of decrease, and this can be found by subtracting the two home values we are given. $\$225,000 - \$205,500 = \$19,500$ (this will be the number that goes in the numerator of the fraction). Secondly, we need to know the original amount. This is \$225,000 (this will be the number that goes in the denominator of the fraction) for this problem because that was the original home value before the change. Plugging these values into formula yields the answer: Percent Increase $= \frac{19500}{225000} = 0.0866666 = 8.67\%$.

4.1.4 Exercises: Converting Numbers

Solutions appear at the end of this textbook.

1. Convert the following numbers from decimal form to percent form:
   - 19.78
   - 1.12
   - 0.57
   - 0.0031
2. Convert the following numbers from percent form to decimal form:
   - 25%
   - 148%
   - 0.156%
   - 17.34%
3. Convert the following numbers from fraction form to percent form, rounded to two decimal places as necessary:
   - $\frac{2}{5}$
   - $1\ \frac{7}{8}$
   - $\frac{137}{50}$
   - $\frac{15}{16}$
4. What number is 15% of 200?
5. What number is 34% of 50?
6. 75 is 30% of what number?

7. 48 is 12% of what number?
8. 28 is what percent of 200?
9. If your grocery store total comes to \$153.25 before tax, and the sales tax is 6%, how much sales tax will you need to pay? What will your total be after sales tax?
10. If you purchase a vehicle normally priced at \$23,050 at a discount of 25% off, how much money do you save by buying it on sale? How much will you pay for the vehicle after the discount?
11. If you purchase a laptop normally priced at \$899.99 subject to a discount of 33% and a sales tax rate of 9%, how much do you end up paying for the laptop after the discount and tax?
12. If your investment portfolio increased from \$45,000 to \$55,000 over the last year, what is the percent of increase?
13. If your home's value went from \$157,000 to \$135,000 over the last year, what is the percent of decrease?

4.2 Simple and Compound Interest

4.2.1 Simple Interest

In the world of finance, the concept of interest is quite important to understand. Many of you have student loans which have an interest rate associated with them. Over time, many of you will buy cars or homes using loans from financial institutions, and those loans are likely to be accompanied with an interest rate. Since we will probably all use (or pay) interest over the course of our lives, let's be sure we have a strong understanding of what it is.

When you take out student loans to pay for college, you are using someone else's money (often backed by the federal government) to pay for college now. You sign a promissory document to pay it back over time, but you get the benefit of having college now and paying for it later over time using money that is not yours. Since you are getting this massive benefit of using what you do not currently own, you will end up paying back more than you borrow. This extra amount that you pay back is interest. **Interest** is the cost of being able to use someone else's money now and pay it back slowly over time.

When you take out a loan for college, cars or a home, the amount of interest you pay back is dependent on several variable including how much you borrow, how long you take to pay it back and what the interest rate is. The interest rate is presented in percent form, and the higher the interest rate, the more you will have to pay back over time. Common interest rates for subsidized loans are between 3% and 5% while interest rates for a home loan (or mortgage) are commonly around 4% at the time of writing this text.

Simple Interest: There are two types of interest we will discuss in this lesson, and the first is simple interest. In a simple interest agreement, the borrower will eventually pay back the amount that was originally borrowed (the principal) plus interest calculated only on that original amount. The formula for calculating the simple interest owed on a loan is as follows:

Simple Interest Formula: $I = Prt$

In this formula: I is the total amount of interest P is the original amount borrowed, also known as the principal r is the interest rate (need to be converted to decimal form t is the number of years it takes to pay the loan back

The simple interest formula provides you with the amount of interest that must be repaid. A borrower must also pay the principal back as well, so if you want to determine the total amount that must be repaid, you can add the principal to the interest calculated to find the future value or total value. We have a separate formula for this, and it is as follows:

Future Value Formula for Simple Interest: $A = P(1 + rt)$

In this formula: A is the future value or principal plus interest (also known as the accumulated value) P is the original amount borrowed, also known as the principal r is the interest rate (need to be converted to decimal form t is the number of years it takes to pay the loan back

Example: Use simple interest formula and the future value formula for simple interest to solve the following problems:

1. Suppose you need to borrow money to purchase a car valued at \$4,500. Your local bank offers a 5% simple interest loan that must be paid back in 5 years. How much simple interest will you pay back over the 5 years?

   Solution: Since we are just looking for how much simple interest will be paid back, we will use the simple interest formula. The principal (or original amount borrowed) is \$4,500, so $P = 4500$. The interest rate is 5%, so in decimal form, $r = 0.05$. The amount of time it takes to pay back the loan in years is 5, so $t = 5$. Plugging these values into the simple interest formula we have:

   $I = Prt$

   $I = (4500)(0.05)(5) = 1125$

Thus the amount of interest that must be paid back in this arrangement is $1,125.

2. Suppose you need to borrow money to purchase a car valued at $4,500. Your local bank offers a 5% simple interest loan that must be paid back in 5 years. How much will you pay back over the 5 years in total?

   Solution: Since we are just looking for the total amount that must be paid back (the principal in addition to the interest), we will use the future value formula for simple interest. The principal (or original amount borrowed) is $4,500, so $P = 4500$. The interest rate is 5%, so in decimal form, $r = 0.05$. The amount of time it takes to pay back the loan in years is 5, so $t = 5$. Plugging these values into the future value formula we have:

   $A = P(1 + rt)$

   $A = 4500(1 + (0.05)(5)) = 5625$

   Thus the total amount that must be paid back in this arrangement is $5,625. Notice that we could have determined this number in an alternative manner by adding the interest calculated in the previous question to the principal: $\$1,125 + \$4,500 = \$5,625$.

Another application of the future value formula is transforming the formula so that it is solved for the principal, P. This would allow us to use the formula to set a goal future value and determine the principal required for reaching this goal. Interest can work in your favor, and if you have a savings account with a bank or financial institution, you probably know that you gain interest over time. The future value formula solved for P is as follows:

$$P = \frac{A}{(1 + rt)}$$

Use this formula to solve the following problem:

1. If we want to have $6,000 in 7 years, how much must we deposit into a savings account now subject to a 3% simple interest rate?

Solution: Since we want to know how much money we would need to deposit today, we are looking for the principal. We want to have a future value of \$6,000, so $A = 6000$. The rate is 3%, so in decimal form, $r = 0.03$. We want to save this money for 7 years, so $t = 7$. Plugging these values into the formula, we have:

$$P = \frac{6000}{(1 + 0.03 * 7)} = \frac{6000}{1.21} = 4958.68$$

Thus we would need to deposit \$4,958.68 today in order for it to grow to \$6,000 in 7 years subject to this simple interest rate. This problem illustrates how helpful this formula and future formulas can be when solved for P for planning purposes.

4.2.2 Compound Interest

The second type of interest we will discuss in this lesson is compound interest. While simple interest calculates and charges interest only on the principal amount over time, **compound interest** calculates and charges on the principal as well as interest that builds over time. This is why it is called compound because the interest compounds on itself over time. Compound interest is the more common form of interest that is charged for loans in our current financial world. We will examine and use two formulas for compound interest, and they both give the answer in terms of the total future value, A.

Periodic Compound Interest Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

In this formula:

A is the future value (also known as the accumulated value) P is the original amount borrowed, also known as the principal r is the interest rate (need to be converted to decimal form) t is the number of years it takes to pay the loan back n is the number of times per year the interest is calculated and compounded per year (also known as number of compounding periods per year)

The only new letter we have in this formula is n and while each other number will usually be explicitly given in the problem, n requires a little bit of deciphering. Since n represents the number of times per year the interest is calculated and compounded, words like daily, weekly, monthly, quarterly and biannually will be used. Here is a guide to what n would be in each case:

If the interest is compounded:

- Annually: $n = 1$
- semi-annually: $n = 2$
- Quarterly: $n = 4$
- Monthly: $n = 12$
- Weekly: $n = 52$
- Daily: $n = 365$ (some financial institutions use 360 for this instead. We will stick with 365 for our calculations in this text)

The second compound interest is as follows:

Continuous Compounding Formula: $A = Pe^{rt}$

In this formula:

A is the future value (also known as the accumulated value) P is the original amount borrowed, also known as the principal r is the interest rate (need to be converted to decimal form) t is the number of years it takes to pay the loan back e is actually a number (like pi), and it can be found on your calculator. e s the natural base, and it often will share a key with the natural log (ln) on calculators.

When the problem we are working on states that the interest is compounded continuously or is subject to continuous compounding, that is the key to know when to use this second formula. If you do not see those words in the question, then you always will use the periodic compound interest formula.

Example: Use the periodic compound interest formula and/or continuous compounding formula to solve the following problems:

1. Suppose you invest a sum of \$10,000 subject to compound interest compounded monthly with an interest rate of 5.5%. How much will your investment be worth after 7 years?

   Solution: Before we identify the numbers that will replace the letters in our formula, we need to determine which compound interest formula to use. A quick scan of the problem reveals that the words continuous or continuously do not appear. Thus, we will use the periodic compound interest formula. Now we can identify:

   $P = 10000$, since this is the original amount invested $r = 0.055$ in decimal form $t = 7$

   $n = 12$ since the interest is compounded each month (or 12 times a year)

   Plugging these values into the formula yields: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

   $A = 10000\left(1 + \dfrac{0.055}{12}\right)^{12*7} = 14683.22$

   Therefore, in this situation, our original investment of \$10,000 would have grown to \$14,683.22 after 7 years. Now let's do this problem with the same numbers and change to continuous compounding to see how similar the answers are.

2. Suppose you invest a sum of \$10,000 subject to continuous compounding with an interest rate of 5.5%. How much will your investment be worth after 7 years?

   Solution: Before we identify the numbers that will replace the letters in our formula, we need to determine which compound interest formula to use. A quick scan of the problem reveals that the word continuous does appear. Thus, we will use the continuous compounding formula. Now we can identify:

$P = 10000$, since this is the original amount invested $r = 0.055$ in decimal form $t = 7$

Plugging these values into the formula yields: $A = Pe^{rt}$

$A = 10000e^{0.055*7} = 14696.14$

Therefore, in this situation, our original investment of \$10,000 would have grown to \$14,696.14 after 7 years. As you can see, this answer is very similar to the answer we arrived at in question 1 where we compounded the interest each month. In general, the more times you compound the interest, the more money you will have after the given time. Thus, if we chose to do this problem again but decided to compound the interest quarterly (4 times per year), we would arrive at an answer that is less than either we have done so far since the interest would be compounded less times per year. One final note to keep in mind is that the most you can compound the interest is continuously, so we would expect this to be the largest dollar amount answer possible.

4.2.3 Exercises: Simple and Compound Interest

Solutions appear at the end of this textbook.

1. Suppose you deposit \$500 into a bank account offering 4% simple interest. If you leave the money in the account for 15 years, how much interest will you accrue? How much money will be in the account total after the 15 years?

2. Suppose you borrow \$1,000 from your parents to purchase supplies for your dorm and a tablet. If you promise to pay the money back after 5 years subject to 3% simple interest, how much interest do you end up paying back? How much do you end up paying back in total?

3. If you want to have \$3,000 in a savings account that offers 6% simple interest after 10 years, how much should you deposit into the account today?

4. If you want to have \$12,000 in a savings account that offers 2% simple interest after 15 years, how much should you deposit into the account today?

5. If you borrow \$3,200 from a friend and promise to pay them back \$4,000 after 4 years, what simple interest rate are you agreeing to pay back? (Hint: Use the Future Value Formula for Simple Interest and solve for the unknown letter)

6. Which of the following investment options is better:
   Option 1: Investing \$5,000 into an account offering 6% simple interest for 7 years
   Option 2: Investing \$5,000 into an account offering 7% simple interest for 6 years

7. If you invest \$10,000 into an account offering 5% compound interest for 8 years, what is the future value of the account subject to the following compounding periods/options:

   - Biannually:
   - Annually:

- Monthly:
- Weekly:
- Daily:
- Compounded Continuously:

8. Which of the following options is a better investment AND by how much?

   Option 1: \$7,500 invested for 8 years subject to 6.75% interest compounded monthly

   Option 2: \$7,700 invested for 8 years subject to 6.5% interest compounded continuously

9. If you need to have \$14,000 in 9 years, how much should you invest today in an account offering 5.5% interest compounded weekly?

10. If you need to have \$150,000 in 5 years, how much should you invest today in an account offering 10% interest compounded continuously?

4.3 Annuities and Loans

4.3.1 Annuities

Strategies and concepts that support building wealth is a very important and practical topic in this chapter. Who does not want to build wealth over their career in order to provide security during retirement and quite frankly the means to have some fun and enjoy life? Many careers provide retirement plans such as a 401k or a pension that is meant to provide financial means when you retire. You can also have your own personal retirement mechanisms on the side to accelerate your wealth growth. In this lesson, we are going to discuss a myriad of vocabulary and concepts that can help you understand how investing and wealth growth works.

A great place to get started is with annuities. An **annuity** is technically a series of equal payments that are made over equal time periods. The word is used in a variety of applications including payouts for insurance settlements or lottery winnings. In these situations, a large sum of money is distributed to the winner or beneficiary little bits at a time in equal amounts over equal time periods. We are going to discuss annuities in terms of investments as equal amounts of money we contribute to a retirement account over time.

It is very common to put aside a set amount of money each month into a retirement or investment account. Let's say that you budget $200 a month every month over your career for investments. That ends up being only $2,400 per year ($200 for 12 months), and if you do this for 40 years during your career, you end up saving and investing $96,000. After the 40 years though, you would have an unbelievable amount of money if you invested this money in even moderately safe places like an S&P 500 index fund. Historically (dating back to the early 1900's), the S&P 500 has averaged return of around 10%.

You can think of return in the same way we considered interest rates in previous lessons, but now we are gaining money instead of paying it. If we invested this $200 a month for

40 years subject to 10% return, we would have \$1,264,815.92! Keep in mind that we only contributed/saved a little less than \$100,000 ourselves, but this money has grown to over one million dollars. This represents the power of compounding money over long periods of time. All you have to do in this scenario is be disciplined and invest \$200 a month. The formula for an annuity that allowed us to calculate this is as follows:

$$\textbf{Annuity Formula: } A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}$$

In this formula A is the future value (also known as the accumulated value)
P is the amount you deposit each equal period of time (usually monthly in practice)
r is the return rate (need to be converted to decimal form)
t is the number of years you invest
n is the number of times per year you invest the set amount (again, in practice, this is normally 12 for monthly, but you could adjust this if you wanted to invest with greater or lesser frequency)

Using this formula, we can see where our calculation came from earlier:

Investing \$200 each month means $P = 200$ and $n = 12$. Investing in this way for 40 years means $t = 40$. Finally, the 10% return rate means $r = 0.10$. Plugging this into the annuity formula yields:

$$A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)} = \frac{200\left[\left(1+\frac{0.10}{12}\right)^{(12*40)} - 1\right]}{\left(\frac{0.10}{12}\right)} = \$1,264,815.92$$

It is very important to work through this calculation carefully. There are quite a few things going on in the formula, so it may be helpful to break it down into several individual calculations and arrive at the answer over a series of steps.

Example: Use the annuity formula and/or to solve the following problem:

Suppose you invest $300 into an IRA (individual retirement account) each month. If you average 12% return on your investments, how much money will be in the IRA after 30 years? How much of that total is pure return/growth?

Solution: Investing $300 each month means $P = 300$ and $n = 12$. Investing in this way for 30 years means $t = 30$. Finally, the 12% return rate means $r = 0.12$. Plugging this into the annuity formula yields:

$$A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)} = \frac{300\left[\left(1+\frac{0.12}{12}\right)^{(12*30)} - 1\right]}{\left(\frac{0.12}{12}\right)} = \$1,048,489.24$$

Thus we will have $1,048,489.24 in the IRA after 30 years.

In order to calculate how much of this total is pure return/growth, we need to first calculate how much we actually contributed and then subtract that from the total. We can find this by multiplying $300(12 months)30 years=$108,000.

Subtracting the amount we actually saved/invested from the total yields:

$\$1,048,489.24 - \$108,000 = \$940,489.24$ in pure return.

4.3.2 Investments

Now that we have discussed the power of investing consistently over time, it is reasonable to discuss types of investments available. Most investments have a particular amount of risk and reward associated with them. In general, the more risk associated with an investment, the greater the potential reward. It is also generally true that less risk associated with an investment means there is less potential reward. The types of investments we will discuss are CDs, bonds, individual stocks (equities), mutual funds and index funds.

A **CD** is a certificate of deposit, and it represents a very safe investment with a very low return. Various banks and credit unions offer CDs, and under such an agreement, you would deposit and leave an amount of money in an account, and over time, your money would grow according to a set interest rate that the financial institution would pay you. After the set time, you could retrieve your money plus any interest that you earned. The downside is that most CD's do not pay any interest if you withdraw the money early.

Bonds are similar to CDs in that you would give a set amount of money (depending on how many bonds you purchase) for a set amount of time. After that time period ends, you would be given back your original investment plus a fixed or variable amount of return that was agreed upon originally. Bonds are offered by companies, municipalities and various governments, and they represent a moderately safe investment with a relatively low return. When a company or government is selling bonds, they are technically raising capital (money) for various purposes, so buying bonds is essentially loaning money.

Individual **Stocks** are also known as equities, and they represent ownership of small piece of companies. When you purchase stock in Apple (AAPL), you then own a very tiny part of the company. These individual tiny parts of companies are known as shares, and the more you own, the more of the company you own. Since you can buy shares of any publicly traded company (a company that is not privately owned), the risk factor for stocks can vary, but in general it is higher than most other investments. The opportunity for return is higher though. When you purchase stock in a company, you pay the currently agreed upon price for each share. This number is rapidly changing as the agreed upon price is basically what millions of individual traders and investment firms are willing to pay at that exact moment.

If you buy a share of a company today for $50, good news, bad news and speculation could cause that share to be worth $60 tomorrow. Your stock is now worth $10 more than when you bought it, so you have made a profit or capital gain of $10. If you sell that stock, you then have made money. It you hold onto the stock though, it could easily grow to be worth more over time, but it could also drop in value over time. This is why buying individual stocks can be risky. Some companies have more stable prices while others can change dramatically from day to day.

Buying shares of a company low and then selling them high represents growth called capital gains. There is another way to make money from stocks though. Some companies offer dividends, and this is an amount of money you are paid (generally each quarter) just for owning the stock. Dividends are basically little bits of a company's profits that they distribute to the shareholders, because if you own part of a profitable company after all, you should benefit from its growth.

Mutual Funds are collections of stocks (and sometimes bonds) that you can buy into. Mutual funds are often managed by a fund manager who actively buys and sells stocks to maximize profits. Since individual stock buying can be quite risky, buying into mutual funds is a clever way to minimize this risk while still making relatively high returns on your money. If you own an individual stock that has a terrible day or month, you can lose a lot of money. If you have your money in a mutual fund however, individual stocks that have bad days and months should be kept in check by the other stocks in the fund so that everything hopefully balances out. You also benefit from not having to make the hard decisions and do constant research as the fund manager does that for you. Most mutual funds do have a fee associated with them since they are actively managed, so this is the main downside.

Index Funds are yet another way to buy stocks without the risk of having to choose individual stocks yourself. Index funds are collections of large numbers of stocks that seek to match the return of the S&P 500 or NASDAQ indices. The S&P 500 is a collection of 500 companies base in the USA, so you are mitigating much of the risk associated with stocks. We discussed before that the S&P 500 has historically yielded a 10% return on average, so S&P 500 index funds are fairly safe while offering decent return. Index funds generally have extremely small fees associated with them, so you do not cut into your profit nearly as much as you might with a mutual fund.

When choosing investments, you want to be sure to **diversify**. By sure you spread your investments around so that your investments do not get wiped out when things go bad in one area or sector. Consistent and diversified investment over many years can yield enormous returns for retirement, so be sure to educate yourself through research and do not leave opportunity on the table.

4.3.3 Loans and Payments

As we just saw from the previous section, building wealth over time through investing and saving is critical to financial independence and freedom later in life. While it is our sincere hope that each of you will take these principles and prosper over time, the reality is that we all must do this while still balancing other aspects of life. Before you find a career, you likely need to complete the college degree you are currently working on. In order to commute to school and a career, you will likely need a reliable vehicle. It is also helpful to have a place you call home, and most of you will own a home at some point in your lives. While it is possible to pay for each of these things up front, many of us will take out loans for one or more of these (and other) items. A **loan** is an amount of money that you borrow from a lender to pay for something now and that you pay back over time to that lender in what are called installments.

Statistics from the federal reserve (The Fed - Student Loans and Other Education Debt (federalreserve.gov)) suggest that more than half of you will take out student loans to pay for part or all of college. After you graduate, you will begin to pay the borrowed amount back in the form of monthly payments. Each payment will be broken down such that a portion pays down the principle (the outstanding debt left to be paid) and a portion pays interest (as discussed before, you can think of this as the cost of using someone else's money in the present). Student loans are an example of **installment loans**, and other examples include

auto loans and **mortgages**. For each of these types of loans, normally the installment is paid monthly, commonly referred to as the monthly payment

The formula for calculating a monthly payment amount is very similar to the annuity formula used in Lesson 3, and fortunately it uses the same letters having the same meaning that we are used to.

Monthly Payment Formula: $MonthlyPayment = \dfrac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}$ In this formula:

The $MonthlyPayment$ is the amount you pay back each month P is the original loan amount r is the annual interest rate (needs to be converted to decimal form) t t is the number of years you take to pay back the loan n is the number of times per year you make a payment, and since we are discussing monthly payments, we will always use 12. If you wish to cater this formula to another payment schedule like quarterly, annually, etc., you can use the values for n that correspond to those terms.

Example: Use the Monthly Payment Formula to solve the following problems:

1. Suppose you wish to take out a mortgage to buy a home valued at \$150,000. If this particular loan does not require a down payment, has an interest rate of 4%, and has a term of 30 years, a) what would be your monthly payment? b) How much money do you end up paying back total over the 30 years, and c) how much of that is pure interest?

   Solution: a) The loan amount (value of the house) is \$150,000, meaning $P = 150,000$. Since we are calculating a monthly payment, $n = 12$. The term of the loan being 30 years means $t = 30$. Finally, the 4% interest rate means $r = 0.04$. Plugging this into the monthly payment formula yields:

   $$\text{Monthly Payment} = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}} = \frac{150000\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-12*30}} = 716.12.$$

Thus we will have to make a monthly payment of \$716.12 each month for the next 30 years to own this home.

b) In order to calculate how much money we end up paying for this home over 30 years, we can multiply the monthly payment by 12 (twelve months in a year) and then by 30 (the number of years we are making these payments).

Total Amount Paid = 716.12*12 months*30 years = \$257,803.20

c) In order to calculate how much of this total is pure interest, we can subtract the original value of the house from this grand total. \$257,803.20 - \$150,000 = \$107,803.20 in pure interest.

2. Suppose you wish to purchase a vehicle by taking out an auto loan, and your budget will allow \$250 per month as a payment. If this particular auto loan does not require a down payment, has an interest rate of 7%, and has a term of 8 years, a) how expensive of a vehicle can you afford? b) How much money do you end up paying back total over the 8 years, and c) how much of that is pure interest?

Solution: a) The auto loan amount is unknown, meaning we will be solving for P. Since we are calculating a monthly payment, $n = 12$. The term of the loan being 8 years means $t = 8$. Finally, the 7% interest rate means $r = 0.07$. In this problem, we know the Monthly Payment Amount to be \$250. Plugging this into the Monthly Payment Formula yields:

$$250 = \frac{P\left(\frac{0.07}{12}\right)}{1 - \left(1 + \frac{0.07}{12}\right)^{-12*8}} = P(0.0136337)$$

$P = \dfrac{250}{0.0136337} = \$18,336.92$ is the amount of vehicle you can afford.

b) In order to calculate how much money we end up paying for this vehicle over 8 years, we can multiply the monthly payment by 12 (twelve months in a year) and then by 8 (the number of years we are making these payments).

Total Amount Paid = 250*12 months*8 years=$24,000

c) In order to calculate how much of this total is pure interest, we can subtract the original value of the car from this grand total. $\$24,000 - \$18,336.92 = \$5,663.08$ in pure interest.

A term that we have been using in the previous examples is down payment. A **down payment** is an amount of money (usually calculated as a percentage of the loan) that you pay up front. Many loan types including most mortgages require a down payment. Conventional mortgage loans require a 20% down payment at closing, and the remaining 80% of the value of the home is what is broken up into monthly payments (plus interest, homeowner's insurance, property taxes, and potentially other fees). In order to calculate a down payment, you merely change the percentage required to decimal form and multiply it by the cost of the item you are purchasing.

Example: Use the monthly payment formula and down payment calculations to solve the following problems:

1. If you wish to purchase a home valued at $200,000 by taking out a mortgage that requires a 20% down payment, what is the required down payment due at closing?

   Solution: In order to calculate this down payment, we first need to change 20% into decimal form. $20\% = 0.20 or 0.2$. Now we need to multiply this by the value of the home we wish to purchase:

   $DownPayment = 0.2 * 200,000 = \$40,000$ Thus the down payment required is $40,000.

2. If you wish to purchase a home valued at $200,000 by taking out a 30-year mortgage that requires a 20% down payment (which we just calculated to be $40,000), what is the monthly payment if the interest rate is 4.5%?

   Solution: The loan amount can be found by taking the original value of the home and subtracting the down payment (since we are paying that up front).

$LoanAmount = \$200,000 - \$40,000 = \$160,000$

The loan amount is \$160,000, meaning $P = 160,000$. Since we are calculating a monthly payment, $n = 12$. The term of the loan being 30 years means $t = 30$. Finally, the 4.5% interest rate means $r = 0.045$. Plugging this into the monthly payment formula yields:

$$MonthlyPayment = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}} = \frac{160000\left(\frac{0.045}{12}\right)}{1-\left(1+\frac{0.045}{12}\right)^{-12*30}} = 810.70.$$

Thus we will have to make a monthly payment of \$810.70 each month for the next 30 years to own this home in addition to the \$40,000 that we paid up front as a down payment.

4.3.4 Exercises: Annuities and Loans

Solutions appear at the end of this textbook.

1. Suppose you invest \$5,000 into a 401k retirement account each year. If you average 8% return on your investments, how much money will be in the IRA after 30 years? How much of that total is pure return/growth?

2. Suppose you invest \$250 into an IRA (individual retirement account) each month. If you average 11% return on your investments, how much money will be in the IRA after 40 years? How much of that total is pure return/growth?

3. Suppose you invest \$50 into an IRA (individual retirement account) each month. If you average 12% return on your investments, how much money will be in the IRA after 35 years? How much of that total is pure return/growth?

4. List the following investments in order form least risk to highest risk: Bonds, single stocks, index funds, CD's.

5. Suppose your budget will allow you to make a monthly payment of \$900 toward a mortgage to buy a home.

   a) How expensive of a home could you afford if you wish to take out a 15-year loan subject to an interest rate of 3.5%?

   b) How much do you end up paying for the home over the 15-year term of the loan?

   c) How much of the total amount paid back over the 15-year term is pure interest?

6. Suppose you wish to purchase a home valued at \$315,000 by taking out a 30-year mortgage subject to a 5% interest rate. Suppose this loan requires a 20% down payment.

   a) How much is the down payment for this loan?

   b) How much is the monthly payment for this mortgage?

c) How much do you end up paying in total for this house over the 30-year term of the mortgage?

d) How much of the total amount paid back is pure interest?

Chapter 5

Modeling From Data

Written by Robert Burnham

5.1 Function Concepts

5.1.1 Relations

A function is a particular type of relation. So what is a relation? A **relation** is a set of ordered pairs, typically represented by (x, y). The first component x is called the domain and y is called the range. The values from the domain are also called the input or independent variable, while the values from the range are called the output or dependent variable.

Example: Consider the relation

$$\{(-9, 3), (-5, -1), (0, 5), (2, 1), (7, 4)\}$$

a. Identify the domain.

Solution: Since this relation is given as a set of ordered pairs, we will use set notation to state the domain. We will list the distinct x values in order from smallest to largest. The domain in this case is $\{-9, -5, 0, 2, 7\}$

b. Identify the range.

Solution: Again, because this relation is given as a set of ordered pairs, we will use set notation to state the range. We will list the distinct y values in order from smallest to largest. The range in this case is $\{-1, 1, 3, 4, 5\}$.

5.1.2 Functions

A **function** is a relation in which each x is associated with exactly one y. To determine whether a given relation is a function use the following steps:

1. Identify the domain, i.e., the set of x values.

2. Identify the range, i.e., the set of y values.

3. Look for any x that is associated with more than one y. If there are any such x values, then the relation is not a function, else it is a function.

Example: Determine if the following relations are functions.

a. $\{(-2, 3), (0, 2), (-1, 5), (7, -5), (0, 3)\}$

Solution: This relation is not a function because $x = 0$ is associated with multiple y values, $y = 2$ and $y = 3$.

b. $\{(-4, -2), (-2, 0), (1, 3), (5, 7)\}$

Solution: This relation is a function because every x-value is associated with exactly one y-value.

Vertical Line Test

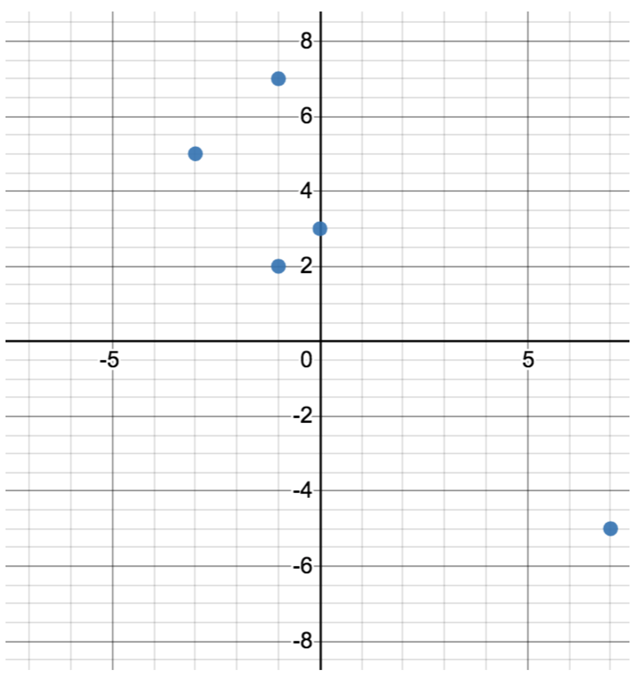
Another way to determine if a given relation is a function, is by using the **Vertical Line Test** or **VLT**. This is a graphical approach to determine if a relation is a function.

How to Use the VLT

Given the graph of a relation, the graph represents a function if and only if no vertical line can intersect the graph at more than one location for any value of x.

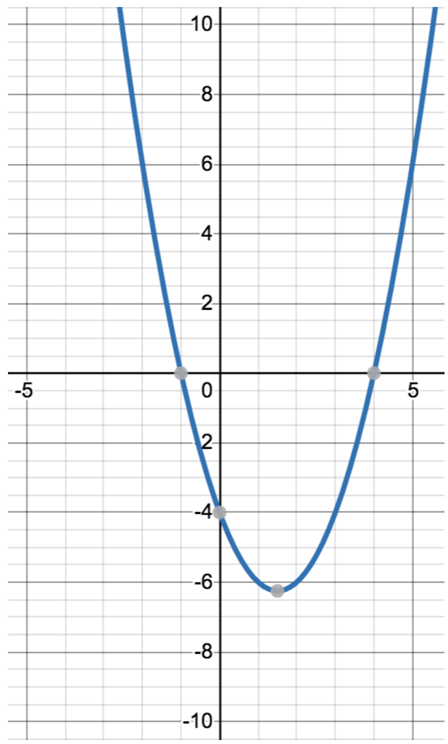
Example: Determine if the graphs represent a function.

a:



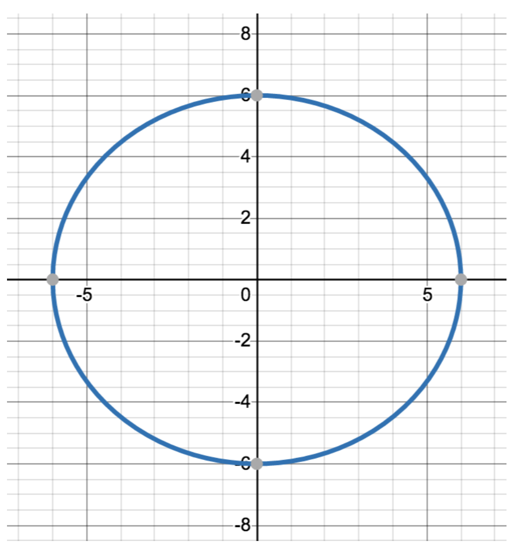
Solution: This graph does not represent a function because it fails the vertical line test at $x = -1$.

b:



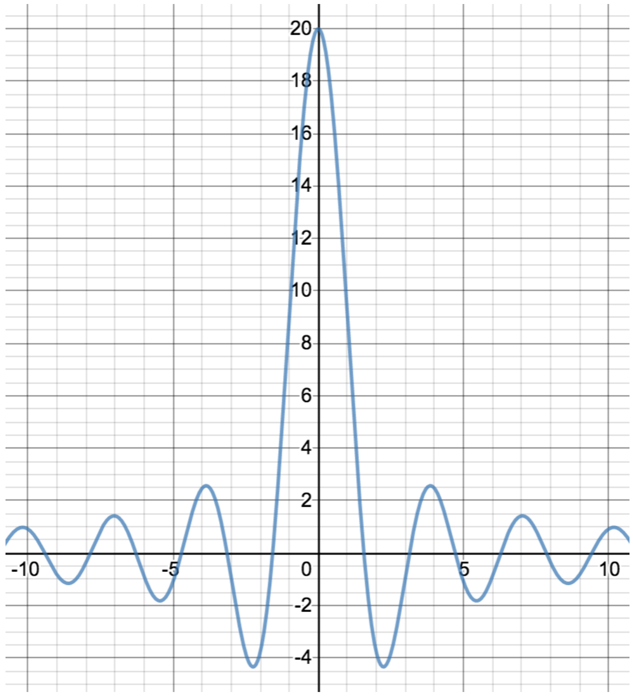
Solution: This graph does represent a function because it passes the vertical line test.

c:



Solution: This graph does not represent a function because it fails the vertical line test at multiple values of x.

d:



Solution: This graph does represent a function because it passes the vertical line test.

5.1.3 Function Notation

Now we will shift our focus to function notation. The statement $y = f(x)$ names a function f whose input (or independent variable) is x and output (or dependent variable) is y or $f(x)$. In the following examples we will get some experience finding particular output values for given input values. To compute these output values we will plug in the given input value x into the given function.

Example: Evaluate $f(x) = 5x + 2$ for the given values of x.

a $f(1)$

Solution:

$$f(1) = 5(1) + 1 = 5 + 2 = 7$$

b: $f(-2)$

Solution:

$$f(-2) = 5(-2) + 2 = -10 + 2 = -8$$

Example: Evaluate $g(x) = x^2 + 2x - 3$ for the given values of x.

a. $g(-1)$

Solution:

$$\begin{aligned} g(-1) &= (-1)^2 + 2(-1) - 3 \\ &= 1 - 2 - 3 \\ &= -4 \end{aligned}$$

b. $g(4)$

Solution:

$$\begin{aligned} g(4) &= (4)^2 + 2(4) - 3 \\ &= 16 + 8 - 3 \\ &= 21 \end{aligned}$$

Example: Evaluate $h(x) = x^3 + 7x$ for the given values of x.

a. $h(1)$

Solution:

$$h(1) = (1)^3 + 7(1) = 1 + 7 = 8$$

b. $h(-2)$

Solution:

$$h(-2) = (-2)^3 + 7(-2) = -8 - 14 = -22$$

Example: Evaluate $y = 2^x$ for the given values of x.

a. $y(2)$

Solution:

$y(2) = 2^2 = 4$

b. $y(-3)$

Solution:

$y(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$

Example: Evaluate $h(x) = \sqrt{x-3}$ for the given values of x.

a $h(4)$

Solution:

$h(4) = \sqrt{4-3} = \sqrt{1} = 1$

b: $h(2)$

Solution:

$h(2) = \sqrt{2-3} = \sqrt{-1} = \text{Undefined}$

Example: Evaluate $f(x) = \frac{5}{x-3}$ for the given values of x.

a $f(0)$

Solution:

$$f(0) = \frac{5}{0-3} = -\frac{5}{3}$$

b: $f(4)$

Solution:

$$f(4) = \frac{5}{4-3} = \frac{5}{1} = 5$$

Example: Evaluate $y = |x - 3|$ for the given values of x.

a $y(-1)$

Solution:

$y(-1) = |-1-3| = |-4| = 4$

b: $y(5)$

Solution:

$y(5) = |5-3| = |2| = 2$

****Try this on your own**: For the function $h(t) = 4\sqrt{5-t} - 1$, find $h(1), h(4)$, and $h(-11)$.

5.1.4 Exercises: Functions

Solutions appear at the end of this textbook.

1. One of the statements below is false and the other is true. Which one is which, and explain why. "All functions are relations." "All relations are functions."

2. List out the domain and range of the following relations. Do they represent functions?

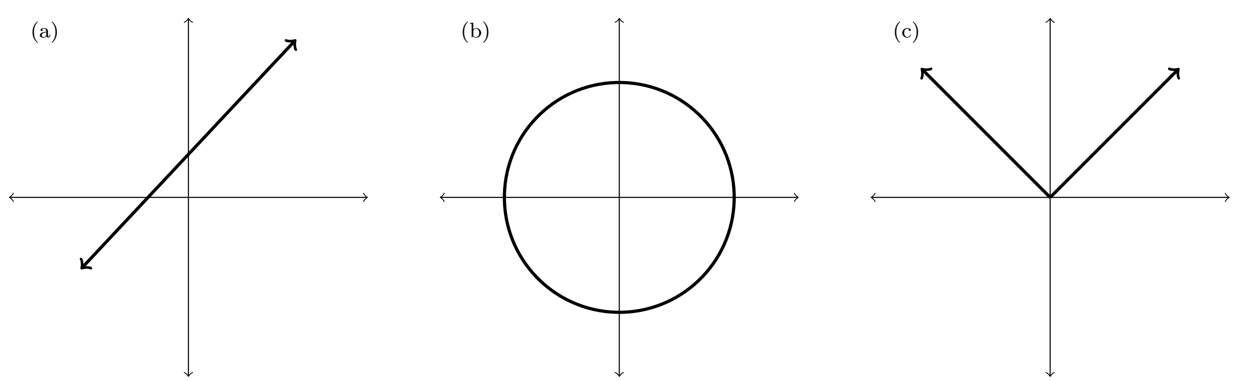
   (a) $\{(2,6),(3,6),(4,7),(5,7),(6,3)\}$

   (b) $\{(-1,-1),(0,0),(2,2),(4,4),(278,278)\}$

   (c) $\{(\$2, taco),(\$3, burrito),(\$2, enchilada),(\$5, fajitas)\}$

3. For the function $f(x) = x^2 - 3x + 1$, find $f(0)$, $f(-2)$, and $f(\frac{1}{4})$.

4. Use the vertical line test to determine if the graphs below are functions.



5.2 Scatter Plots

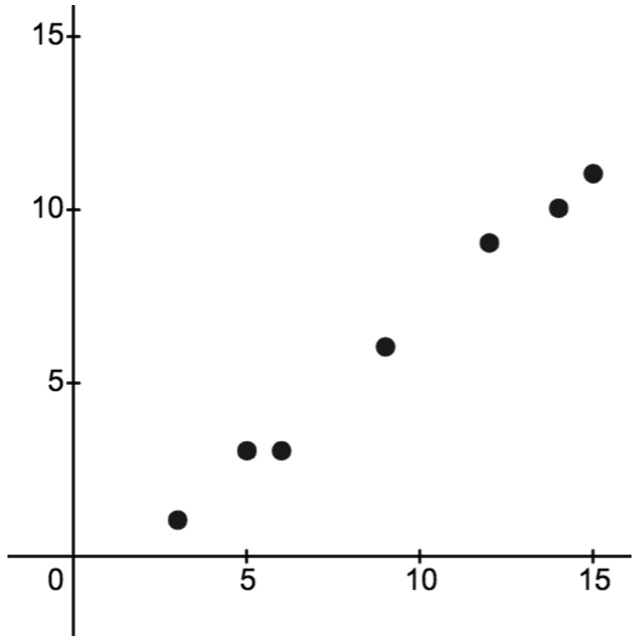
5.2.1 What are Scatter Plots?

One way to investigate the relationship between two variables is via a scatter plot. To create a scatter plot we graph ordered pairs for corresponding values of x and y, i.e., (x, y) in the cartesian plane and look for patterns.

Example: Two week ago Ben started a new job as a car salesman. His supervisor gives him the advice that the more test drives per day he gets his customers to take the more sales he will make per day. He records the following data over the past week.

x (Number of Test Drives Per Day)	y (Number of Sales Per Day)
3	1
5	3
6	3
9	6
12	9
14	10
15	11

When we plot this set of data as a scatter plot we get the following graph.



This clearly shows there is a relationship between the two variables. As x increases we see that y also increases. This shows there is what's called a positive linear correlation between the two variables.

5.2.2 Correlation

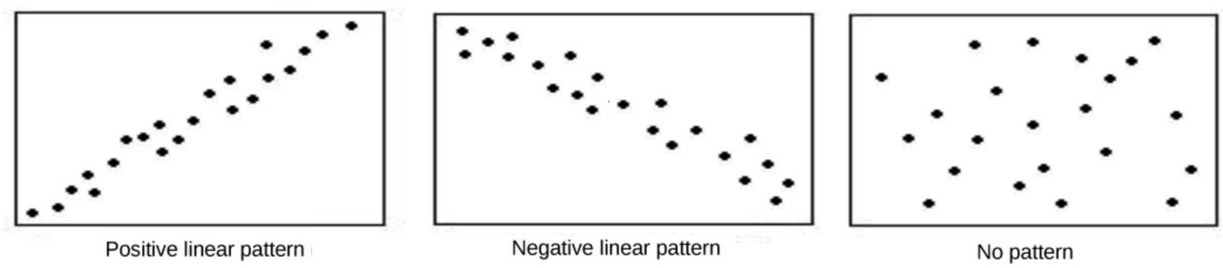
Specifically, linear correlation, is where there appears to be a linear relationship between the two variables. There are three types of correlation:

- Positive Correlation. As x increases y increases.

- Negative Correlation. As x increases y decreases.

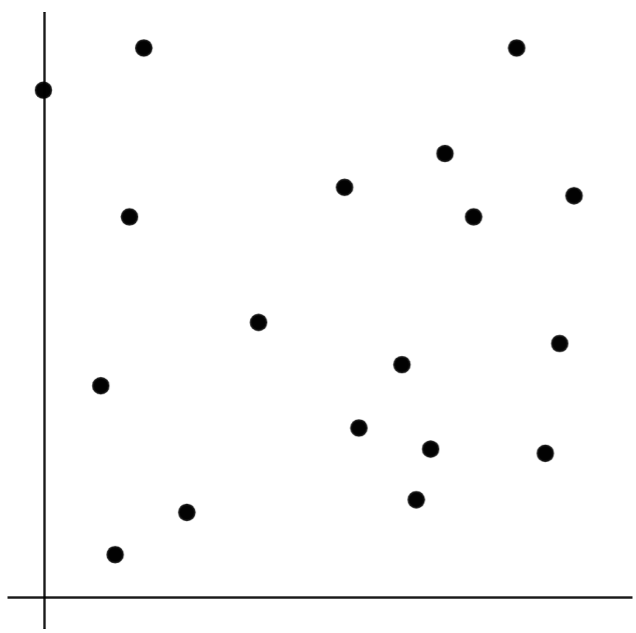
- Zero Correlation. There is no relationship.

Graphically, these cases look like the following.



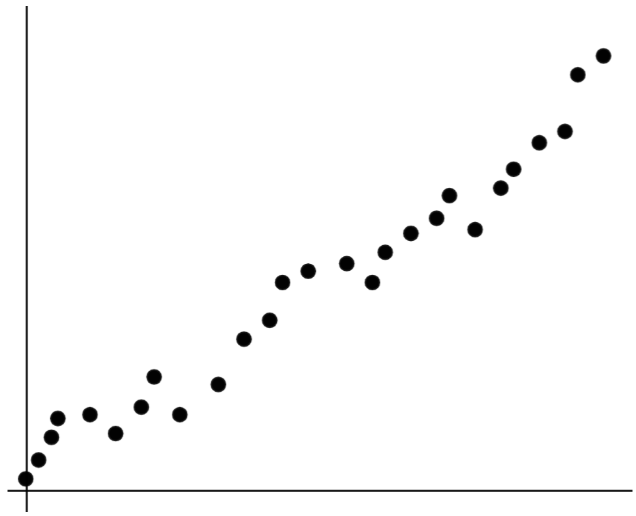
Example: Determine if the following scatter plots exhibit positive linear correlation, negative linear correlation, or no linear correlation.

a.)



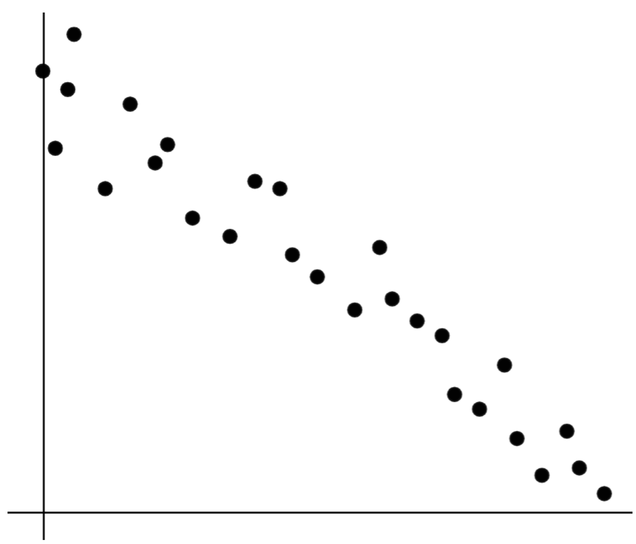
Solution: The graph shows no pattern. In this case there is no correlation.

b.)



Solution: The graph shows the pattern that as x increases, y increases. This is positive linear correlation.

c.)



Solution: The graph shows the pattern that as x increases, y decreases. This is negative linear correlation.

5.2.3 The Correlation Coefficient

A way to measure the strength of the correlation between two variables is to compute the Correlation Coefficient r. The easiest way to compute this value is via a graphing or statistical calculator. Some of these calculators will have operations built into them that will compute this value for you. Here are some facts about the Correlation Coefficient r.

Facts About the Correlation Coefficient r

- A positive value of r indicates positive correlation.
- A negative value of r indicates negative correlation.
- There is no linear relationship between x and y if $r = 0$.
- The value of the Correlation Coefficient r is always between -1 and $+1$, inclusive.
- If r is near -1 or $+1$ it then there is a stronger linear relationship between x and y.
- There is perfect positive linear correlation, if $r = 1$. The data points lie on a straight increasing line.
- There is perfect negative linear correlation, if $r = -1$. The data points lie on a straight decreasing line.

Example: Interpret the following correlation coefficients. State whether they indicate positive, negative, or no correlation. Indicate whether the correlation is strong or weak.

a $r = -0.27$

Solution:

Negative Correlation because r is negative.

The relationship is weak because $|r|$ is

is close to 0.

b: $r = 0.8$

Solution:

Positive Correlation because r is positive.

The relationship is strong because $|r|$ is

close to 1

Computing the Correlation Coefficient

As mentioned previously, the easiest way to compute the Correlation Coefficient r is via a graphing or statistical calculator. However, we can compute this value by hand by using the following formula.

Correlation Coefficient Formula

The Correlation Coefficient is computed using the following formula.

$$r = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right) - \left(\sum x\right)^2} \cdot \sqrt{n\left(\sum y^2\right) - \left(\sum y\right)^2}}$$

Where n is the number of data points, $\sum x$ is the sum of the x-values, $\sum y$ is the sum of the y-values, $\sum xy$ is the sum of the product of x and y values in each data point, $\sum x^2$ is the sum of the squares of the x-values, and $\sum y^2$ is the sum of the squares of the y-values.

Example: Let's reconsider our previous example with Ben the car salesman. Compute the Correlation Coefficient r for the data set.

x (Number of Test Drives Per Day)	y (Number of Sales Per Day)
3	1
5	3
6	3
9	6
12	9
14	10
15	11

Solution:

x	y	xy	x^2	y^2
3	1	3	9	1
5	3	15	25	9
6	3	18	36	9
9	6	54	81	36
12	9	108	144	81
14	10	140	196	100
15	11	165	225	121
$\sum x = 64$	$\sum y = 43$	$\sum xy = 503$	$\sum x^2 = 716$	$\sum y^2 = 357$

We have 7 data points so $n = 7$. So lets compute r.

$$r = \frac{7\,(503) - (64)\,(43)}{\sqrt{7\,(716) - (64)^2} \cdot \sqrt{7\,(357) - (43)^2}} = \frac{769}{\sqrt{916} \cdot \sqrt{650}} \approx 0.99660239$$

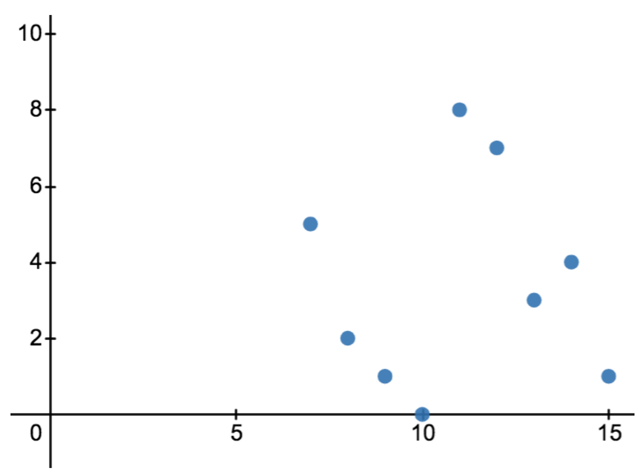
Since the correlation coefficient r is nearly 1, this shows that there is a strong linear correlation between x and y.

Example: Ashley has decided to collect the shoe size and number of hours playing videos games from her 7th grade class. The data below is collected.

x (Shoe Size)	y (Hours Playing Video Games Per Week)
7	5
8	2
9	1
10	0
11	8
12	7
13	3
14	4
15	1

Graph the Scatter Plot of the ordered pairs. Interpret the results. Describe the correlation. Is it negative, positive, or no correlation? Is it strong or weak? Describe the correlation coefficient. Make your judgement based off the scatter plot.

Solution:



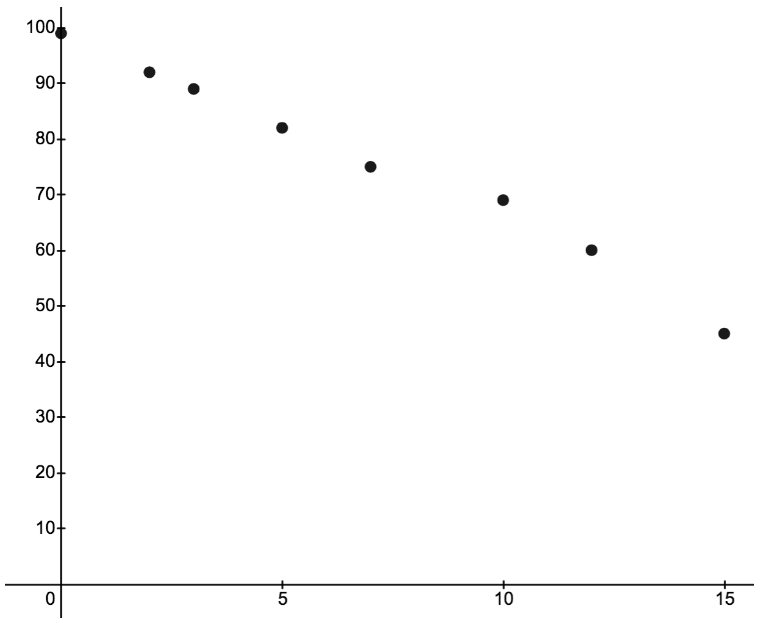
There is no pattern from the scatter plot. There is no correlation. r will be near 0.

Example: Dr.Jones is a professor at UWG. She's looking over her grades for the previous semester and notices that there is a relationship between the number of absences a student has and their numerical grade in the class. She organized this data in the table below.

x Number of Absences	y Numerical Grade
0	99
2	92
3	89
5	82
7	75
10	69
12	60
15	45

Graph the Scatter Plot of the ordered pairs. Interpret the results. Describe the correlation. Is it negative, positive, or no correlation? Is it strong or weak? Describe the correlation coefficient. Make your judgement based off the scatter plot.

Solution:



These two variables have a strong negative linear correlation because as x increases, y decreases and the scatter plot has a linear shape. The correlation coefficient r is negative and closer to -1 than 0.

****Try this on your own**: Create a scatterplot for the data below. Is the linear correlation positive or negative, weak or strong?

X	1	2	4	4	5	6	7	9	10
Y	6	4	6	8	10	8	9	11	9

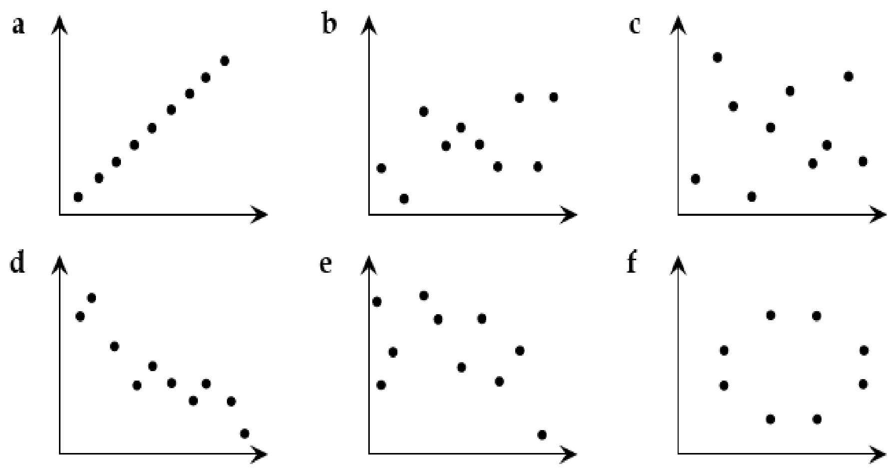
5.2.4 Exercises: Scatter Plots

Solutions appear at the end of this textbook.

1. Explain what correlation is, and the difference between positive and negative correlation.

2. Match the most likely linear correlation values to the graphs below.

   $r = +0.7 \quad r = +0.99 \quad r = -0.4 \quad r = +0.15 \quad r = -0.86 \quad r = 0$



3. For the data below, draw a scatter plot and use the formula to calculate the correlation coefficient r.

X	2	5	7	10	12	14
Y	2	6	7	9	11	14

4. For the data below, use the calculator to find the correlation r. Is the r correlation positive or negative, weak or strong?

SAT math	643	558	703	512	552	430	605
College GPA	3.52	2.91	3.63	2.21	3.02	2.80	3.18

5.3 Data Patterns, Linear Models and Regression

If we are going to create regression or best fit lines, then it is important that we recognize when it is appropriate

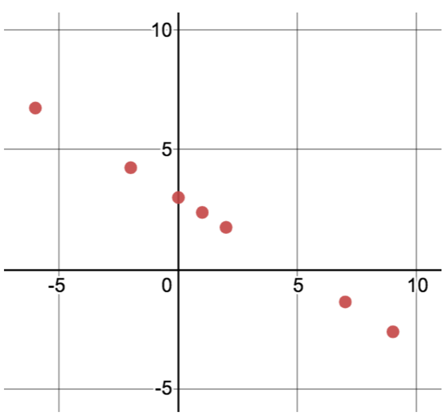
Regression Lines or best fit lines are used to model linear data. Therefore it is important that we are able to recognize the pattern of linear data. Here we will also consider other patterns of data before getting practice creating linear regression equations.

5.3.1 Types of Data Patterns

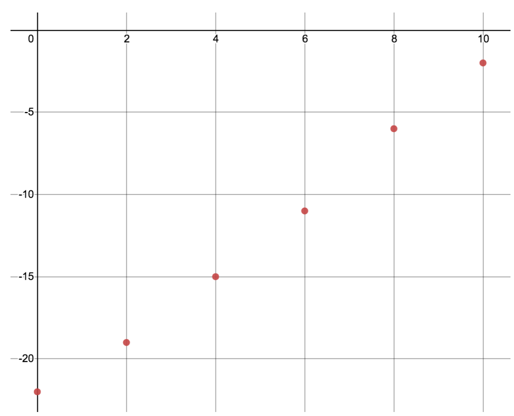
Linear Data follows fairly close to a straight line. The points may fluctuate up and down some, but still follow the trend line. If the points increase from left to right, then the data follows a positive trend line. If the points decrease from left to right, then the data follows a negative trend line.

Examples: Here are some examples of data that are linear in pattern.

a:



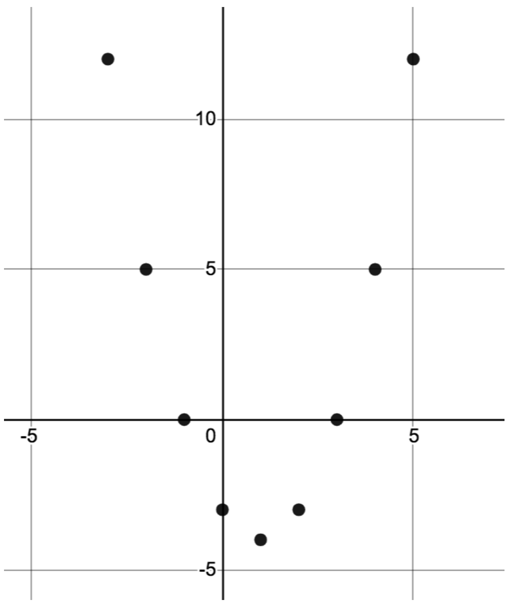
b:



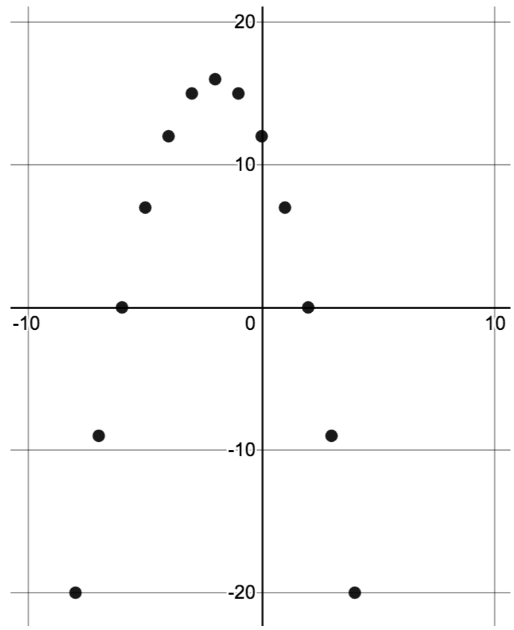
Quadratic Data follows fairly close to a parabola "U" shape. The parabola could be upside down as well.

Examples: Here are some examples of data that are quadratic in pattern.

a:



b:



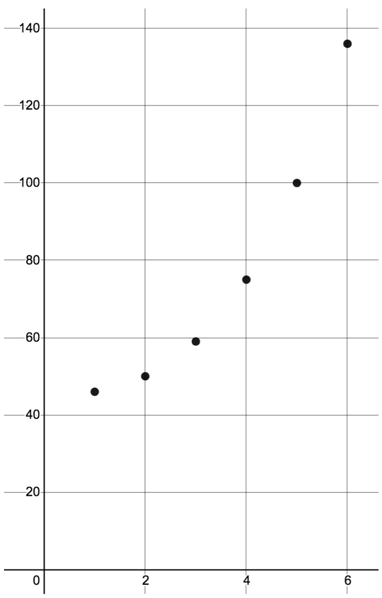
The first scatter plot follows a regular upward parabola "U".

The second one follows an upside down parabola shape, but it is still the quadratic shape.

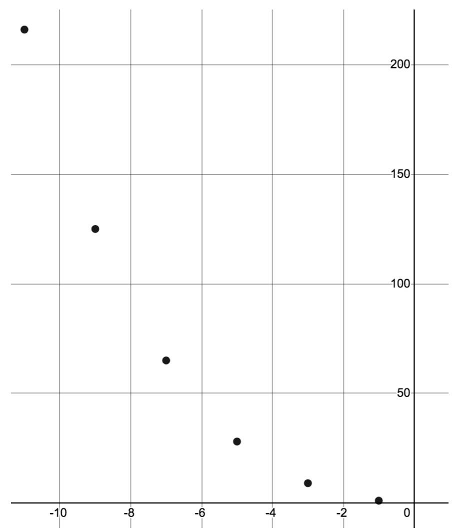
Exponential Data follows fairly close to an upward curve "J" shape. The curve could be backwards or upside down as well.

Examples: Here are some examples of data that are exponential in pattern.

a:



b:



The first scatter plot follows a regular upward "J" curve.

The second one follows backwards (decreasing) "J" curve.

5.3.2 Linear Models and Regression Lines

If we have determined that two variables have a linear relationship, then we can create the equation of the regression line.

Definition: The **Equation of the Regression Line** is $y = mx + b$, where

$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2}, \qquad b = \frac{\sum y - m\left(\sum x\right)}{n}$$

n is the number of data points, $\sum x$ is the sum of the x-values, $\sum y$ is the sum of the y-values, $\sum xy$ is the sum of the product of x and y values in each data point, and $\sum x^2$ is the sum of the squares of the x-values.

Example: Let's reconsider our previous example with Ben the car salesman. Two week ago Ben started a new job as a car salesman. His supervisor gives him the advice that the more test drives he gets his customers to take the more sales he will make. He records the following data over the past week.

x (Number of Test Drives)	y (Number of Sales)
3	1
5	3
6	3
9	6
12	9
14	10
15	11

Find the equation of the regression line. Round the slope m and the intercept b to two decimals.

Solution:

x	y	xy	x^2
3	1	3	9
5	3	15	25
6	3	18	36
9	6	54	81
12	9	108	144
14	10	140	196
15	11	165	225
$\sum x = 64$	$\sum y = 43$	$\sum xy = 503$	$\sum x^2 = 716$

We have 7 data points so $n = 7$. Lets compute the slope of the regression line m.

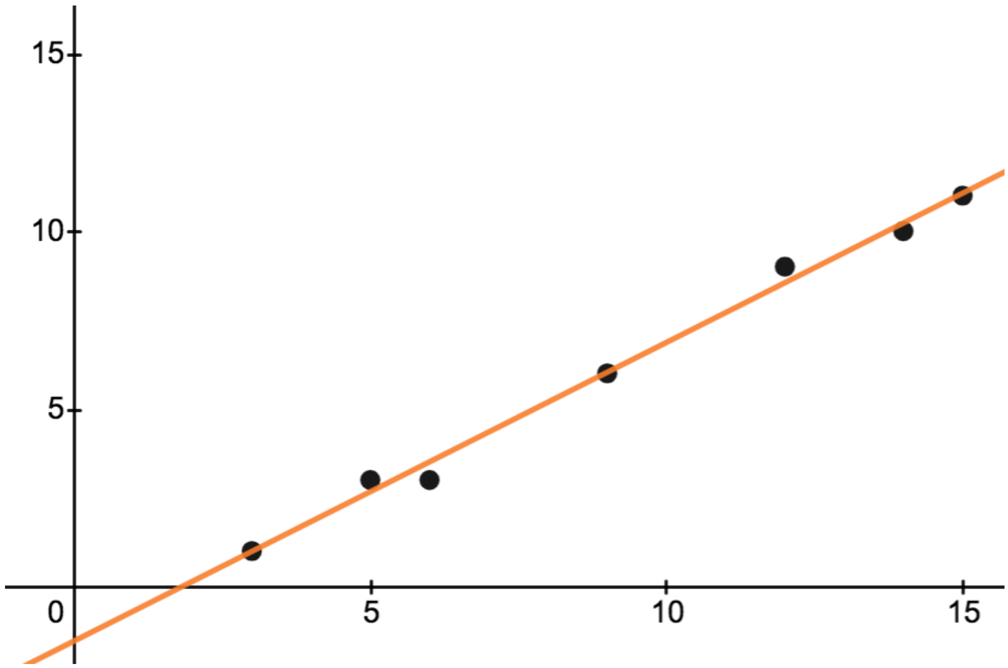
$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} = \frac{7(503) - (64)(43)}{7(716) - (64)^2} = \frac{769}{916} \approx 0.83951965$$

So $m = 0.84$. Now we will compute the y-intercept b.

$$b = \frac{\sum y - m\left(\sum x\right)}{n} = \frac{43 - (0.83951965)(64)}{7} \approx -1.53275109$$

This gives $b = -1.53$. The regression line for this data set is $y = 0.84x - 1.53$.

Here is the scatter plot of the data set with the regression line that we've created.



Example: Construct the linear regression line for the following data set. Round your slope and y-intercept to two decimal places.

x	13	16	19	23	28	30
y	38	42	51	54	58	62

Solution:

x	y	xy	x^2
13	38	494	169
16	42	672	256
19	51	969	361
23	54	1242	529
28	58	1624	784
30	62	1860	900
$\sum x = 129$	$\sum y = 305$	$\sum xy = 6861$	$\sum x^2 = 2999$

We have 6 data points so $n = 6$. Lets compute the slope of the regression line m.

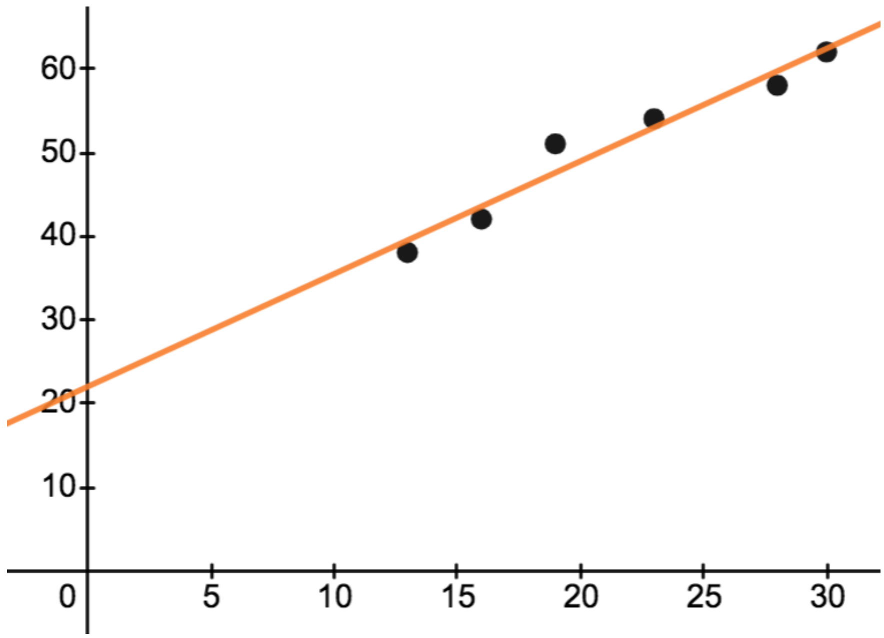
$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} = \frac{6(6861) - (129)(305)}{6(2999) - (129)^2} = \frac{1821}{1353} \approx 1.345898$$

So $m = 1.35$. Now we will compute the y-intercept b.

$$b = \frac{\sum y - m\left(\sum x\right)}{n} = \frac{305 - (1.345898)(129)}{6} \approx 21.89652624$$

This gives $b = 21.90$. The regression line for this data set is $y = 1.35x + 21.9$.

Here is the scatter plot of the data set with the regression line that we've created.



Example: Adam is a personal trainer with 6 clients. He's reviewing their progress and notices the more cardio days per week (x) his clients have, the lower the body fat percent (y) they have. He records this data in the table below.

x (Cardio Days Per Week)	1	2	2	3	4	4
y (Body Fat Percent)	23	18	16	12	9	4

Construct the linear regression line for the data set. Round your slope and y-intercept to two decimal places.

Solution:

x	y	xy	x^2
1	23	23	1
2	18	36	4
2	16	32	4
3	12	36	9
4	9	36	16
4	4	16	16
$\sum x = 16$	$\sum y = 82$	$\sum xy = 179$	$\sum x^2 = 50$

We have 6 data points so $n = 6$. Lets compute the slope of the regression line m.

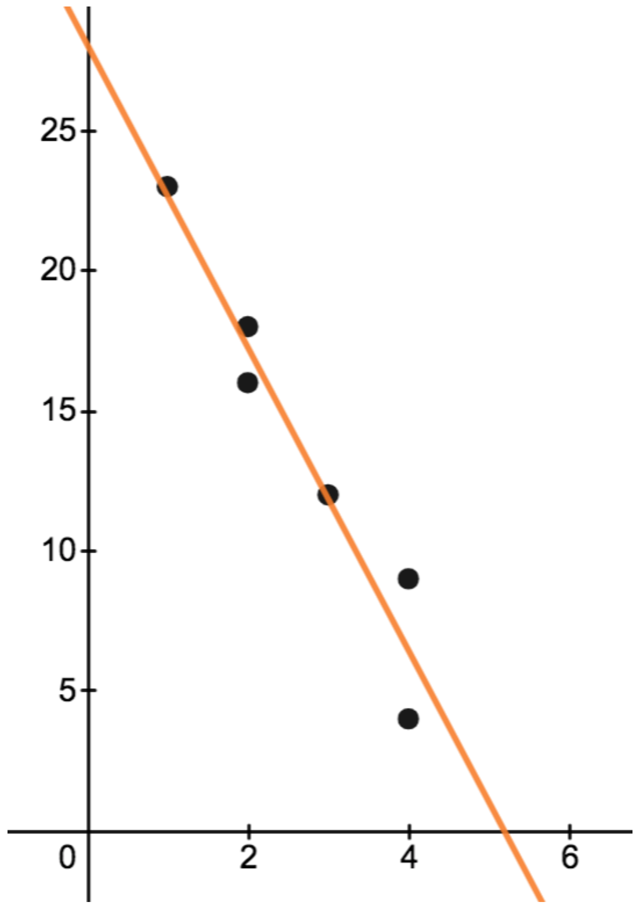
$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} = \frac{6(179) - (16)(82)}{6(50) - (16)^2} = \frac{-238}{44} \approx -5.4090909$$

So $m = -5.41$. Now we will compute the y-intercept b.

$$b = \frac{\sum y - m\left(\sum x\right)}{n} = \frac{82 - (-5.4090909)(16)}{6} \approx 28.0909091$$

This gives $b = 28.09$. The regression line for this data set is $y = -5.41x + 28.09$.

Here is the scatter plot of the data set with the regression line that we've created.



The regression equation can be used to predict or forecast output values for given input values. If the input value falls in the range of the original data, it is called **Interpolation**. Interpolation usually gives reasonable and realistic predictions, since the inputs are within the data range.

If the input value falls outside the range of the original data, it is called **Extrapolation**. Beware of using predictions with extrapolation. For example, if you have sales data from 1999 – 2009, predicting sales for 2024 is too far into the future to be reasonable.

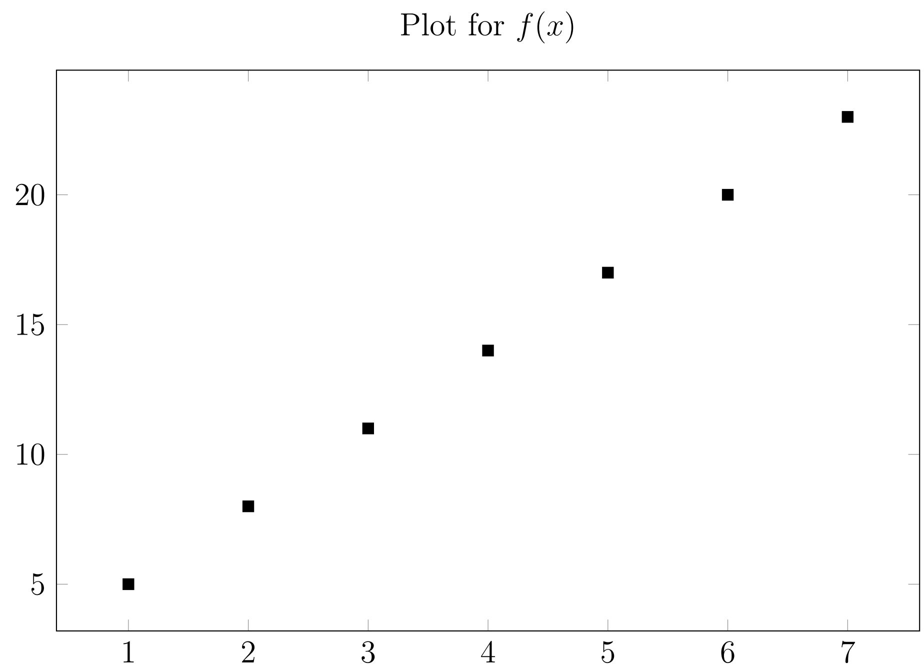
****Try this on your own**: The table below shows data from ten people of their average monthly spending on fast food, as well as their average number of days of exercise each month. Create a scatterplot and find the correlation coefficient r. Then find the regression equation and use it to forecast the number of days of exercise output for the fast food value \$70. Is that prediction interpolation or extrapolation? Do you notice a pattern between fast food spending and exercise? What could explain the pattern?

fast food \$	20	40	58	50	140	30	90	45	100	120
exercise days	20	15	13	11	3	26	7	18	12	1

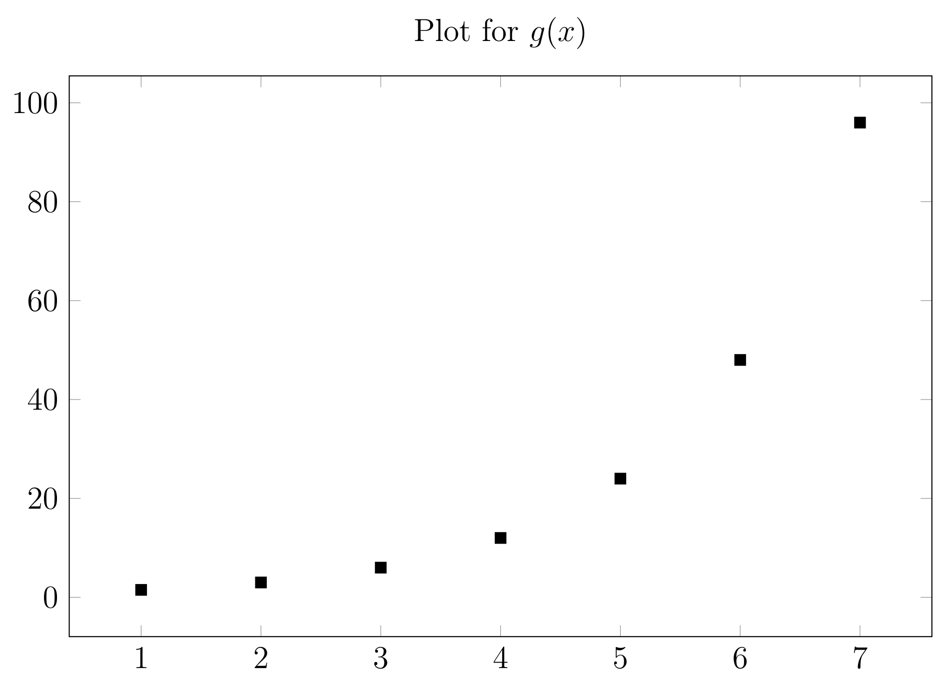
5.3.3 Exercises: Models and Regression

Solutions appear at the end of this textbook.

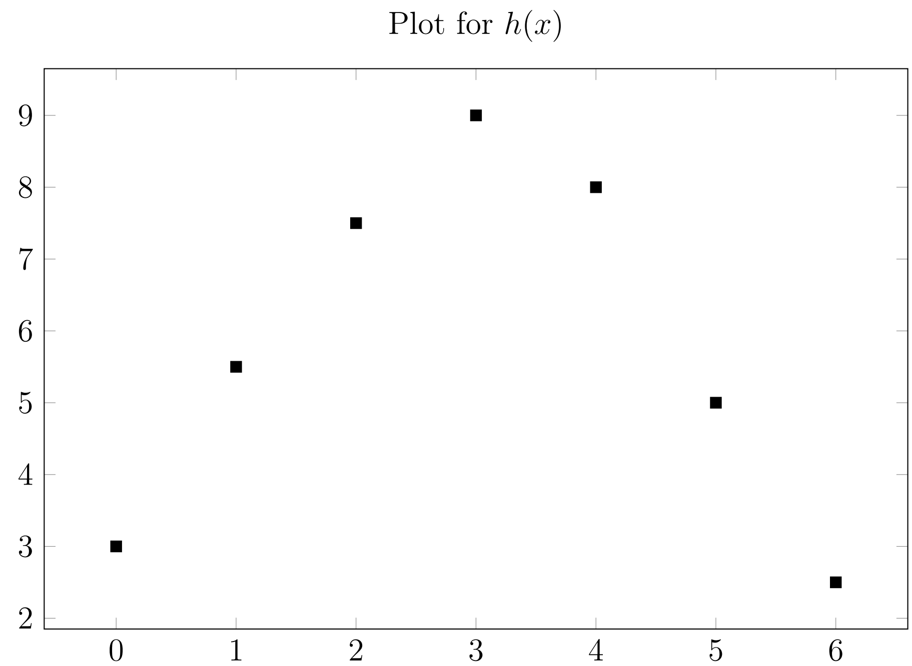
1. Which shape does the data pattern best fit with? Linear, quadratic, or exponential?



2. Which shape does the data pattern best fit with? Linear, quadratic, or exponential?



3. Which shape does the data pattern best fit with? Linear, quadratic, or exponential?

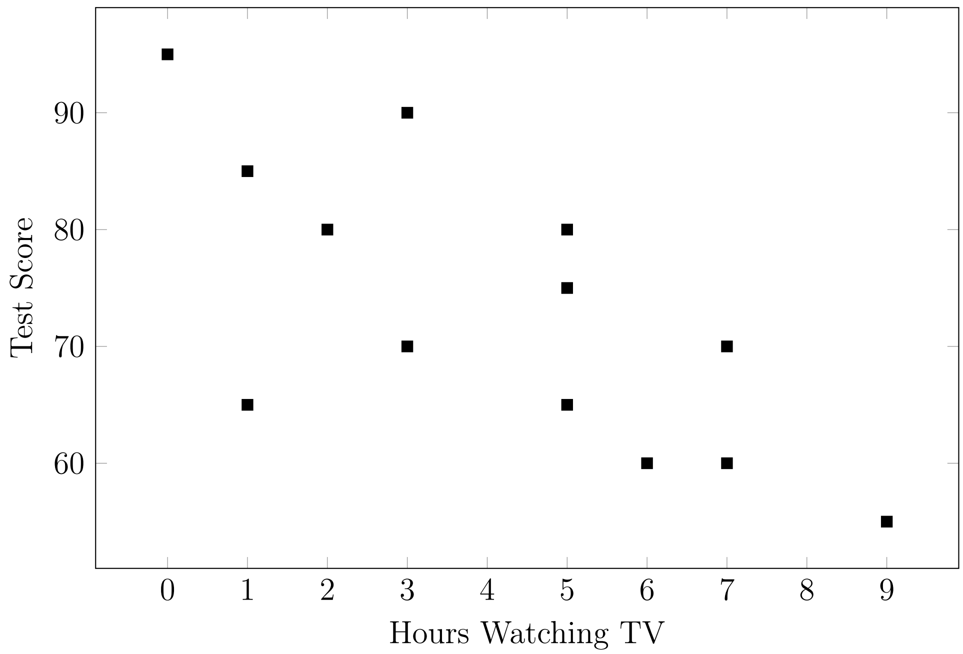


4. For the data below create the scatterplot. Find the regression equation for the SAT/GPA data. Plot the regression line on the scatterplot. Does it fit well?

SAT math	643	558	703	512	552	430	605
College GPA	3.52	2.91	3.63	2.21	3.02	2.80	3.18

5. Use the regression line from the previous problem to predict GPA for math scores of 760 and 500. Are these predictions interpolation or extrapolation?

6. Draw a good estimate of the regression line for the scatterplot below. What relationship do you notice between test scores and TV watching? Which one do you think causes the other, or could there be a hidden variable causing both?



7. What is the relationship between a linear correlation coefficient r and the slope of the corresponding regression line?

Chapter 6

Measurement and Units

Written by James Bellon

6.1 The Metric System

In many of the problems in the previous sections, you had measurements that had specific units, such as a person 5 feet 6 inches tall, or a baby weighing 4 kilograms, etc. Units are very important in most fields, especially science, engineering, business, construction, transportation, and most aspects of real life. It is important to be able to understand different units and how to convert between them. Most measurements in the U.S. use what is known as the **English System of Units**.

Some of the common units used in the English System are as follows:

- For length: inch, foot, yard, mile.
- For area: acre, square inch, square yard.
- For volume: cubic foot, cup, quart, gallon, tablespoon.
- For weight: ounce, pound, ton.

6.1.1 The Metric System

The English System may be familiar to you, but in the international community, it is very confusing. It uses many different names of units and many different size ratios.

Several scientists got together and came up with a simple system of units called the **Metric System**. It is used as the primary system in most countries (in the USA it is used secondary). It is based on powers of ten and uses common prefixes to represent larger and smaller variations of basic units. Once you become familiar with it, it is much easier to use and convert units in the metric system.

One of the great features of the metric system is that no matter how small or large an object is, the units used have the same base name, just with added prefixes which signify the size. On the next page is a table with the common prefixes used in the metric system.

Prefix	Symbol	Meaning
giga	G	$1,000,000,000$ times base unit
mega	M	$1,000,000$ times base unit
kilo	k	$1,000$ times base unit
hecto	h	100 times base unit
deka	da	10 times base unit
deci	d	$\frac{1}{10}$ or 0.1 of base unit
centi	c	$\frac{1}{100}$ or 0.01 of base unit
milli	m	$\frac{1}{1{,}000}$ or 0.001 of base unit
micro	μ	$\frac{1}{1{,}000{,}000}$ or 0.000001 of base unit
nano	n	$\frac{1}{1{,}000{,}000{,}000}$ or 0.000000001 of base unit

For example, the basic unit for linear measure (length) in the metric system is the **Meter**. To make larger or smaller units of length, we just need to place an appropriate prefix in front of meter. So 100 meters is also known as a hectometer. One-tenth of a meter is also known as a decimeter.

To change from larger units to smaller units involves multiplying by ten repeatedly. To change from smaller units to larger units involves dividing by ten repeatedly. It also helps to have an understanding of about how big (or small) the metric units are. When doing conversions, make sure the answer makes sense. Here are some simple benchmarks to use as approximations.

- One nanometer is approximately the diameter of the largest atom
  (only visible under an electron microscope).
- One millimeter is approximately the width of the lead in a pencil.
- One centimeter is approximately the width of an average pinky finger.
- One meter is approximately the length of a large man's arm.
- One dekameter is approximately the length of a tractor-trailer truck (18 wheeler).
- One kilometer is approximately nine football fields end to end (including end-zones).
- One Megameter is approximately the radius of the planet Earth.

6.1.2 Unit Fractions

When converting units, it is recommended that you setup the units as **Unit Fractions**, also known as **Conversion Ratios**. Unit fractions are ratios that show two measurements which are equal in size, but have different values with different units.

For example, 1 foot is the same as 12 inches, and can be written as a unit fraction $\frac{1\ ft}{12\ in}$ or $\frac{12\ in}{1\ ft}$. Some other important lengths you should know are 1 yard = 3 feet and 1 mile = 5,280 feet.

To convert a measurement to a different unit, multiply by the appropriate unit fraction(s). The given unit of measurement that you wish to cancel, should appear in the denominator of the unit fraction, so that this unit cancels upon multiplication. The unit measurement you wish to convert into should appear in the numerator of the fraction, so that this unit will be retained upon multiplication.

Even for simple conversions, it is a good habit to use unit fractions, so you will be able to handle the more complicated conversions. The benefit of unit fractions is that it keeps everything organized and it makes it clear as to which numbers multiply and which divide. Also, if the units don't cancel properly, then you know it is setup incorrectly.

For example, it is fairly obvious that if one yard is 3 feet and each foot is 12 inches, then 1 yard is 36 inches. With unit fractions, this would look like the following:

$$1\ yard \left(\frac{3\ ft}{1\ yd}\right)\left(\frac{12\ in}{1\ ft}\right) = 1\ \cancel{yard}\left(\frac{3\ \cancel{ft}}{1\ \cancel{yd}}\right)\left(\frac{12\ in}{1\ \cancel{ft}}\right) = \frac{1(3)(12)\ in}{1(1)} = 36\ in$$

Example: Use unit fractions to convert 40,000 inches into miles.

Solution: Setup unit fractions as

$$\begin{aligned} 40,000\ in \left(\frac{1\ ft}{12\ in}\right)\left(\frac{1\ mile}{5,280\ ft}\right) &= 40,000\ \cancel{in}\left(\frac{1\ \cancel{ft}}{12\ \cancel{in}}\right)\left(\frac{1\ mile}{5,280\ \cancel{ft}}\right) \\ &= \frac{40,000(1)(1)\ miles}{12(5,280)} \\ &= 0.63\ miles \end{aligned}$$

Example: Convert 504.7 meters to kilometers and 27 meters to nanometers.

Solution: Setup unit fractions as

$$504.7\ m\left(\frac{1\ km}{1,000\ m}\right) = 504.7\ \cancel{m}\left(\frac{1\ km}{1,000\ \cancel{m}}\right) = \frac{504.7(1)(1)\ km}{1,000} = 0.5047\ km$$

$$\begin{aligned} 27\ m\left(\frac{1,000,000,000\ nm}{1\ m}\right) &= 27\ \cancel{m}\left(\frac{1,000,000,000\ nm}{1\ \cancel{m}}\right) \\ &= (27\ nm)(1,000,000,000) = 27,000,000,000\ nm \\ &= 2.7 \times 10^{10}\ nm \end{aligned}$$

****Try this on your own**: Convert 7 yds to inches and 310,000 cm to hectometers.

6.1.3 Converting Between Systems

There are many approximate English to Metric Equivalents, but only one which is exact. It is 1 inch = 2.54 cm. To be accurate, we must go through this exact conversion to get the answer. If we just want an estimate, we can use one of many approximate conversions easily found in books or on the internet. Such as 1 mile = 1.6 km, etc.

Example: Convert 125 miles to kilometers, and 26,800 millimeters to inches.

Solution: For the first conversion, we start with 125 miles and multiply by unit fractions to get to feet and inches, then convert over to centimeters, then to meters and finally kilometers. Notice that the answer makes sense, because one kilometer is slightly more than half a mile, so 125 miles will be slightly less than double in kilometers.

$$
\begin{aligned}
&125\ miles\left(\frac{5,280\ ft}{1\ mile}\right)\left(\frac{12\ in}{1\ ft}\right)\left(\frac{2.54\ cm}{1\ in}\right)\left(\frac{1\ m}{100\ cm}\right)\left(\frac{1\ km}{1,000\ m}\right)\\
&= 125\,\cancel{miles}\left(\frac{5,280\,\cancel{ft}}{1\,\cancel{mile}}\right)\left(\frac{12\,\cancel{in}}{1\,\cancel{ft}}\right)\left(\frac{2.54\,\cancel{cm}}{1\,\cancel{in}}\right)\left(\frac{1\,\cancel{m}}{100\,\cancel{cm}}\right)\left(\frac{1\ km}{1,000\,\cancel{m}}\right)\\
&= \frac{125(5,280)(12)(2.54)\ km}{100(1,000)}\\
&= 201.168\ km
\end{aligned}
$$

For the second conversion, we start with millimeters and multiply by unit fractions to get to centimeters, then convert over to inches. Inches are much bigger than millimeters, so the answer should be much smaller than 26,800.

$$
\begin{aligned}
&26,800\ mm\left(\frac{1\ cm}{10\ mm}\right)\left(\frac{1\ in}{2.54\ cm}\right)\\
&= 26,800\,\cancel{mm}\left(\frac{1\,\cancel{cm}}{10\,\cancel{mm}}\right)\left(\frac{1\ in}{2.54\,\cancel{cm}}\right)\\
&= \frac{26,800\ in}{10(2.54)}\\
&= 1,055.12\ in
\end{aligned}
$$

****Try this on your own**: Convert 8 yds to centimeters using the conversion $1ft = 30.5cm$

6.1.4 Exercises: Length and the Metric System

Solutions appear at the end of this textbook.

1. Convert 237,600 feet to miles.
2. Convert 22 yards into inches.
3. What are the metric prefixes that mean one-tenth and one Hundred?
4. Convert 7 kilometers to centimeters.
5. Convert 45,000 millimeters to meters.
6. Convert 4 km to yards (round to nearest 0.1).
7. Convert 18 feet to decimeters (round to nearest 0.01).

6.2 Area and Volume

6.2.1 Area

As you probably learned in Geometry, **Area** is the amount of space enclosed inside a shape along a flat surface. Areas can be measured in square units. That is, we can still use linear units (lengths) but square them to get area.

For example, one square foot can be represented as a square shape that measures one foot on each side. Then each side is also 12 inches. This means that one square foot $= 12\ in \cdot 12\ in = 144$ square inches. However, an area of one square foot, does not have to be a square shape. It could be a long skinny rectangle (5 feet by 0.2 feet), a circle, or even an odd irregular shape. Just as long as the enclosed area is the same size.

To convert areas, we can still use conversion ratios (unit fractions), but we just have to remember that when we go from regular units to squared units, we square the numbers in the ratios as well. For example, the unit fraction $\frac{1\ ft}{12\ in}$ becomes $\frac{1\ ft^2}{144\ in^2}$, after being squared.

Example: Use unit fractions to convert 1,000 square feet to square yards.

Solution: Setup unit fractions as

$$\begin{aligned} 1,000\ ft^2 \left(\frac{1\ yd}{3\ ft}\right)^2 &= 1,000\ \cancel{ft^2} \left(\frac{1\ yd^2}{9\ \cancel{ft^2}}\right) \\ &= \frac{1,000\ yd^2}{9} \\ &= 111.1\ yd^2 \end{aligned}$$

Example: Use unit fractions to convert 7 square meters to square centimeters.

Solution: Setup unit fractions as

$$\begin{aligned} 7\ m^2 \left(\frac{100\ cm}{1\ m}\right)^2 &= 7\ \cancel{m^2} \left(\frac{10,000\ cm^2}{1\ \cancel{m^2}}\right) \\ &= \frac{7(10,000)\ cm^2}{1} \\ &= 70,000\ cm^2 \end{aligned}$$

When converting between the metric and English systems for area, make sure you setup unit fractions and don't forget the square them (including the numbers) to end up with the appropriate units.

Example: Convert 13 square miles to square meters.

Solution: We start with 13 square miles and multiply by unit fractions to get to feet and inches, then convert over to centimeters, then to meters.

$$13\ miles^2 \left(\frac{5,280\ ft}{1\ mile}\right)^2 \left(\frac{12\ in}{1\ ft}\right)^2 \left(\frac{2.54\ cm}{1\ in}\right)^2 \left(\frac{1\ m}{100\ cm}\right)^2$$

$$= 13\ miles^2 \left(\frac{27,878,400\ ft^2}{1\ mile^2}\right) \left(\frac{144\ in^2}{1\ ft^2}\right) \left(\frac{6.4516\ cm^2}{1\ in^2}\right) \left(\frac{1\ m^2}{10,000\ cm^2}\right)$$

$$= 13\ \cancel{miles^2} \left(\frac{27,878,400\ \cancel{ft^2}}{1\ \cancel{mile^2}}\right) \left(\frac{144\ \cancel{in^2}}{1\ \cancel{ft^2}}\right) \left(\frac{6.4516\ \cancel{cm^2}}{1\ \cancel{in^2}}\right) \left(\frac{1\ m^2}{10,000\ \cancel{cm^2}}\right)$$

$$= \frac{13(27,878,400)(144)(6.4516)\ m^2}{10,000}$$

$$= 33,669,845\ m^2$$

Both systems have special units for land areas. The English system uses a unit called an **Acre**, and 1 acre $= 43,560$ square feet. The metric system uses a unit called an **Are** (pronounced like the word "air"), and 1 $are = 100\ m^2$. Since this is too small for land, they typically use a hectare which is 100 $are = 10,000\ m^2 = 0.01\ km^2$.

Example: The University of West Georgia campus is 644 acres. How many square feet is the campus? How many hectares?

Solution: The first one is a straight conversion from acres to square feet. Notice the unit fraction does not have to be squared, since it already has area units. Acres are large, so the answer is a large number of square feet.

$$644\ acres\left(\frac{43,560\ ft^2}{1\ acre}\right) = 644\ \cancel{acres}\left(\frac{43,560\ ft^2}{1\ \cancel{acre}}\right) = 644(43,560)\ ft^2 = 28,052,640\ ft^2$$

The second one has many conversions.

$$644\ acres\left(\frac{43,560\ ft^2}{1\ acre}\right)\left(\frac{12\ in}{1\ ft}\right)^2\left(\frac{2.54\ cm}{1\ in}\right)^2\left(\frac{1\ m}{100\ cm}\right)^2\left(\frac{1\ are}{100\ m^2}\right)\left(\frac{1\ hectare}{100\ ares}\right)$$

$$= 644\ acres\left(\frac{43,560\ ft^2}{1\ acre}\right)\left(\frac{144\ in^2}{1\ ft^2}\right)\left(\frac{6.4516\ cm^2}{1\ in^2}\right)\left(\frac{1\ m^2}{10,000\ cm^2}\right)\left(\frac{1\ are}{100\ m^2}\right)\left(\frac{1\ hectare}{100\ ares}\right)$$

$$= 644\ \cancel{acres}\left(\frac{43,560\ \cancel{ft^2}}{1\ \cancel{acre}}\right)\left(\frac{144\ \cancel{in^2}}{1\ \cancel{ft^2}}\right)\left(\frac{6.4516\ \cancel{cm^2}}{1\ \cancel{in^2}}\right)\left(\frac{1\ \cancel{m^2}}{10,000\ \cancel{cm^2}}\right)\left(\frac{1\ \cancel{are}}{100\ \cancel{m^2}}\right)\left(\frac{1\ hectare}{100\ \cancel{ares}}\right)$$

$$= \frac{644(43,560)(144)(6.4516)\ hectares}{10,000(100)(100)}$$

$$= 260.62\ hectares$$

6.2.2 Volume

Another geometry concept you should have learned is **Volume**, which is the amount of space inside a three dimensional shape. Think of it as how much liquid it would take to fill the space. Volumes can be measured in cubic units. That is, we can still use linear units (lengths) but raise them to the third power to get volume.

For example, one cubic yard can be represented as a cube shape that measures one yard on each side. Then each side is also 3 feet. This means that one cubic yard $= 3\ ft \times 3\ ft \times 3\ ft = 27$ cubic feet. To convert, we can still use conversion ratios (unit fractions). We just have to remember that when we go to cubic units, we cube the numbers as well.

Example: Use unit fractions to convert 900,000 cubic yards to cubic miles.

Solution: Setup unit fractions as

$$900,000\ yd^3 \left(\frac{3\ ft}{1\ yd}\right)^3 \left(\frac{1\ mile}{5,280\ ft}\right)^3$$

$$= 900,000\ \cancel{yd^3} \left(\frac{27\ \cancel{ft^3}}{1\ \cancel{yd^3}}\right) \left(\frac{1\ mile^3}{147,197,952,000\ \cancel{ft^3}}\right)$$

$$= \frac{900,000(27)\ mile^3}{147,197,952,000}$$

$$= 0.000165\ mile^3$$

Surprising that so many cubic yards is only a small fraction of a cubic mile. This means that a cubic mile is HUGE! As a reference, it is the amount of water inside Devil's Lake in North Dakota, which is a large lake with fishing, boating, and a state park on an island in the middle of it.

Example: Convert 2 km^3 to cm^3.

Solution: Setup unit fractions as

$$2\ km^3 \left(\frac{1,000\ m}{1\ km}\right)^3 \left(\frac{100\ cm}{1\ m}\right)^3$$

$$= 2\ \cancel{km^3} \left(\frac{1,000,000,000\ \cancel{m^3}}{1\ \cancel{km^3}}\right) \left(\frac{1,000,000\ cm^3}{1\ \cancel{m^3}}\right)$$

$$= 2(1,000,000,000)(1,000,000)cm^3$$

$$= 2 \times 10^{15}\ cm^3$$

When converting between the metric and English systems for volume in cubic units, make sure you setup unit fractions and don't forget the cube them (including the numbers) to end up with the appropriate units.

Example: Convert $7,000\ mm^3$ to in^3.

Solution: Setup unit fractions as

$$7,000\ mm^3 \left(\frac{1\ cm}{10\ mm}\right)^3 \left(\frac{1\ in}{2.54\ cm}\right)^3$$

$$= 7,000\ \cancel{mm^3} \left(\frac{1\ \cancel{cm^3}}{1,000\ \cancel{mm^3}}\right) \left(\frac{1\ in^3}{16.387\ \cancel{cm^3}}\right)$$

$$= \frac{7,000}{1,000(16.387)} in^3$$

$$= 0.43\ in^3$$

Both systems have special units for measuring volume. The English system uses many different units. Some of the common units, their abbreviations and their equivalents are:

1 teaspoon (tsp)		
1 tablespoon (Tbsp)	= 3 teaspoons	
1 fluid ounce (fl.oz.)	= 2 Tbsp	
1 cup	= 8 fl.oz.	
1 pint	= 2 cups	= 16 fl.oz.
1 quart	= 2 pints	= 32 fl. oz.
1 gallon	= 4 quarts	= 128 fl. oz.
1 cubic foot	= 7.5 gallons	(rounded)

Example: How many cups in a gallon?

Solution: Setup unit fractions as

$$1\ gal \left(\frac{128\ fl.oz}{1\ gal}\right)\left(\frac{1\ cup}{8\ fl.oz}\right)$$

$$= 1\ \cancel{gal} \left(\frac{128\ \cancel{fl.oz}}{1\ \cancel{gal}}\right)\left(\frac{1\ cup}{8\ \cancel{fl.oz}}\right)$$

$$= \frac{1(128)}{8}\ cups$$

$$= 16\ cups$$

Example: How many quarts in 50 cubic feet?

Solution: Setup unit fractions as

$$50\ ft^3 \left(\frac{7.5\ gal}{1\ ft^3}\right)\left(\frac{4\ qts}{1\ gal}\right)$$

$$= 50\ \cancel{ft^3} \left(\frac{7.5\ \cancel{gal}}{1\ \cancel{ft^3}}\right)\left(\frac{4\ qts}{1\ \cancel{gal}}\right)$$

$$= \frac{50(7.5)(4)}{1}\ qts$$

$$= 1500\ quarts$$

The base unit of volume in the metric system is called a **Liter** (abbreviated as L). The Liter was set to be the exact volume of a cube, 10 centimeters on each side. On a smaller scale, 1 milliliter = 1 cubic centimeter, which is also known as a "cc". The same prefixes and factors that were used for Meters, are also used for Liters. This is another reason that the Metric System is easy to use.

Example: Convert 3.5 kiloliters to milliliters.

Solution: Setup unit fractions as

$$
\begin{aligned}
&3.5\ kL\left(\frac{1,000\ L}{1\ kL}\right)\left(\frac{1,000\ mL}{1\ L}\right)\\
&= 3.5\ \cancel{kL}\left(\frac{1,000\ \cancel{L}}{1\ \cancel{kL}}\right)\left(\frac{1,000\ mL}{1\ \cancel{L}}\right)\\
&= 3.5(1,000)(1,000)\ mL\\
&= 3,500,000\ mL
\end{aligned}
$$

We can also switch between standard volume units and cubic units in the metric system. In the next example, notice that one kiloliter (1,000 Liters) turns out to be the same as one cubic meter.

Example: Convert 4 m^3 to Liters.

Solution: We convert to cubic centimeters, which are same as milliliters, then to Liters.

$$
\begin{aligned}
&4\ m^3\left(\frac{100\ cm}{1\ m}\right)^3\left(\frac{1\ mL}{1\ cm^3}\right)\left(\frac{1\ L}{1,000\ mL}\right)\\
&= 4\ \cancel{m^3}\left(\frac{1,000,000\ \cancel{cm^3}}{1\ \cancel{m^3}}\right)\left(\frac{1\ \cancel{mL}}{1\ \cancel{cm^3}}\right)\left(\frac{1\ L}{1,000\ \cancel{mL}}\right)\\
&= \frac{4(1,000,000)}{1,000}\ L\\
&= 4,000\ L
\end{aligned}
$$

There are some useful conversions between English and metric standard volume units. They are not exact, but are very close and good enough for most purposes. The main conversion that we will use is $1\ L = 33.8\ fl.oz.$

Example: How many Liters in 1 gallon?

Solution: We convert to fluid ounces, then to Liters. Notice that a Liter is turns out to be slightly more than a quart. You should be familiar with this from drinking bottled beverages.

$$1\ gal\left(\frac{128\ fl.oz}{1\ gal}\right)\left(\frac{1\ L}{33.8\ fl.oz}\right)$$

$$= 1\ \cancel{gal}\left(\frac{128\ \cancel{fl.oz}}{1\ \cancel{gal}}\right)\left(\frac{1\ L}{33.8\ \cancel{fl.oz}}\right)$$

$$= \frac{1(128)}{33.8}\ L$$

$$= 3.79\ L$$

Example: How many teaspoons in 1 Liter?

Solution: We convert a Liter to fluid ounces, then to tablespoons, then finally teaspoons. Consider that it should take quite a few teaspoons to fill a Liter bottle.

$$1\ L\left(\frac{33.8\ fl.oz}{1\ L}\right)\left(\frac{2\ Tbsp}{1\ fl.oz}\right)\left(\frac{3\ tsp}{1\ Tbsp}\right)$$

$$= 1\ \cancel{L}\left(\frac{33.8\ \cancel{fl.oz}}{1\ \cancel{L}}\right)\left(\frac{2\ \cancel{Tbsp}}{1\ \cancel{fl.oz}}\right)\left(\frac{3\ tsp}{1\ \cancel{Tbsp}}\right)$$

$$= 1(33.8)(2)(3)\ tsp$$

$$= 202.8\ tsp$$

****Try this on your own**: Convert 5 gallons into tablespoons.

6.2.3 Exercises: Area and Volume

Solutions appear at the end of this textbook.

1. How many square inches in a square yard?
2. Convert 3 square miles to square kilometers.
3. If a property is 1.5 acres, how many square feet is it? How many metric ares?
4. Convert 3 in^3 to mm^3.
5. How many tablespoons in a pint?
6. How many cubic inches in a gallon?
7. Convert 12,500 mm^3 to Liters.

6.3 Weight and Temperature

6.3.1 Mass and Weight

All of the previous units and conversions dealt with the size of objects, or how much space they took up. Two other characteristics of objects are **Mass** and **Weight**. Mass is a measure of how much stuff an object is made of, and weight is the measure of how strongly gravity pulls on an object.

On the planet earth, mass and weight have a specific relationship, and so their units can be converted. However, that relationship changes in different gravity (in space, on the moon, etc.).

For example, a man who weighs about 150 pounds on Earth, would see 25 pounds on a scale on the moon and weigh nothing in space. On the other hand, mass is constant everywhere, unless you change the amount of stuff you are made of (build muscle, get fatter, etc.).

The common weight units in the English System are **Ounces** (oz.) for very light objects (bag of pretzels), **Pounds** (lbs.) for heavier objects (people, bag of potatoes), and **Tons** for extremely heavy objects (elephants, trucks).

Simple conversions are one pound is 16 ounces, and 1 ton is 2,000 pounds.

Example: Some fire trucks weigh about 20 tons. How many ounces is that?

Solution: Setup unit fractions as

$$20\ tons\left(\frac{2000\ lbs}{1\ ton}\right)\left(\frac{16\ oz}{1\ lb}\right)$$

$$= 20\ \cancel{tons}\left(\frac{2000\ \cancel{lbs}}{1\ \cancel{ton}}\right)\left(\frac{16\ oz}{1\ \cancel{lb}}\right)$$

$$= 20(2,000)(16)\ oz$$

$$= 640,000\ oz$$

In the metric system, weights are not typically used, except in science. More often the mass of an object in relation to its weight on Earth is used. The base unit of mass is the **Gram**, abbreviated as "g". One gram is about the amount of metal in a paper clip. It is very small and light.

The same prefixes are used, milli, kilo, etc., with abbreviations such as mg for milligrams and kg for kilograms. On the Earth's surface, an object that weighs 2.2 pounds (rounded), has a mass of one kilogram. This is the conversion factor we will use.

Scientists set the mass units so that one gram is the mass of pure water that has a volume of one milliliter (same as one cubic centimeter). Due to the simple nature of the metric system, one Liter of pure water has a mass of one kilogram.

Example: A common dosage of the medication Thyroxine is 88 milligrams.
Convert that to grams.

Solution: Setup unit fractions as

$$88\ mg\left(\frac{1\ g}{1,000\ mg}\right) = 88\ \cancel{mg}\left(\frac{1\ g}{1,000\ \cancel{mg}}\right) = \frac{88}{1,000}\ g = 0.088\ g$$

Example: If a man weighs 180 pounds on Earth, how much mass does he have in kilograms?

Solution: Setup unit fractions as

$$180\ lbs\left(\frac{1\ kg}{2.2\ lb}\right) = 180\ \cancel{lbs}\left(\frac{1\ kg}{2.2\ \cancel{lb}}\right) = \frac{180}{2.2}\ kg = 81.8\ kg$$

In case you are wondering how weight is measured in the metric system, it uses a unit called Newtons (named after the famous scientist Isaac Newton who discovered gravity). Newtons are actually a measure of force, and gravity is a force that attracts objects.

Metric mass has a special unit called a metric **Tonne**, abbreviated as "T", which is 1,000 kilograms. It is slightly larger than an English ton (of course only on Earth). It is equivalent to one megagram (1,000,000 grams), but the word tonne is more commonly used.

Example: How many metric tonnes of mass does a fire trucks weighing 20 tons have?

Solution: Setup unit fractions as

$$20\ tons\left(\frac{2000\ lbs}{1\ ton}\right)\left(\frac{1\ kg}{2.2\ lbs}\right)\left(\frac{1\ T}{1,000\ kg}\right)$$

$$= 20\ \cancel{tons}\left(\frac{2000\ \cancel{lbs}}{1\ \cancel{ton}}\right)\left(\frac{1\ \cancel{kg}}{2.2\ \cancel{lbs}}\right)\left(\frac{1\ T}{1,000\ \cancel{kg}}\right)$$

$$= \frac{20(2,000)}{2.2(1,000)}\ T$$

$$= 18.2\ T$$

6.3.2 Temperature

The last topic in this section is the relationship between temperature scales. Temperature is a numerical measure of heat intensity. There are different number scales for temperature, most of which are relative scales, meaning that they don't measure the amount of heat, just the relative difference in heat intensity. Hotter objects have higher temperatures and colder objects have lower temperatures.

In the English system, temperature is measured on the **Fahrenheit** scale. This scale is set so that the freezing point of water is 32 degrees and the boiling point of water is 212 degrees. The abbreviation for degrees Fahrenheit is $^\circ F$. Degrees on the Fahrenheit scale can range from -459 (in deep space) up to millions (inside a star).

In the metric system, temperature is measured on the **Celsius** scale. This scale is set so that the freezing point of water is 0 degrees and the boiling point of water is 100 degrees. The abbreviation for degrees Celsius is $^\circ C$. The Celsius scale is sometimes referred to as the Centigrade scale. Degrees on the Celsius scale can range from -273 (in deep space) up to millions (inside a star).

On a technical note, the freezing and boiling points of water used as reference for temperature scales are measured at sea-level on Earth. They can differ in other places. The minimum temperature is known as <u>Absolute Zero</u>, it is where the molecules essentially stop moving and there is no heat being generated.

There is a simple formula to do conversions between Fahrenheit and Celsius. It is

$$F = \frac{9}{5}C + 32$$

Example: The highest temperature ever recorded at the North Pole is $5°C$. Convert this to Fahrenheit.

Solution: $F = \frac{9}{5}(5°) + 32 = 9 + 32 = 41°F$

Example: Room temperature is defined as $72°F$ (comfortable to the average person). Convert this to Celsius.

Solution: We can setup the equation and then use algebra to rearrange and solve for C.

$$\begin{aligned} 72 &= \frac{9}{5}C + 32 \\ 72 - 32 &= \frac{9}{5}C \\ 40\left(\frac{5}{9}\right) &= \left(\frac{\cancel{5}}{\cancel{9}}\right)\frac{\cancel{9}}{\cancel{5}}C \\ \frac{200}{9} &= C \\ &= 22.2° \end{aligned}$$

****Try this on your own**: Convert $20°F$ to Celsius. Round to whole degree.

6.3.3 Exercises: Weight and Temperature

Solutions appear at the end of this textbook.

1. Convert 12 pounds into ounces.

2. Convert $13,600$ centigrams into kilograms.

3. Convert $9,000$ pounds into tons.

4. Convert 6.75 grams into milligrams.

5. Convert one metric tonne into ounces.

6. Convert 86 pounds into kilograms.

7. Convert $28°C$ into Fahrenheit.

8. Convert $-10°F$ into Celsius.

Chapter 7

Propositional Logic

Written by Nathan Rehfuss

7.1 Statements

Math is consistent, meaning that no matter what method you use to solve a problem, you will get the same answer. We know this because when a mathematician creates a new method, they must prove it is correct. In the early 20$^{\text{th}}$ century mathematicians faced a problem. Our methods worked for the most part. But our proofs were long bouts of deductive reasoning that we accepted just on the basis of them "seeming correct". As proofs grew longer and more complicated we could no longer rely on this intuition. We needed to perform a mathematical study of proof itself. A proof is a series of statements, each of which should follow from the previous in a logical fashion. In order to validate our reasoning we will investigate a branch of mathematics where the basic objects of study are not numbers but statements! This branch is known as propositional logic.

Definition: A **statement** is a sentence that makes a clear and unambiguous factual claim. Every statement has a **truth value**: a label of either **true** or **false** that tells whether it is an accurate statement about the world as it is.

Example: For each sentence, tell whether it qualifies as a statement.

1. The sun is a star.
2. The moon is made of cheese.
3. How far away is Mars?
4. Help me set up the telescope.
5. Woah, space, dude like woah!
6. The telescope is pointed at a planet.
7. That star is many light years away.
8. That planet has life on it!

Solution: We will check each sentence against the definition of a statement.

1. Yes. This sentence makes a clear and unambiguous factual claim. The claim has a truth value: it is **true**.
2. Yes. The claim here is silly, but it is still clearly a claim: either the moon is made of cheese or it isn't. The truth value of this statement is **false**.
3. No. This sentence does not make a claim and it has no truth value. We cannot say that "How far away is Mars?" is **true** or **false**. The answer to this question will be a statement but questions themselves are not statements.

4. No. This request refers to objects in the world but does not make its own factual claim about them. It does not make sense to say that this sentence is **true** or **false**. Commands and requests are not statements because they do not have truth values.

5. No. This sentence is an expression of emotion and personal experience, but it does not make any clear factual claim.

6. Yes. This sentence is making a claim. In order to say whether the claim is true we would need to know the position of the telescope, nevertheless at any given moment the claim is either **true** or **false**. The truth value still exists even if it is changeable.

7. Yes. If the star in question is our sun then this claim is **false**; the sun is only 8 light-minutes away. If it is any other star the claim is **true**. The truth value depends on which star is in question, but this claim always has a truth value.

8. Yes. Again the truth value depends on which planet is in question, but for any given planet the claim is either **true** or **false**.

Now we have been introduced to our object of study, the statement. In the rest of this section we will consider a few common variations on the concept of a statement and begin to see the types of discoveries we will make in propositional logic.

7.1.1 Statements with Parameters

We are going to develop some formulas in the mathematics of propositional logic. First we must introduce some basic notation.

Definition: A **variable** is a letter used to refer to a quantity or mathematical object without having to state its value. When we use variable names to refer to statements, the most common choices are p, q, r and s.

For example, if we let $p =$ "The sun is hot", then when we wish to refer to the proposition that the sun is hot we may simply call it by the name p.

In the previous section, examples (7) and (8) were statements whose truth value depended on precisely which object was the subject of the statement. We will now formalize this notion.

Definition: A **parameter** is one word or phrase within a statement that can be changed out to create a similar statement about a different subject. A **parameterized statement** is a statement with an explicit parameter. We use the notation $p(x)$ to describe a statement p with a parameter x. We can choose different values of x from a category of possibilities. We plug the chosen value of x into the statement by inserting it into the statement where indicated.

Example: For each statement, plug in each of the given parameters. Then tell whether the resulting statement is true or false.

1. $p(x) =$ "The species of pet x is a mammal."
   (a) $x =$ cat
   (b) $x =$ dog
   (c) $x =$ hamster
2. $q(n) =$ " n is an even number."
   (a) $n = 2$
   (b) $n = 7$
   (c) $n = 122$
3. $r(x) =$ "Paris is the capital of the country x."
   (a) $x =$ Russia
   (b) $x =$ Germany

(c) $x =$ France

Solution: Put the indicated value of the parameter into the sentence in place of the variable name.

1. As long as the meaning is equivalent, we can word the statement how we like.

   (a) $p\,(cat) =$ "This species of pet, the cat, is a mammal." **true**.

   (b) $p\,(dog) =$ "The dog is a species of pet that is a mammal." **true**.

   (c) $p\,(hamster) =$ "The pet hamster is a mammal." **true**.

2. Statements where the parameter is a number are the most common type of parameterized statement. Their truth value may depend on which number is chosen.

   (a) $q\,(2) =$ "Two is an even number." **true**.

   (b) $q\,(7) =$ "Seven is an even number." **false**.

   (c) $q\,(122) =$ "One hundred and twenty-two is an even number." **true**.

3. The parameter does not need to be the subject of the sentence.

   (a) $r\,(Russia) =$ "Paris is the capital of Russia." **false**.

   (b) $r\,(Germany) =$ "Paris is the capital of Germany." **false**.

   (c) $r\,(France) =$ "Paris is the capital of France." **true**.

7.1.2 Equivalent statements

In algebra we cannot make equations without first having a concept of equality. Two numbers are equal when they have the same value. We now introduce the corresponding notion in propositional logic. Essentially we will say that two statements are the same if they always agree.

Definition: Two statements are **equivalent** if they always have the same truth value. There is no situation real or hypothetical where they could have different truth values. We will use the symbol $\equiv$ for equivalence, so that $p \equiv q$ reads " p is equivalent to q "

Definition: Two parameterized statements $p(x)$ and $q(x)$ are **equivalent** if they have the same truth value for every possible value of x .

In arithmetic the next task is to get a single name for each number. It's no good to talk about "one hundred" and "a ten of tens" and "the number after ninety-nine" as if they are all different. We identify all of these as a single number because they all have the same value.

Definition: Now when we refer to statement p we will mean the sentence we defined as *p or any equivalent rephrasing* of that sentence. Remember, equivalent statements must always have the same truth value.

So we will consider "I have a headache" and "I've got a headache" and "My head is aching" as merely three English representations or phrasings of the same statement. It will not matter in a formula which one we are referring to because they all have the same meaning.

Example: For each pair of statements, tell whether the two statements are equivalent.

1. $p:$ This is an oak tree. $q:$ This tree is an oak.

2. $p:$ I didn't leave the house. $q:$ It is not true that I left the house.

3. $p:$ Insects have 6 legs. $q:$ Insects do not have 8 legs.

4. $p(x):$ This whole number is even. $q(x):$ This whole number is not odd.

5. $p(x):$ This rational number is even. $q(x):$ This rational number is not odd.

Solution: Check each pair of statements against the definition of *equivalent.*

1. Yes, $p \equiv q$. Since these two statements have the same meaning and only differ in word order, they must both be true or both be false.

2. Yes, $p \equiv q$. p and q are two different ways of saying that the statement "I left the house" is false. If I left the house, then p and q will both be false. If I did not leave the house, then p and q will both be true. So p and q will always have the same truth value.

3. No, $p \not\equiv q$. Both statements are true of our insects. But they do not necessarily stand and fall together. If insects had 10 legs like lobsters instead of their 6, then q would be true and p would be false. Since p and q would have different truth values in this case, they are not equivalent.

4. Yes, $p(x) \equiv q(x)$. A whole number must be either even or odd. $p(x)$ and $q(x)$ are both true when x is even. $p(x)$ and $q(x)$ are both false when x is odd. Since $p(x)$ and $q(x)$ have the same truth value no matter what parameter x is chosen, they are equivalent.

5. No! $p(x) \not\equiv q(x)$. The rational numbers include not only whole numbers but also fractional numbers. Mixed numbers like $3\frac{1}{2}$ are neither even nor odd. $p\left(3\frac{1}{2}\right)$ = "The rational number $3\frac{1}{2}$ is even" is a false statement. $q\left(3\frac{1}{2}\right)$ = "The rational number $3\frac{1}{2}$ is not odd" is a true statement. Since $p(x)$ and $q(x)$ can have different truth values at certain parameters, they are not equivalent.

7.1.3 Negations of statements

If two statements always agree in truth value, they are equivalent. Now we will define a term for two statements that always *disagree* in truth value. This will formalize the notion of two sentences having opposite meaning.

Definition: The **negation** of a statement is a corresponding statement that always has the opposite truth value from the original. Whenever a statement is true its negation must be false. Whenever a statement is false its negation must be true. A statement and its negation can never both be true or both be false.

Definition: We will use the symbol $\sim$ to represent negation in our formulas. So $\sim p$ could be read "the negation of p" or simply "not p".

Example: For each statement, write a negation of the statement.

1. $p =$ "My car is red."
2. $q =$ "The glass is empty."
3. $r =$ "It is not true that the store is closed."
4. $s(x) =$ "The number x is greater than 4."

Solution:

1. There are many possibilities. The phrasing can vary as long as the statement is **false** whenever p is **true** and **true** whenever p is **false**. Any of the following will do:
   $\sim p =$ "My car is not red."
   $\sim p =$ "My car isn't red."
   $\sim p =$ "It is not true that my car is red."
   All three of these sentences are equivalent. We can say that p only has one negation, because each way to phrase the negation is equivalent to the others.

2. This is **not** a correct negation: $v =$ "The glass is full." q and v are exclusive of each other so they cannot both be true. But if the glass is half full and half empty then q and v will both be false at the same time! One correct way to state the negation is: $\sim q =$ "The glass is not empty."

3. r is the negation of the statement "the store is closed", so "the store is closed" is the negation of r ! From this we see that statements and their negations come in pairs; the negation of a negation is the original statement. $\sim r =$ "The store is closed."

4. $\sim s(x) =$ "The number x is less than or equal to 4." We have to phrase this negation carefully to ensure that it is false with every parameter that makes $s(x)$ true and true for every parameter that makes $s(x)$ false. Without the words "equal to" both statements would be false when $x = 4$.

The negation is our first example of a logical **operator**. It operates on our statements: when we take the negation of a statement, a new statement results. In arithmetic the operator most like negation is the minus sign: when we apply a minus sign to a number we get a new number with the opposite value.

In example (3) in this section we noticed that the negation of a negation is the original statement. In this way negation is very much like a minus sign: the negative of negative three is simply three! Two of either operator will undo each other. We can write a formula to this effect:

$$\sim\sim p \equiv p.$$

7.1.4 Quantified statements

A large amount of mathematical thought is dedicated to showing that some property or another is always true, that is, that the property holds in every possible case. We will now describe this endeavor in the language of propositional logic. The property will be a statement and the possible cases will be its parameters. We now define a way to talk about statements that are true for some or all possible parameters.

Definition: A **quantifier** is a way to turn a parameterized statement into a general statement about all possible parameters. The two quantifiers are $\forall$, meaning "for all" , and $\exists$, meaning "there exists" . So $\forall x\ p(x)$ could be read, "For all values of the parameter x , the statement $p(x)$ is true." The other quantified statement $\exists x\ p(x)$ reads, "There exists at least one value for x that makes $p(x)$ true." Often times we will abbreviate these expressions to "For all $x, p(x)$ " and "There exists x such that $p(x)$."

Example: Given the parameterized statement $p(x)$, write English representations for the quantified statements $\forall x\ p(x)$, $\exists x\ p(x)$, $\forall x \sim p(x)$ and $\exists x \sim p(x)$.

1. $p(x) =$ "This person (x) has some good in them."

2. $q(x) =$ "The prime number x is greater than 1."

Solution: We will give both a transliteration and a more natural phrasing for each quantified statement.

1. $\forall x\ p(x) =$ "For any person, that person has some good in them." or
   $\forall x\ p(x) =$ "Every person has some good in them."

   $\exists x\ p(x) =$ "There exists a person such that that person has some good in them." or
   $\exists x\ p(x) =$ "There is a person with some good in them."

$\forall x \sim p(x) =$ "For any person, that person does not have some good in them" or

$\forall x \sim p(x) =$ "Nobody has any good in them."

$\exists x \sim p(x) =$ "There exists a person such that that person does not have good in them." or

$\exists x \sim p(x) =$ "There are people who don't have any good in them."

2. $\forall x\ q(x) =$ "All prime numbers are greater than 1."

   $\exists x\ q(x) =$ "There exist prime numbers greater than 1."

   $\forall x \sim q(x) =$ "All prime numbers are less than or equal to 1."

   $\exists x \sim q(x) =$ "Prime numbers less than or equal to 1 exist."

7.1.5 Interpreting English sentences as quantified statements

Example: For each quantified statement, tell what parameterized statement it is made from and what the parameter represents. Then write the quantified statement in symbolic form.

1. All dogs go to heaven.
2. Some snakes are venomous.
3. No horse can talk.
4. Not all prime numbers are odd. There is an even prime number.
5. All triangles have three sides. There is no triangle that does not have three sides.

Solution:

1. Let $p(x) =$ "this dog goes to heaven" where the parameter x represents which dog. Then

   $\forall x \; p(x) =$ "For every dog, that dog goes to heaven"

   which is equivalent to the given statement.

2. Let $q(x) =$ "this snake is venomous" where x represents which snake. Then

   $\exists x \; q(x) =$ " There exists a snake that is venomous"

   which is equivalent to the given statement.

3. Let $r(x) =$ "this horse can talk" where x represents which horse. Then

   $\sim \exists x \; r(x) =$ "There does not exist a horse which can talk"

   which is equivalent to the given statement.

4. Let $p(x) =$ "the prime number x is odd." Then

   $\sim \forall x \; p(x) =$ "It is not true that all prime numbers are odd."

   $\exists x \sim p(x) =$ "There exists a prime number that is not odd."

5. Let $q(t) =$ "the triangle t has three sides." Then

   $\forall t \; q(t) =$ "All triangles have three sides."

   $\sim \exists t \sim q(t) =$ "There does not exist a triangle that does not have three sides."

The two quantified statements in (4) are equivalent to each other and the two quantified statements in (5) are equivalent to each other as well. In fact, a pair of statements with the structure above will always be equivalent. The negation of a "for all" statement is the same as the existence of a parameter value which makes the statement false. We can write this observation as a **logical identity**. The following equation is true for any statement $p(x)$:

$$\sim \forall x \; p(x) \equiv \exists x \; \sim p(x) .$$

In words, this identity expresses that a generalization that p is always true is incorrect *if and only if* there exists a parameter that makes the statement false. That parameter is called a **counterexample**.

7.1.6 Counterexamples to quantified statements

Example: Write a counterexample to each statement.

1. All cats have fur.

2. Every even number is divisible by four.

3. No president served more than two terms.

Solution: We will use the identity from last section to manipulate the symbolic forms of these statements.

1. Let $p(x) =$ "The cat x has fur." Then $\forall x\ p(x) =$ "All cats have fur." By our identity the negation of $\forall x\ p(x)$ is equivalent to $\exists x \sim p(x) =$ "There exists a cat that does not have fur." One such cat is the sphinx cat. So $\sim p(\text{sphinx cat}) =$ "The sphinx cat does not have fur." is a counterexample to the given statement.

2. Let $q(x) =$ "This even number is divisible by four." Then $\forall x\ q(x) =$ "Every even number is divisible by four." By our identity the negation of $\forall x\ q(x)$ is equivalent to $\exists x \sim q(x) =$ "There exists an even number that is not divisible by four." One such even number is six. So $\sim q(6) =$ "The even number 6 is not divisible by four." is a counterexample to the given statement.

3. Let $r(x) =$ "President x served more than two terms." Then $\forall x \sim r(x) =$ "Each president served no more than two terms." The negation of $\forall x \sim r(x)$ is equivalent to

$\exists x\ r(x) =$ "There exists a president who served more than two terms." Franklin Delano Roosevelt is the only such president, but one counterexample is enough to disprove a general statement. So $r(FDR) =$ "President FDR served more than two terms" is the desired counterexample.

7.1.7 Negations of quantified statements

If a "for all" generalization is false then there exists a counterexample. If there does not exist a counterexample then the "for all" statement is true. We can express this fact as follows:

$$\sim \forall x\ p(x) \equiv \exists x \sim p(x)$$

$$\sim \exists x\ p(x) \equiv \forall x \sim p(x)$$

We have discovered a law of algebra for propositional logic: we can pass a negation through a quantifier by changing the quantifier. So it is also true that:

$$\sim \forall x \sim p(x) \equiv \exists x\ p(x)$$

$$\sim \exists x \sim p(x) \equiv \forall x\ p(x)$$

Since $\sim\sim p(x) \equiv p(x)$.

We can use these formulas to negate any quantified statement.

Example: Negate each quantified statement.

1. All of the test runs went smoothly.

2. Some Mersenne primes are even.

3. Some people just can't be reasoned with.

4. Nobody knows who stole the painting.

5. None of the marbles are black.

Solution: We will negate each statement by considering its symbolic form.

1. Let $p(x)$ = "Test run x went smoothly." Then the given statement is $\forall x\ p(x)$. Its negation $\sim \forall x\ p(x)$ is equivalent to $\exists x \sim p(x)$ = "One or more of the test runs did not go smoothly."

2. Let $q(x)$ = "Mersenne prime x is even." Then the given statement is $\exists x\ q(x)$. Its negation $\sim \exists x\ q(x)$ is equivalent to $\forall x \sim p(x)$ = "Every Mersenne prime is odd."

3. Let $p(x)$ = "The person x can be reasoned with." Then the given statement is $\exists x \sim p(x)$. Its negation $\sim \exists x \sim p(x)$ is equivalent to $\forall x\ p(x)$ = "Everyone can be reasoned with."

4. Let $q(x)$ = " x knows who stole the painting." Then the given statement is $\forall x \sim q(x)$. Its negation $\sim \forall x \sim q(x)$ is equivalent to $\exists x\ q(x)$ = "Somebody knows who stole the painting."

5. Let $p(x)$ = "This marble (x) is black." Then the given statement is $\forall x \sim p(x)$. Its negation $\sim \forall x \sim p(x)$ is equivalent to $\exists x\ p(x)$ = "At least one of the marbles is black."

So as a rule, when negating quantified statements, the opposite of "all" is "some are not" and the opposite of "some" is "none".

7.1.8 Exercises: Statements

Solutions appear at the end of this textbook.

1. Is this a statement? Magnets have a north pole and a south pole.

2. Is this a statement? Be careful not to let magnets come together too quickly.

3. Are these two statements equivalent? p: This tree is 20 years old. q: This tree is older than 19 years.

4. Are these two statements equivalent? p: Three lefts make a right turn. q: A right turn is made from three left turns.

5. Suppose $p(x) = x$ is greater than 5. What is $p(3)$? Is this statement true or false?

6. Suppose $p(x) =$ A rectangle with height 2 inches and width x inches has an area of 8 square inches. What is $q(4)$? is this statement true or false?

7. Write the negation: This bolt is over-tightened.

8. Write the negation: I saw him finish the cake.

9. Suppose $p(x) =$ Part x is missing from the kit. Write English statements for the formulas

   (a) $\forall x\ p(x)$

   (b) $\exists x\ p(x)$

   (c) $\forall x\ {\sim}p(x)$

   (d) $\exists x\ {\sim}p(x)$

10. Write the negation: All of us have taken the training.

11. Write the negation: Some of the beams have rotted.

12. Write the negation: None of this is relevant.

13. Write the negation: Some of your packages could not be delivered.

7.2 Logical Operators

7.2.1 The conjunction "*and*"

All of the most interesting operators in arithmetic are binary, meaning they take two inputs and combine them into one output. The binary operators include addition, subtraction, multiplication, and division. When we say that $10 + 2 = 12$, ten and two are the inputs, twelve is the output, and addition is the operator that turns one into the other. We will study four binary operators for propositional logic; these operators will add more descriptive ability and rich detail to our model of logical reasoning.

Our first binary operator is the conjunction "*and*". We will use $\wedge$ as its symbol in formulas. When two input statements are connected with an '"and" , the resulting output statement declares that both input statements are true. For example if $p =$ "dogs have four legs" and $q =$ "humans have two legs" , then the output of $p \wedge q$ is "Dogs have four legs and humans have two legs", which says that both p and q are true.

In other words the new statement $p \wedge q$ has the truth value "true" only when both p and q are true.

If a statement can be written as the output of one or more binary operators we call it a *compound statement*. If it cannot we call it a simple statement. We will now practice decomposing compound statements into the results of operations on simple statements.

Example: Translate the English sentence into a propositional logic formula. Define simple statements as needed for components.

1. I read it and I remembered it.

2. I was there, but I didn't see anything.

3. I don't know you but I can help you.

4. I'm not infallible and I don't mind being wrong.
5. War and Peace is a novel by Leo Tolstoy.
6. This number is even and bigger than 5.
7. This number is prime but not odd.
8. There is a number that is prime but not odd.

Solution: We will rewrite these statements as conjunctions of simple statements.

1. There are two independent clauses here connected by the word "and" . Let

   p: "I read it."

   q: "I remembered it."

   Each of these is a statement: each of p and q makes a clear factual claim on its own. From these building blocks we can make the following propositional logic formula:

   $p \wedge q$: "I read it *and* I remembered it."

2. The word "but" in this sentence fills the same logical role as "and" did in the last: it asserts that the clause before and the clause after are both true. So if we let

   p: "I was there."

   q: "I didn't see anything."

   then this sentence is equivalent to $p \wedge q$. Alternatively since q contains the negated verb "didn't" we could simplify by letting

   r: "I saw something." Our compound statement asserts that p is true and r is false, so we have

   $p \wedge \sim r$: "I was there *but* I did*n't* see anything."

3. By the same reasoning as in example (2), if we let

   p: "I know you."

   q: "I can help you."

   then the given statement is equivalent to $\sim p \wedge q$.

4. Again we have two independent clauses, each forming a complete statement on its own, connected by an "and". This time both simple statements are negated. Let

   p: "I'm infallible."

   q: "I mind being wrong."

   Then the given statement is equivalent to $\sim p \wedge \sim q$.

5. English can be deceiving! In this case "War and Peace" is the title of the book. The word "and" does not express a conjunction here because it does not split the sentence into two complete thoughts. The given sentence is already a simple statement, and we have nothing to do here.

6. This sentence makes two assertions about the number: that it is even and that it is bigger than five. The same subject and verb, "This number is", are shared between both assertions. Let

   p: "This number is even."

   q: "This number is bigger than five."

   Then the given statement is equivalent to $p \wedge q$. There is one more thing we can add to this propositional logic formula to give it more power in modeling the given sentence. Since each simple statement makes a claim about "this number", and "this number" can refer to different subjects, we can represent it with a parameter x. Let

   $p(x)$: "This number (x) is even."

   $q(x)$: "This number (x) is bigger than five."

   $r(x)$: "This number (x) is even and bigger than five."

   Then $r(x) \equiv p(x) \wedge q(x)$.

7. Following our reasoning in (2) and (6), let

    $p(x)$: "This number is prime."

    $q(x)$: "This number is odd."

    Then the given statement is equivalent to $p(x) \wedge \sim q(x)$.

8. The given sentence says that a number that satisfies the statement in (7) exists. Using the same simple statements as before, since the statement in (7) is $p(x) \wedge \sim q(x)$, this statement is equivalent to $\exists x p(x) \wedge \sim q(x)$.

7.2.2 Truth values for conjunctions

We will now use some parameterized statements about arithmetic to explore the truth values of compound statements built using "and".

Example: Using the simple statements

$p(x)$: "This number is even."

$q(x)$: "This number is divisible by 10."

Write English sentences to represent

1. $p \wedge q$
2. $p \wedge \sim q$
3. $\sim p \wedge q$
4. $\sim p \wedge \sim q$

Then tell the truth value of each statement (1)-(4) at each of the following parameters:

a) $x = 3$

b) $x = 6$

c) $x = 10$

Solution:

1. $p \wedge q$: This number is even and divisible by 10.

2. $p \wedge \sim q$: This number is even but not divisible by 10.

3. $\sim p \wedge q$: This number is not even but it is divisible by 10.

4. $\sim p \wedge \sim q$: This number is neither even nor divisible by 10.

	$p(x)$	$q(x)$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$
$x = 3$	F	F	F	F	F	**T**
$x = 6$	**T**	F	F	**T**	F	F
$x = 10$	**T**	**T**	**T**	F	F	F

Compound statements built from p and q can distinguish between three types of numbers:

- Numbers like 3 for which both statements are false

- Numbers like 6 for which p is true but q is false

- Numbers like 10 for which both statements are true

If there were a fourth type of number for which p were false and q were true, these compound statements could distinguish it from the others. However no such number exists.

Statement p must be true as a condition for q to be true. We will discuss this type of relationship, called a *conditional*, in a later section.

Notice that each of the four "and" statements we can form is true for only one type of number. The conjunction *and* is highly selective. We will study a less selective operator *or* in the next subsection.

7.2.3 The disjunction " *or* "

"And" and "or" are the two essential words in English for connecting thoughts together. Whereas "and" requires all of the conditions it connects to be met, "or" only requires that at least one be met. When we introduce an operator in propositional logic for "or", we must first be aware that the word in English has two contradictory meanings.

The *exclusive or* means "one or the other but not both. Choose one only." When a jilted lover declares "It's me or him" he is delivering an ultimatum. You must choose one, you cannot keep both. By contrast the *inclusive or* means "one or the other or both. At least one." When your cashier asks if you will pay by cash or credit, they are not asking you to pick only one. You must pay, but whether it is by cash, credit, or a mix of both is up to you.

We need our operators to have definite unambiguous meanings, so we must choose one of these two senses of the word "or" to represent with our new binary operator. In this text we will follow mathematical tradition and choose the inclusive or.

The *disjunction* "or" is a binary operator with the symbol $\vee$ whose output is a statement that indicates one or both of the input statements are true. In other words $p \vee q$ is true so long as p is true, q is true, or both p and q are true. The output $p \vee q$ is only false when both p and q are false.

Example: Given the following simple statements

$p(x)$: "This animal has wings"

$q(x)$: "This animal has an exoskeleton"

$r(x)$: "This animal is red"

For each propositional logic formula, write an English sentence that represents the formula and give an example of an animal for which the statement is true.

1. $p \wedge q$
2. $p \vee q$
3. $p \wedge \sim q$
4. $p \vee \sim q$
5. $p \wedge q \wedge r$
6. $p \vee q \vee r$
7. $p \wedge \sim q \wedge r$
8. $p \wedge (q \vee r)$
9. $(p \wedge q) \vee r$
10. $\sim (p \vee q)$
11. $\sim p \wedge \sim q$

Solution:

1. "This animal has wings and an exoskeleton." A bee, for example.

2. "This animal has wings or an exoskeleton." A bee still qualifies, as does any other animal that satisfies (1). A bird or an ant would qualify as well, since they satisfy at least one of the two conditions.

3. "This animal has wings but no exoskeleton." A bat, perhaps.

4. "This animal has wings or has no exoskeleton." A bat still qualifies. So does a beetle, which has wings, or an octopus, which has no exoskeleton.

5. "This animal has wings and an exoskeleton and is red." With only one type of operator, the order of the simple statements does not matter, so "This animal is red and has an exoskeleton and wings" is also correct.

   The animal we choose must satisfy all three conditions: this statement with multiple *and*s is very selective. A ladybug would qualify.

6. "This animal has wings, or has an exoskeleton, or is red." This statement with multiple *or*s is very permissive, an animal need only meet one of the three criteria to satisfy it, though an animal who meets two or more also qualifies. So a blue bird, black spider, or red herring fits, as does a red mite or a ladybug. Only an animal which matches none of the three criteria is excluded.

7. "This animal has wings, has no exoskeleton, and is red." Once again this *and* statement is quite selective, the animal must match all three specifications. A cardinal will do it.

8. $p \wedge (q \vee r)$: "This animal has wings, and it is either red or has an exoskeleton." Only animals with wings can be considered, in addition the animal must match one or more of the other two descriptors. So a cardinal, who has wings and is red, or a grasshopper, who has wings and an exoskeleton, will do.

9. $(p \wedge q) \vee r$: The location of the parentheses is crucial! Now we can meet one or the other of two conditions: Either the animal must have wings and an exoskeleton, or it

must be red. A grasshopper meets the first condition, a red fish meets the second, so both are acceptable. The red fish would not have satisfied the statement in (8).

10. "It is not true that this animal has wings or an exoskeleton" If the animal has wings or an exoskeleton or both then the statement "This animal has wings or an exoskeleton" will be true. We need to find an animal with neither. A monkey, for example.

11. "This animal does not have wings and it does not have an exoskeleton." So we must find an animal with neither wings nor an exoskeleton. A monkey, for example. This statement is equivalent to the statement in (10) - we will prove this equivalence and more in a later section when we discuss *De Morgan's Laws*.

7.2.4 Writing formulas for statements with *and, or,* and *not*

Example: Translate each English sentence into a propositional logic formula. Define simple statements as needed for components.

1. We can go to the park or the museum.

2. I've seen John today but not Shirley.

3. We can use metric or imperial units but not both.

4. This customer is grandfathered in, or they have enough purchases and rewards points.

5. I don't want any balloons or confetti but a banner is okay.

Solution:

1. Let

   p : 6 "We can go to the park."

   q : "We can go to the museum."

   Then the given statement is $p \vee q$.

2. Remember that "but" is another word to indicate the conjunction. Let

   p : "I've seen John today."

   q : "I've seen Shirley today."

   Then the given statement is $p \wedge \sim q$.

3. Let

   p : "We can use metric units."

   q : "We can use imperial units."

   This sentence expresses an exclusive or: one or the other but not both. We need to encode this into a propositional logic formula. "One or the other" can be written $p \vee q$. "Not both" can be written $\sim (p \wedge q)$. Again "but" means "and" here, so "one or the other but not both" is

   $$(p \vee q) \wedge \sim (p \wedge q)\,.$$

4. Let

   p : "This customer is grandfathered in."

   q : "This customer has enough purchases."

   r : "This customer has enough rewards points."

   Since this sentence uses both "and" and "or" , we need parentheses to specify which operation is applied first. (Recall problems (8) and (9) from previous example) The comma tells us where the parentheses go: p stands alone while r and q are grouped together. The given statement is equivalent to

   $$p \vee (q \wedge r)\,.$$

5. Let

p : "I want balloons."

q : "I want confetti."

r : "A banner is OK."

Like in (4) this statement uses both $\wedge$ and $\vee$. This time we know p and q are grouped together because in the original sentence "balloons" and "confetti" share the verb "want" and the negation "don't" . So our answer is

$$\sim (p \vee q) \wedge r.$$

The distinctions between complex formulas like the ones in (4) and (5) are subtle. In the next section we will study an analytical tool that helps us elucidate and precisely describe these distinctions.

7.2.5 Exercises: Logical Operators

Solutions appear at the end of this textbook.

1. Given the simple statements p: This item is damaged. q: This item is being sold at a discount.

   Write English statements for the following formulas.

   (a) $p \wedge q$

   (b) $\sim p \wedge q$

   (c) $p \wedge \sim q$

2. Suppose p and q are both true. What is the truth value of $p \wedge q$?

3. Suppose p is false and q is true. What is the truth value of $p \wedge \sim q$?

4. Suppose p is true and q is false. What is the truth value of $p \wedge \sim q$?

5. Given the simple statements p: The toner is low. q: The paper tray is empty. r: The paper is jammed.

   Write English statements for the following formulas.

   (a) $p \vee q$

   (b) $\sim p \wedge (q \vee r)$

   (c) $\sim r \wedge q \wedge p$

7.3 Truth Tables

7.3.1 Truth tables

If we know the truth values for a function's inputs, then we can determine the truth value of the output. A *truth table* is a diagram that lists the truth value of a formula's output for each possible set of truth values for the inputs.

This is a powerful tool. If two formulas have the same truth value for every combination of inputs, then they are equivalent. Truth tables allow us to demonstrate this. So, using truth tables, we can demonstrate logical *identities*: pairs of formulas that always have the same meaning.

Here is our first truth table, for the unary operator *not.*

p	$\sim p$
T	F
F	T

The top row are the column headers. It lists all of the simple statements and formulas used in the table. The first column lists all possible truth values for p , the second column lists the truth values for $\sim p$.

Each row after the first corresponds to one possible truth value for the input. There are as many rows as there are possible inputs. Since the statement p must be either true or false, there are two rows, one for each state.

In the first row the input p is true. Since $\sim p$ must be false whenever p is true, $\sim p$ is false in the first row. In the second row the input p is false. Since $\sim p$ must be true whenever p is false, $\sim p$ is true in the second row. We can see from the truth table that p and $\sim p$ can never both be true, and they can never both be false. We can use this table as a complete definition of the negation, if we wish.

Here is the truth table for the disjunction *or*.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Since *or* has two inputs instead of one, this table has four rows. We have two choices for the truth value of p and two choices for the truth value of q. Therefore by the fundamental counting principle there are four unique ways to assign truth values to the pair of inputs. For ease of comparison we will always list the four cases in the order shown in columns 1 and 2 above in this text.

The third column gives us our definition for this operator. The symbol $\vee$ is an *inclusive or*, so its output is true whenever one or both inputs are true. In row 1 both inputs p and q are true, so $p \vee q$ is also true. In row 2 input p is true but q is false; since one of the inputs is true $p \vee q$ is true. In row 3 input p is false but q is true; since one of the inputs is true $p \vee q$ is true. In row 4 neither p nor q is true, so $p \vee q$ is false.

Here is the truth table for the conjunction *and*.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

By definition $p \wedge q$ is a statement that asserts that both p and q are true. Therefore $p \wedge q$ is true in row 1 where both p and q are true, and false in rows 2, 3 and 4 where this is not the case. We can now see visually what we meant in the previous section when we said that *or* is the more permissive operator and *and* is more selective: $p \vee q$ is true in three of four possible cases, while $p \wedge q$ is only true in one.

Example: Fill in truth tables for each of the following formulas.

1. $\sim (p \wedge q)$

2. $q \wedge (p \vee q)$

Solution: These formulas contain two operations each. Rather than guess at the final answer for each row we will complete the truth tables one operation at a time. First we will perform the operation in parentheses, then we will use that result as an input for the other operation.

1. We start with the truth table for *and*.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	
T	F	F	
F	T	F	
F	F	F	

   Now we proceed one row at a time. Column 4 is the negation of column 3, so we will apply the definition of the negation using the truth values in column 3 as input.

   Since $p \wedge q$ is true in row 1, its negation is false in row 1.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	
F	T	F	
F	F	F	

Since $p \wedge q$ is false in row 2, its negation is true in row 2.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	
F	F	F	

Similarly $p \wedge q$ is false in rows 3 and 4 so its negation is true there as well.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

With that the truth table is complete. Now we can use the fourth column to summarize what we have learned about $\sim (p \wedge q)$: it is false when both inputs are true, and true when at least one of the inputs is false. This conclusion is correct no matter what p and q actually state! For example suppose that

$p =$ "Gorillas are primates."

$q =$ "Piranhas are herbivores."

Since p is true and q is false we know from row 2 of the above table that $\sim (p \wedge q)$ will be true without having to construct it! Indeed

$\sim (p \wedge q) =$ "It is not true that gorillas are primates and piranhas are herbivores" is a true statement.

2. We start with the truth table for *or*.

p	q	$p \vee q$	$q \wedge (p \vee q)$
T	T	T	
T	F	T	
F	T	T	
F	F	F	

As in (1) we proceed one row at a time. This time we will perform the operation *and* using column 2, q, and column 3, $p \vee q$, as inputs. This will produce the correct truth values for their conjunction $q \wedge (p \vee q)$. In row 1, both column 2 and column 3 are true. A conjunction is true whenever both of its inputs are true, so column 4 is true here as well.

p	q	$p \vee q$	$q \wedge (p \vee q)$
T	T	T	T
T	F	T	
F	T	T	
F	F	F	

In row 2, column 2 is false and column 3 is true. Since they are not both true, their conjunction in column 4 is false.

p	q	$p \vee q$	$q \wedge (p \vee q)$
T	T	T	T
T	F	T	F
F	T	T	
F	F	F	

In row 3, columns 2 and 3 are both true like in row 1, so column 4 is true here as well. In row 4, both column 2 and column 3 are false, so their conjunction in column 4 must be false.

p	q	$p \vee q$	$q \wedge (p \vee q)$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	F

Now we notice something remarkable: in each of the four rows, $q \wedge (p \vee q)$ has the same truth value as q. That means that no matter what p and q actually state and no matter what their truth values are, the formula $q \wedge (p \vee q)$ is always logically equivalent to just q. Any time we see a statement of the form $q \wedge (p \vee q)$, we can replace it with q to simplify our argument.

For example, if a proof contains the line, "this number must be even, and it must be square or even", we can replace this with, "this number must be even", omitting the reference to square numbers without changing the meaning of the argument. The equation

$$q \wedge (p \vee q) \equiv q$$

is a logical *identity* that represents this result. In a later section, we will discover and apply additional logical identities.

7.3.2 Filling truth tables

In order to create truth tables for more complicated formulas we will progress through the formulas one operator at a time, as in the example above. We need a way to decide which operators are applied in which order: this is the *order of operations* for propositional logic. By convention the ordering is as follows:

- Operations in parentheses are performed first.
- Negations are performed second.
- The conjunction and disjunction are performed third, and neither takes precedence over the other.

The third point above means that when multiple conjunctions and disjunctions are used, we must use parentheses to indicate their ordering. Compare $(p \wedge q) \vee r$ versus $p \wedge (q \vee r)$ in the example below. When multiple operations all of the same type are used we do not need parentheses. $p \wedge q \wedge r$ and $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$ are all equivalent: each is only true when all three inputs are true.

Example: Create a truth table for each formula. Start by filling in the top row using the order of operations. Then fill each column using the definition of the operator for that step.

1. $p \wedge \sim q$
2. $(\sim p \wedge \sim q) \vee (p \wedge q)$
3. $p \wedge \sim p$
4. $p \vee \sim p$
5. $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$
6. $(p \wedge q) \vee r$
7. $p \wedge (q \vee r)$

Solution:

1. Negations come before conjunctions in the order of operations. We will negate q first, then take the conjunction of p and $\sim q$. So the top row should look like this:

p	q	$\sim q$	$p \wedge \sim q$
T	T		
T	F		
F	T		
F	F		

Now we fill in the column for $\sim q$. The negation $\sim q$ is false in rows 1 and 3 where q in column 2 is true, and true in rows 2 and 4 where q is false, so the third column looks like this:

p	q	$\sim q$	$p\wedge \sim q$
T	T	F	
T	F	T	
F	T	F	
F	F	T	

The operation for column 4 is a conjunction of columns 1 and 3. The conjunction is true when both input statements are true. This is only the case in row 2. Therefore we put a T in row 2 and F everywhere else.

p	q	$\sim q$	$p\wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

2. The formula $(\sim p\wedge \sim q)\vee(p \wedge q)$ contains five operations, so there will be five additional columns after the inputs in its truth table. The negations inside the parentheses must go first, followed by the conjunctions in parentheses, followed by the disjunction. So the top row could look like this:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \wedge q$	$(\sim p \wedge \sim q) \vee (p \wedge q)$
T	T					
T	F					
F	T					
F	F					

Columns 3 and 4 are negations, so we can fill them with the opposites of the truth values from columns 1 and 2 respectively.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \wedge q$	$(\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

Column 5 is the conjunction of columns 3 and 4. It will only be true when both inputs are true. This is only the case in row 4, so we put a T there and F elsewhere.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \wedge q$	$(\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	F		
T	F	F	T	F		
F	T	T	F	F		
F	F	T	T	T		

Column 6 is the conjunction of columns 1 and 2. Since we always arrange the four input cases in the same order, we can copy this column from the truth table for $p \wedge q$ at the beginning of this section.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \wedge q$	$(\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	F	T	
T	F	F	T	F	F	
F	T	T	F	F	F	
F	F	T	T	T	F	

Finally column 7 is the disjunction of columns 5 and 6. It will be true whenever one or both inputs are true. The second input is true in row 1, so the disjunction is true there. The first input is true in row 4, so the disjunction is true there as well. In rows 2 and 3 neither input is true, so the disjunction is false in these rows.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \wedge q$	$(\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	T	F	T

3. The formula $p \wedge \sim p$ only uses one simple statement, so this truth table only needs two rows: one where p is true and one where p is false.

p
T
F

Negations come before conjunctions, so the other two columns look like this:

p	$\sim p$	$p\wedge \sim p$
T		
F		

The negation of p is true when p is false and false when p is true.

p	$\sim p$	$p\wedge \sim p$
T	F	
F	T	

The third column is the conjunction of the first two. Conjunctions are only true when both inputs are true. In each row, only one input is true, so the conjunction here is never true.

p	$\sim p$	$p\wedge \sim p$
T	F	F
F	T	F

The formula $p\wedge \sim p$ is false in every row. No matter what the input p actually states, any statement of the form $p\wedge \sim p$ is automatically false. A statement and its negation can never both be true.

4. We can proceed in much the same way as in the previous example. We will need a column for p with two rows and a column for $\sim p$. Only the final column is different.

p	$\sim p$	$p\vee \sim p$
T	F	
F	T	

Column 3 is the disjunction of columns 1 and 2. A disjunction is true whenever one or more of its inputs is true. The first input is true in row 1 and the second is true in row 2, so this disjunction is true in both rows.

p	$\sim p$	$p\vee \sim p$
T	F	T
F	T	T

A formula that is always true is called a *tautology*; this is an example of such a formula. No matter the truth value or content of the simple statement p, either it must be true or its negation must be true. By the definition of negation, a statement and its negation can never both be false.

5. The formula $(p \wedge q) \vee (p\wedge \sim q) \vee (\sim p \wedge q)$ has two simple statements and seven operators, so this table will have nine columns. The negations in parentheses must be performed first, followed by the conjunctions in parentheses, followed by the disjunctions. The two disjunctions can be performed in either order. The table could be set up this way:

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge q) \vee (p \wedge \sim q)$	$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$
T	T							
T	F							
F	T							
F	F							

Now we can save some effort by copying columns 3 and 4 from example (2), column 5 from the table for *and*, and column 6 from example (1). This is again possible because the first two columns of all of these tables are set up identically. Row 1 describes the same configuration of inputs in this table as it does any other four-row table in this book.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge q) \vee (p \wedge \sim q)$	$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	T	F			
T	F	F	T	F	T			
F	T	T	F	F	F			
F	F	T	T	F	F			

Column 7, $\sim p \wedge q$, is the conjunction of columns 3 and 2. Both inputs are true only in row 3, so we put a T there and F elsewhere.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge q) \vee$ $(p \wedge \sim q)$	$(p \wedge q) \vee (p \wedge \sim q) \vee$ $(\sim p \wedge q)$
T	T	F	F	T	F	F		
T	F	F	T	F	T	F		
F	T	T	F	F	F	T		
F	F	T	T	F	F	F		

Column 8 is the disjunction of columns 5 and 6. At least one input is true in rows 1 and 2 but not in rows 3 and 4.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge q) \vee$ $(p \wedge \sim q)$	$(p \wedge q) \vee (p \wedge \sim q) \vee$ $(\sim p \wedge q)$
T	T	F	F	T	F	F	T	
T	F	F	T	F	T	F	T	
F	T	T	F	F	F	T	F	
F	F	T	T	F	F	F	F	

Similarly column 9 is the disjunction of columns 8 and 7. In fact we could have skipped column 8 here entirely: column 9 is the disjunction of columns 5, 6 and 7, so it is true whenever at least one of the three is true.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge q) \vee$ $(p \wedge \sim q)$	$(p \wedge q) \vee (p \wedge \sim q) \vee$ $(\sim p \wedge q)$
T	T	F	F	T	F	F	T	T
T	F	F	T	F	T	F	T	T
F	T	T	F	F	F	T	F	T
F	F	T	T	F	F	F	F	F

Notice that the last column of this table is identical to the truth table for *or*. Why is this the case? If we attempt to read $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$ out loud we might say, "p and q are both true, or p is true but not q, or q is true but not p." In other words, one or the other or both of the inputs are true, which is the definition of the disjunction! So it is no wonder that

$$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \equiv p \vee q.$$

6. The formula $(p \wedge q) \vee r$ uses three simple statements as input. Each input can have one of two truth values, and $2 \cdot 2 \cdot 2 = 8$, so by the fundamental counting principle this table must have eight rows below the header. We will arrange them as follows:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Now the conjunction $p \wedge q$ is only true where both inputs are true: rows 1 and 2. Elsewhere it is false.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	
T	T	F	T	
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	F	
F	F	T	F	
F	F	F	F	

The final column is the disjunction of columns 4 and 3. It is true in every row where at least one of those inputs is true. In rows 4, 6 and 8 both inputs are false, so the disjunction is false there.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

7. This formula's truth table is assembled in the same way as the last. We will make the first 3 columns identical as we did for two-input tables. Now the disjunction is in parentheses, so it is performed before the conjunction.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Column 4 is the disjunction of columns 2 and 3, so it is true whenever at least one of its inputs is true.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	F	
F	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	F	

Column 5 is the conjunction of columns 1 and 4 and is only true when both inputs are true.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Now we can see that the difference between $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ is in rows 5 and 7. When p is false and r is true, $(p \wedge q) \vee r$ is true and $p \wedge (q \vee r)$ is false. The location of the parentheses does in fact make a difference.

7.3.3 Proving Logical Identities

Now we will try to discover some logical identities by testing pairs of formulas with truth tables to see if they agree for every combination of inputs.

Example: Use truth tables to determine whether the two formulas given are equivalent.

1. $\sim\sim p$ and p
2. $\sim (p \wedge q)$ and $\sim p \wedge \sim q$
3. $\sim (p \wedge q)$ and $\sim p \vee \sim q$
4. $\sim (p \vee q)$ and $\sim p \wedge \sim q$

Solution: For each pair we will set up a truth table containing both formulas. Then if the columns for the two formulas agree in every row, they must be equivalent.

1. These formulas have only one simple statement, so our truth table will have two rows. We can set it up like this:

p	$\sim p$	$\sim\sim p$
T		
F		

The negation of p is false whenever p is true and true whenever p is false, so we invert the truth values in column 1 to fill column 2.

p	$\sim p$	$\sim\sim p$
T	F	
F	T	

The negation of $\sim p$ is false whenever $\sim p$ is true and true whenever $\sim p$ is false, so we invert the truth values in column 2 to fill column 3.

p	$\sim p$	$\sim\sim p$
T	F	T
F	T	F

Now p is represented by column 1 and $\sim\sim p$ is represented by column 3. Since these two columns agree in both rows, there is no statement p, true or false, for which they can have different truth values. Therefore they are equivalent: $p \equiv\sim\sim p$. A pair of negation operators cancel each other out, in the same way that a pair of negative signs cancel in arithmetic. An operator with this property is called an *involution*.

For example, if p is the statement "this number is prime" then $\sim p$ is "this number isn't prime" and $\sim\sim p$ is "it is not true that this number isn't prime", which has the same meaning as p.

2. We will set up a table that includes both $\sim (p \wedge q)$ and $\sim p \wedge \sim q$. We will first perform the operations in parentheses, then the negations, then the conjunction in the second formula, so the table header looks like this:

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T					
T	F					
F	T					
F	F					

We can fill column 3 using the truth table for *and* from a previous section.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	T				
T	F	F				
F	T	F				
F	F	F				

Columns 4, 5 and 6 are the negations of columns 1, 2 and 3 respectively. We will change every T to an F and every F to a T.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	
T	F	F	F	T	T	
F	T	F	T	F	T	
F	F	F	T	T	T	

Column 7 is the conjunction of columns 4 and 5. It is true only when both inputs are true.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	T

Since columns 6 and 7 do not Agree in rows 2 and 3, these two formulas are not equivalent. We can write $\sim (p \wedge q) \sim p \wedge \sim q$.

3. Some of our work is already done: we can copy the truth values for $\sim (p \wedge q)$, $\sim p$ and $\sim q$ from the previous table.

p	q	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	F	
T	F	F	T	T	
F	T	T	F	T	
F	F	T	T	T	

All that is left is the disjunction of columns 3 and 4, which will be true whenever at least one of the inputs is true, namely in rows 2, 3 and 4.

p	q	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Since columns 5 and 6 are identical in every row, the two formulas are equivalent. We can write

$$\sim (p \wedge q) \equiv \sim p \vee \sim q.$$

This result is one of the two *De Morgan's laws*, named after their discoverer. We will discuss the significance of this identity in a later section.

4. We can copy the truth values for $\sim p \wedge \sim q$ from the truth table in example (1), so all that remains is to build a truth table for $\sim (p \vee q)$. We will perform the disjunction in parentheses first, followed by the negation.

p	q	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F		
T	F	F		
F	T	F		
F	F	T		

We can copy the truth values for column 4 from the table for the disjunction at the beginning of section 8.3.

p	q	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F	T	
T	F	F	T	
F	T	F	T	
F	F	T	F	

Column 5 is the negation of column 4, so we replace all Ts with Fs and vice versa.

p	q	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	F	T	F
F	F	T	F	T

Since columns 3 and 5 are identical in every row, the two formulas are equivalent. We can write

$$\sim (p \vee q) \equiv \sim p \wedge \sim q.$$

This is De Morgan's other law.

7.3.4 De Morgan's Laws

In the previous subsection we discovered two new logical identities:

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

Together these are known as De Morgan's laws. They allow us to negate statements containing conjunctions and disjunctions and to rewrite such statements in new forms. This provides us some much-needed versatility in rewriting mathematical proofs.

Example: Use De Morgan's laws to write the negations of the following statements and formulas.

1. This number is divisible by 10 or smaller than 100.
2. My dog is less than a year old and over 100 pounds.
3. It isn't vegetarian and it isn't kosher.
4. This number is not divisible by 3, nor is it divisible by 4.
5. This number is even and not square.
6. It's either not trendy or not affordable.
7. $p \vee q \vee r$
8. $p \vee (q \wedge r)$

Solution: We will identify each statement with one of the formulas in De Morgan's laws, and then rewrite it using the other side of the same identity.

1. This is an *or* statement. If we let

   $p:$ "This number is divisible by 10"

   $q:$ "This number is smaller than 100"

   then the given statement is equivalent to $p \vee q$. This formula shows up as part of the left side of the identity

   $$\sim (p \vee q) \equiv \sim p \wedge \sim q.$$

   This identity states that the negation of the formula $p \vee q$ is equivalent to $\sim p \wedge \sim q$. Inserting our simple statements into the formula $\sim p \wedge \sim q$ gives us, "This number is not divisible by 10 and is not smaller than 100." This is the desired negation of the original statement.

2. This is an *and* statement. If we let

   $p:$ "My dog is less than a year old"

   $q:$ "My dog is over 100 pounds"

   then the given statement is equivalent to $p \wedge q$. De Morgan's second law

   $$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

   states that the negation of our statement is equivalent to $\sim p \vee \sim q$. We can write $\sim p$ as "my dog is over a year old" and $\sim q$ as "my dog is under 100 pounds", so $\sim p \vee \sim q$ will read "my dog is over a year old or under 100 pounds." This is the desired negation of the original statement.

3. If we let

   $p:$ "It is vegetarian"

$q:$ "It is kosher"

then the given statement "it isn't vegetarian and it isn't kosher" is equivalent to $\sim p \wedge \sim q$. According to De Morgan's second law this is equivalent to $\sim (p \vee q)$, so its negation is equivalent to $p \vee q$. So the negation of the given statement is "it is vegetarian or kosher."

4. In this context the word "nor" is another way to write a conjunction; it asserts that both statements surrounding it are true. If we let

   $p:$ "This number is divisible by 3"

   $q:$ "This number is divisible by 4"

   then the given statement is equivalent to $\sim p \wedge \sim q$. According to De Morgan's first law this formula is equivalent to $\sim (p \vee q)$, so its negation is equivalent to $p \vee q$. Thus the desired negation is "This number is divisible by 3 or 4."

5. This is an *and* statement. Let

   $p:$ "The number is even"

   $r:$ "The number is square."

   So the given statement is equivalent to $p \wedge \sim r$. This formula does not appear in either of De Morgan's laws itself, but if we let $q = \sim r$ then $p \wedge \sim r$ becomes $p \wedge q$ by substitution. De Morgan's second law $\sim (p \wedge q) \equiv \sim p \vee \sim q$ states that the negation of $p \wedge q$ is equivalent to $\sim p \vee \sim q$.

   Since we substituted $q = \sim r$, the formula $\sim p \vee \sim q$ becomes $\sim p \vee \sim\sim r$. In the previous subsection we proved that $\sim\sim p \equiv p$. We apply this identity here so that

$$\sim p \vee \sim\sim r \equiv \sim p \vee r.$$

So the negation of $p \wedge \sim r$ is $\sim p \vee r$, which states "This number is not even or it is square."

6. If we let

   p : "It is trendy"

   q : "It is affordable"

   Then the given statement is equivalent to $\sim p \vee \sim q$. According to De Morgan's second law this is equivalent to $\sim (p \wedge q)$, so its negation is equivalent to $p \wedge q$ which states, "it is trendy and affordable."

7. We will use De Morgan's first law twice to form the negation of this formula. First we group q and r together:

$$p \vee q \vee r \equiv p \vee (q \vee r)\,.$$

   De Morgan's first law states that

$$\sim (p \vee s) \equiv \sim p \wedge \sim s.$$

   Letting $s = (q \vee r)$ we obtain

$$\sim (p \vee (q \vee r)) \equiv \sim p \wedge \sim (q \vee r)\,.$$

   Now we use De Morgan's law again in the form $\sim (q \vee r) \equiv \sim q \wedge \sim r$ so that

$$\sim p \wedge \sim (q \vee r) \equiv \sim p \wedge \sim q \wedge \sim r.$$

   Therefore the negation of $p \vee q \vee r$ is $\sim p \wedge \sim q \wedge \sim r$. We can write

$$p \vee q \vee r \equiv \sim p \wedge \sim q \wedge \sim r.$$

This is De Morgan's first law for three simple statements. In fact, De Morgan's laws hold for any number of simple statements, so for example

$$p \vee q \vee r \vee s \vee t \equiv \sim p \wedge \sim q \wedge \sim r \wedge \sim s \wedge \sim t.$$

8. This mixed expression can be negated using one application of each of De Morgan's two laws. First, using $\sim (p \vee s) \equiv \sim p \wedge \sim s$ and letting $s = (q \vee r)$ we obtain

$$\sim (p \vee (q \wedge r)) \equiv \sim p \wedge \sim (q \wedge r) \, .$$

Next using $\sim (q \wedge r) \equiv \sim q \vee \sim r$ we get

$$\sim p \wedge \sim (q \wedge r) \equiv \sim p \wedge (\sim q \vee \sim r) \, .$$

Note that the use of parentheses on the right side of this equation is necessary. Putting these two identities together we see that

$$\sim (p \vee (q \wedge r)) \equiv \sim p \wedge (\sim q \vee \sim r) \, .$$

This is the desired negation.

7.3.5 Exercises: Truth Tables

Solutions appear at the end of this textbook.

1. Create and fill out a truth table for each formula.

   (a) $p \vee \sim(p \vee q)$

   (b) $\sim(\sim p \wedge \sim q)$

   (c) $(p \wedge q) \vee (p \wedge \sim q)$

   (d) $\sim(\sim(p \wedge q) \vee \sim r)$

2. Use De Morgan's law to write the negation: We went to the theater but not the museum.

3. Use De Morgan's law to write the negation: The ball landed in quadrant three or four.

4. Use De Morgan's law to rewrite the statement: I don't have money and I don't have fame.

5. Use De Morgan's law to rewrite the statement: It's not true that the album is too short and too loud.

7.4 Conditionals

7.4.1 The conditional and biconditional

We have two more operators to cover in propositional logic: the **conditional** $\rightarrow$ and the **biconditional** $\leftrightarrow$. The conditional is used to link a pair of simple statements together as cause and effect: $p \rightarrow q$ can be read "p implies q″, "if p then q″, or "whenever p is true, q must also be true.″ We will call the first clause, p, the condition and the second clause, q, the consequence. With the conditional we are able to represent deductions as mathematical objects. This opens the door to a mathematical treatment of mathematical reasoning itself.

The biconditional is shorthand for a double conditional. When we say $p \leftrightarrow q$, we mean that p implies q and q implies p. We can read $p \leftrightarrow q$ as "p is true if and only if q is true″, or "p if and only if q″ for short. Each statement implies the other. When conditionals or biconditionals are used in a formula, we will evaluate them after the other operators. So our complete order of operations for propositional logic is

1. Parentheses
2. $\sim$
3. $\wedge$ and $\vee$
4. $\rightarrow$
5. $\leftrightarrow$

Example: Using the following simple statements, write the English sentence indicated by each formula.

p : This polygon is a rectangle.

q : This polygon is a square.

r : This polygon's sides are all the same length.

1. $q \to p$
2. $\forall x \; q(x) \to p(x)$
3. $q \to r$
4. $r \to q$
5. $\sim r \to \sim q$
6. $p \wedge r \leftrightarrow q$

Solution:

1. The conditional arrow represents an if-then statement. The statement before the arrow is the condition: in this case it is q. The statement after the arrow is the consequence that follows whenever the condition is met. If q, then p. So we will write, "If this polygon is a square, then it is a rectangle.″
2. Remember that $\forall$ is the quantifier "for all″. This quantifier is commonly used with if-then statements to indicate that the conditional is true not just in this case, but in general. Our conditional, q implies p, is the same as in part (1), but now we are speaking for all polygons, not just one. We can read this formula, "For any polygon, call it x, if x is a square then x is a rectangle″. So "this polygon″ in part (1) is replaced with "any specific polygon″. For readability we can simplify our result down to this equivalent statement: "Every polygon that is a square is a rectangle.″
3. The arrow points from q to r, so this formula says, "q implies r″ or "if q then r″. Filling in the simple statements q and r, we get, "If this polygon is a square then its sides are all the same length.″

4. Now the arrow is reversed, pointing from r to q. This formula states, "If this polygon's sides are all the same length then it is a square.″ Note that $r \to q$ does not have the same meaning as $q \to r$! The direction of a conditional matters.

   The formula $q \to r$ is true: every square must have sides all of the same length. However $r \to q$ is false: not every polygon with sides all of the same length is a square. If the polygon is an equilateral triangle, for example, then r is true but q is false, so this conditional fails.

5. Whenever conditionals share a formula with other operators, we evaluate the conditionals last. Here that means we will evaluate the two negations first and handle the conditional afterwards. The negation of r says "this polygon's sides are not all the same length″ and the negation of q is "this polygon is not a square″. So $\sim r \to \sim q$ reads, "If this polygon's sides are not all the same length, then it is not a square.″

   Actually, this formula has the same meaning as $q \to r$ in part (3)! If a polygon is a square it has equal sides, so if it doesn't have equal sides then it is not a square. The relationship between these two formulas is called *contraposition*, and we'll explore it in detail later in this section.

6. As in part (5), we will evaluate the "and″ operator before we apply the biconditional. The formula $p \wedge r$ states, "This polygon's sides are all the same length and it is a rectangle″. The statement q says "This polygon is a square″. The biconditional relates these two sentences together: each is true only if the other is true. We can write, "This polygon's sides are all the same length and it is a rectangle, if and only if this polygon is a square.″

   Since a biconditional works in both directions, the order of the two parts doesn't matter. So our statement has the same meaning as, "This polygon is a square if and only if its sides are all the same length and it is a rectangle.″

The sentiment being expressed is that these two descriptions are equivalent: a square **is** a rectangle whose sides are all the same length. So if a rectangle's sides are all the same length then it is a square, and every square is a rectangle whose sides are all the same length. A polygon either matches both descriptions or neither.

7.4.2 Converting sentences to conditional formulas

Now with conditionals in hand, we are able to examine the bare logical structure of statements that make deductions or inferences.

Example: Write each English sentence as a logical formula built out of simple statements.

1. If a number is odd then it is not divisible by two.
2. I will go to the party if Jake is there.
3. If your pet is a dog or a cat, it is a mammal.
4. A polygon is a triangle if and only if it has three sides.
5. For you to pass the class it is sufficient for you to make above 80 on the final.
6. For you to pass the class it is necessary that you complete your missing work.
7. The presence of a blue crest is necessary and sufficient to identify this species.

Solutions:

1. This is an if-then statement, so we will break it into two simple statements separated by a conditional. If we let

   p: This number is odd.

q: This number is divisible by 2.

Then the given statement is $p \to \sim q$.

2. There are too many ways to write a conditional in English for us to list them all, but we will cover some of the main ones in this example section. In this case the condition is the second part of the sentence "if Jake is there″ and the consequence is the first part. So if

   p: I will go to the party.

   q: Jake is at the party.

   This sentence is equivalent to $q \to p$.

3. The condition here is a compound statement: the pet can be a dog or it can be a cat. So we'll write

   p: Your pet is a dog.

   q: Your pet is a cat.

   r: Your pet is a mammal.

   Then the given sentence is $p \vee q \to r$. Since $\to$ comes after $\vee$ in the order of operations, p and q are grouped together automatically to form the condition and we do not need parentheses.

4. "If and only if″ is the most common way you will see the biconditional written in English. Biconditionals are commonly used to write definitions: the two clauses will be the term and its definition, indicating that they are equivalent and each can replace the other. In this case the term "triangle″ is defined to mean the same thing as "polygon with three sides″ . We will use the simple statements

   p: This polygon is a triangle.

   q: This polygon has 3 sides .

The given statement is equivalent to $p \leftrightarrow q$.

5. The words "necessary" and "sufficient" provide another way to understand the relationship between a condition and its consequence. If $p \rightarrow q$ we say that p is sufficient for q: once we know p is true, that is enough, we can conclude q is true as well.

   Here it is sufficient that you make an 80 on the final. This is the condition. If it is met, we can conclude you will pass the class: that is the consequence. So with

   p: You pass the class.

   q: You make above an 80 on the final.

   The given statement is $q \rightarrow p$.

6. If $p \rightarrow q$ we say that q is necessary for p: whenever p is true q will also be true, so if q is false we cannot get p to be true. If we believe the condition we must necessarily believe the consequence.

   Here we are told that completing missing work is necessary to pass. So if we know that you passed, we can conclude you completed your missing work. Let

   p: You pass the class.

   q: You complete your missing work.

   The given statement is "For p it is necessary that q", which is equivalent to $p \rightarrow q$.

7. Let

   p: A blue crest is present.

   q: The species is identified.

   We are told that p is necessary for q and that p is sufficient for q. When p is necessary for q we write $q \rightarrow p$, when p is sufficient for q we write $p \rightarrow q$. This is the use case for a biconditional: since we are told that $p \rightarrow q$ and $p \leftarrow q$, we put both heads on one arrow and simply write $p \leftrightarrow q$.

7.4.3 Truth tables for conditionals

Now we will extend our analytical tool for statements, the truth table, to cover formulas including conditionals. Truth tables allow us to examine the truth values a formula can take for any possible configuration of inputs. Since conditionals are used to formally represent deductive reasoning, understanding their truth tables will allow us to demonstrate conclusively whether specific forms of deduction are valid. In part 2 of the following example we will prove that *modus ponens* is always valid, and in part 6 we will establish the same for *contraposition*.

First we need to understand the outputs of each operation alone. We will reason by example, starting with the conditional. Suppose we claim, if a number is prime then it is odd. This is a $p \to q$ statement where

p: the number is prime.

q: the number is odd.

Let's say x is the number. Then this table expresses the four possible combinations of truth values for p and q.

x	p	q	$p \to q$
3	T	T	T
2	T	F	F
9	F	T	T
16	F	F	T

When $x = 3$, p is true because 3 is prime and q is true because 3 is odd. Clearly our prediction, "if a number is prime it must be odd," has come true in this case.

When $x = 2$, p is true because 2 is prime but q is false because 2 is not odd. We said that when a number was prime it would be odd, but that did not happen here. In this case our conditional is false: $x = 2$ is called a *counterexample*.

Finally when $x = 9$ or $x = 16$, p is false because those numbers are not prime. The truth value of q no longer matters because our prediction does not apply here: its condition is not met. These cases do not falsify our conditional like the counterexample did, so we mark $p \to q$ true here.

A statement's truth table depends only on the logical structure of its formula, not on its content, so this truth table is valid for any conditional.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Next is the truth table for the biconditional. We will prove that this table is correct in part (1) of the following example.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: For each logical formula, fill out a truth table.

1. $(p \to q) \wedge (q \to p)$

2. $(p \to q) \wedge (q \to r) \to (p \to r)$

3. $p \wedge q \to p \vee q$

4. $q \vee (p \leftrightarrow q) \to {\sim}p \wedge q$

5. ${\sim}p \leftrightarrow {\sim}q$

6. ${\sim}q \to {\sim}p$

Solution:

1. A truth table allows us to compute the truth values of a formula one operation at a time. First, we need to decide in which order we will perform those operations using the order of operations provided in the previous section: first parenthetical expressions, then negations, then the conjunctions and disjunctions together, then the conditional and finally the biconditional.

   For this formula we will first perform each of the conditionals in parentheses, then finish with the conjunction in the middle.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T			
T	F			
F	T			
F	F			

   We copy the values for $p \to q$ from its table above.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T		
T	F	F		
F	T	T		
F	F	T		

We need to be careful with $q \to p$. We can refer to the table above, but the input columns are in the opposite order we are used to. So $q \to p$ is false in row 3 and true elsewhere, because row 3 is the row where the condition q is true and the consequence p is false.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Now all that's left is to apply the conjunction. $(p \to q) \wedge (q \to p)$ will be true only when both of its inputs are true. The inputs are columns 3 and 4, and they are both true in rows 1 and 4.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$p \to q$ is a conditional. When the order of the clauses is reversed to make $q \to p$, we call this statement the *converse*. A statement and its converse are **not** equivalent.

When both are true, we can indicate that using a biconditional. In other words $p \leftrightarrow q$ is defined as shorthand for $(p \to q) \wedge (q \to p)$, and this is its truth table.

2. This is the logical formula for a mode of deduction called *modus ponens*. If the first fact implies the second and the second fact implies the third, then the first fact implies the third.

   Let's look at an example. If an animal is a dog then it is a mammal. If an animal is a mammal then it has hair. Modus ponens is following the implication from the first condition to the final consequence, resulting in the claim, if an animal is a dog then it has hair.

   The truth table for this part will prove that modus ponens is always valid.

   The formula has 3 simple statements, p, q, and r. To list every possible combination of their truth values we need 8 rows because we have two choices for p, two choices for q, and two choices for r. Applying the fundamental counting principle gives $2 \cdot 2 \cdot 2 = 8$. Here are the 8 possibilities:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

   For the order of operations we will take the three conditionals in parentheses first, then the conjunction, and finally the conditional outside the parentheses.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

For these three conditionals we are looking for counterexamples: rows where the condition is true but the consequence is false. The conditionals are false in those rows and true elsewhere.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T		
T	T	F	T	F	F		
T	F	T	F	T	T		
T	F	F	F	T	F		
F	T	T	T	T	T		
F	T	F	T	F	T		
F	F	T	T	T	T		
F	F	F	T	T	T		

A conjunction statement is true only when both of its components are true. We are looking for rows where $p \to q$ and $q \to r$ are both true: this happens in rows 1, 5, 7 and 8, so $(p \to q) \wedge (p \to q)$ is true in those rows and false elsewhere.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r)$ $\to (p \to r)$
T	T	T	T	T	T	T	
T	T	F	T	F	F	F	
T	F	T	F	T	T	F	
T	F	F	F	T	F	F	
F	T	T	T	T	T	T	
F	T	F	T	F	T	F	
F	F	T	T	T	T	T	
F	F	F	T	T	T	T	

Finally we evaluate the conditional that is outside the parentheses. When does $(p \to q) \wedge (p \to q)$ imply $p \to r$? Every row except for those where the condition is true and the consequence is false. In our table the condition $(p \to q) \wedge (p \to q)$ is true in rows 1, 5, 7 and 8 and the consequence $p \to r$ is also true in those rows, so there are no rows where $(p \to q) \wedge (p \to q) \to (p \to r)$ is false. It is true in all 8 rows.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$(p \to q) \wedge (p \to q) \to (p \to r)$ states that if the first conditional $p \to q$ and the second conditional $q \to r$ are both valid, that implies that the chained together conditional $p \to r$ is also valid. So this formula is really just a way of saying that modus ponens works. Since this formula is always true, modus ponens always works!

3. This formula says that $p \wedge q$ implies $p \vee q$. If you have two statements and both are true, that implies that at least one of them is true. Seems obvious, right? This kind of formula is called a *tautology*, a statement that is always true. Tautological formulas allow us to make certain kinds of inferences, like modus ponens in the previous part, without justifying them for each particular use.

   The order of operations tells us that "and" and "or" both come before "implies", but it does not matter which of the two we do first. So we can set up our table like this:

p	q	$p \wedge q$	$q \vee p$	$p \wedge q \to p \vee q$
T	T			
T	F			
F	T			
F	F			

$p \wedge q$ is only true when both p and q are true. $p \vee q$ is true as long as at least one of p or q is true; it is only false when both p and q are false.

p	q	$p \wedge q$	$q \vee p$	$p \wedge q \to p \vee q$
T	T	T	T	
T	F	F	T	
F	T	F	T	
F	F	F	F	

The last operation we perform to construct $p \wedge q \to p \vee q$ is the conditional. The conditional is only false when the condition $p \wedge q$ is true and the consequence $p \vee q$ is false. Since this does not occur in any of our four rows, $p \wedge q \to p \vee q$ is always true. We have confirmed that this statement is a tautology!

p	q	$p \wedge q$	$q \vee p$	$p \wedge q \to p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

4. This formula does not have any particular meaning, but it is good practice with the order of operations. The biconditional comes first because it is in parentheses, fol-

lowed by the negation, the conjunction and disjunction in either order, and finally the conditional. So the table looks like this:

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \rightarrow \sim p \wedge q$
T	T					
T	F					
F	T					
F	F					

We can copy the biconditional table from the beginning of this section.

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \rightarrow \sim p \wedge q$
T	T	T				
T	F	F				
F	T	F				
F	F	T				

For the negation of p, we just reverse the truth values for p in each row.

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \rightarrow \sim p \wedge q$
T	T	T	F			
T	F	F	F			
F	T	F	T			
F	F	T	T			

Next is the conjunction: looking at columns 2 and 4, the conjunction will be true only when both of these are true. This only happens in row 3.

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \to \sim p \wedge q$
T	T	T	F	F		
T	F	F	F	F		
F	T	F	T	T		
F	F	T	T	F		

The disjunction inputs are columns 2 and 3, and it is true whenever at least one of these is true. That is rows 1, 2 and 4.

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \to \sim p \wedge q$
T	T	T	F	F	T	
T	F	F	F	F	F	
F	T	F	T	T	T	
F	F	T	T	F	T	

Finally we can apply the conditional. Again we are looking for rows where the condition $q \vee (p \leftrightarrow q)$ is true but the consequence $\sim p \wedge q$ is false, those are the rows where $q \vee (p \leftrightarrow q) \to \sim p \wedge q$ will be false. So we mark rows 1 and 4 false and rows 2 and 3 true.

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \wedge q$	$q \vee (p \leftrightarrow q)$	$q \vee (p \leftrightarrow q) \to \sim p \wedge q$
T	T	T	F	F	T	F
T	F	F	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	F	T	F

5. This is a biconditional with both clauses negated. We will take the negations first and then apply the biconditional.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T			
T	F			
F	T			
F	F			

To take the negation we just reverse the truth value in each row, so true becomes false and false becomes true.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

The biconditional is true when its inputs have matching truth values and is false when they do not. In row 1 $\sim p$ and $\sim q$ are both false, so $\sim p \leftrightarrow \sim q$ is true. In row 4 $\sim p$ and $\sim q$ are both true, so $\sim p \leftrightarrow \sim q$ is true again. In the other two rows one input is true and the other is false, so $\sim p \leftrightarrow \sim q$ is false.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Notice that the truth values for $\sim p \leftrightarrow \sim q$ are the same as the values for $p \leftrightarrow q$ in every row. That means these two statements are equivalent! They will always have the same meaning. For example if

p: This number is even

q: This number is divisible by 2

then

$p \leftrightarrow q$: A number is even if and only if it is divisible by 2

$\sim p \leftrightarrow \sim q$: A number is not even (odd) if and only if it is not divisible by 2.

These two statements have precisely the same meaning. This is guaranteed because their logical formulas are equivalent.

6. Just like in part 5, we are going to perform the negations first, followed by the conditional. In fact, the first four columns of the table are identical.

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Now we will carefully apply the conditional. The condition is $\sim q$ and the consequence is $\sim p$, so we are looking for a row where $\sim q$ is true but $\sim p$ is false. Row 2 is the only such row, so $\sim q \rightarrow \sim p$ is false in row 2 and true elsewhere.

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Notice that the truth values for $\sim q \rightarrow \sim p$ are identical to the ones for $p \rightarrow q$, indicating that these formulas are equivalent! $\sim q \rightarrow \sim p$ is called the *contrapositive* of $p \rightarrow q$. Replacing a conditional with its contrapositive is called *contraposition*. We've shown that contraposition always produces a new statement that is logically equivalent to the original.

7.4.4 Working with contrapositives

In the previous example we proved the law of contrapositives:

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

This equivalence tells us that whenever we have a conditional statement, we can negate both clauses and reverse their order. This produces a new conditional statement, the **contrapositiveContrapositive**, that is logically equivalent to the original statement. Finding the contrapositive gives us a new perspective we can use to better understand, apply or refute conditionals.

Example: For each conditional statement, write the contrapositive.

1. If there is rain, there must be clouds.
2. If you eat all of that you will get sick.
3. No pain, no gain.
4. Water is necessary for life.
5. We will go outside if it stops raining.

Solution:

1. We will interpret each of these statements as a conditional and identify the two simple statements it is formed out of. Then we will negate both of them and switch their order. In this first part the conditional is easy to recognize: it is an if-then. Let

   p: There is rain.

   q: There are clouds.

   The given statement is $p \to q$, so its contrapositive is $\sim q \to \sim p$, which we can read as, if there are no clouds, then there can be no rain.

2. This statement is $p \to q$ where

   p: You eat all of that.

   q: You get sick.

   So it is equivalent to $\sim q \to \sim p$ which reads, if you didn't get sick then you didn't eat all of that. Notice that we had to change the verbs from future to past tense to make this sentence make sense. That is okay, these statements are still equivalent.

3. We need to be a little creative to understand this sentence as a conditional. We could rewrite it, "If there is no pain then there will be no gain." The negation of "There will be no gain" is "There will be gain", that is our new condition. The negation of "There is no pain" is "There is pain", that's our new consequence. So the desired contrapositive can be stated, "If there will be gain there must be pain." Whether we agree with the sentiment or not, we can see it's equivalent to the original statement.

4. Recall that the words, necessary and sufficient, provide another way to frame conditionals. A consequence necessarily follows from its condition, so when we say water is necessary for life, we are saying life $\to$ water. The contrapositive is no water implies there is no life. If we believe water is necessary for life then we also believe that when there is no water there can be no life. The two beliefs stand or fall together because they are logically equivalent. Evidence against one is evidence against both.

5. Here the condition is written after its consequence. This statement is $p \to q$ where

p: It stops raining.

q: We go outside.

The contrapositive $\sim q \to \sim p$ could be read If we don't go outside, that's because it hasn't stopped raining.

7.4.5 Negating a conditional

We want to find a negation for $p \to q$ that is simple to express. The truth table for $\sim(p \to q)$ shows it is true in row 2 but nowhere else.

p	q	$p \to q$	$\sim(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Since $p \wedge \sim q$ has identical truth values in every row, it is equivalent to $\sim(p \to q)$.

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

$p \wedge \sim q$ states that the condition p is true and the consequence q is false. In other words, it states a counterexample to the conditional! As we saw before, a conditional is false only when it has a counterexample.

There's one more piece to this story: in practice most conditionals come with a "for all" quantifier attached. Earlier we used the example, If a number is prime then it is odd. This is not meant to be a statement about specific numbers! If 3 is prime then 3 is odd, does not seem like a terribly useful statement to make. Rather our conditional is a statement about all numbers: For any number, if that number is prime then it is odd. We can represent this as $\forall x\ p(x) \rightarrow q(x)$ where

$p(x)$: The number x is prime.

$q(x)$: The number x is odd.

Back in a previous section we saw how to negate a quantified statement:

$$\sim\forall x\ r(x) \equiv \exists x\ \sim r(x)$$

Since the negation of $p \rightarrow q$ is $p \wedge \sim q$, the negation of $\forall x\ p(x) \rightarrow q(x)$ is $\exists x\ p(x) \wedge \sim q(x)$.

Example: Write the negation of each statement.

1. If a number is prime then it must be odd.
2. If anyone speeds, they will get a ticket.
3. Anyone who gets into the club must be 21 and show their ID.

Solution:

1. As above, let

   $p(x)$: The number x is prime.

   $q(x)$: The number x is odd.

   Then the negation of this statement is $\exists x\ p(x) \wedge \sim q(x)$, which we can read, There exists a number that is prime and not odd. So when a statement says a consequence always follows its condition, the statement's negation is a counterexample that says there's at least one time that the consequence does not follow.

2. This conditional implicitly claims that speeding results in tickets for all drivers. Let

   $p(x)$: The driver x speeds.

   $q(x)$: The driver x gets a ticket.

   Then the given statement is $\forall x\ p(x) \to q(x)$, so its negation is $\exists x\ p(x) \wedge {\sim} q(x)$, which we can read as There exists a driver who does speed but does not get a ticket.

3. We will use three simple statements:

   $p(x)$: Person x gets into the club.

   $q(x)$: Person x is 21.

   $r(x)$: Person x shows their ID.

   The given statement is

   $$\forall x\ p(x) \to q(x) \wedge r(x)$$

   And its negation is therefore

   $$\exists x\ p(x) \wedge {\sim}(q(x) \wedge r(x))$$

   Since by De Morgan's laws ${\sim}(q(x) \wedge r(x)) \equiv {\sim} q(x) \vee {\sim} r(x)$, we can rewrite the negation

   $$\exists x\ p(x) \wedge ({\sim} q(x) \vee {\sim} r(x))$$

   which reads, There is someone who got into the club who is either not 21 or didn't show their ID.

7.4.6 Exercises: Conditionals

Solutions appear at the end of this textbook.

1. Write English statements for each of the following formulas, given that p: It is after 9 PM. q: The library is closed.

   (a) $p \to q$

   (b) $\sim p \to \sim q$

   (c) $q \leftrightarrow p$

2. Write a formula for each of the following statements, given that p The mango is ripe. q The mango's skin is red. r The mango is hard.

   (a) If it is not hard and the skin is red, then the mango is ripe.

   (b) The mango is not ripe if it is still hard.

   (c) For the skin to turn red, it is sufficient that the mango is ripe.

3. For each formula, write a truth table.

   (a) $(p \leftrightarrow q) \wedge (q \to \sim p)$

   (b) $(q \to p) \wedge q$

4. Write the contrapositive: I can't buy bread if I don't have the money.

5. Write the contrapositive: I I hear bad news I will send you an update.

6. Write the negation to the quantified statement: If anyone saw who did it they would tell me.

7. Write the negation to the quantified statement: If you square any number it gets bigger.

Chapter 8

Set Theory

Written by Nathan Rehfuss

8.1 Describing Sets

8.1.1 What is a set?

Often in math we want to work with lists of objects. Maybe we want to list all possible answers to a question, or all the inputs we can plug into a formula. Maybe we want to compare two lists to see what they have in common. For applications like these we will use sets. A **set** is a container for a collection of objects. The objects inside a set are called **elements**. An element can be in or out: elements cannot be included more than once and the order of elements inside a set doesn't matter. When we describe a set our definition needs to be precise and unambiguous, so that anyone reading can decide exactly which elements are included. The easiest way to do this is with *roster notation*: simply list all the elements of your set in between curly braces. So $\{3, 2, 1\}$ is a set containing the elements 3, 2 and 1.

Example: For each part, tell whether the thing described is a set.

1. $\{1, 3, 6, 10\}$
2. A collection containing every Green Day album.
3. A collection containing the best albums of the 90s.
4. A collection containing the number 4 and two copies of the number 3.
5. $\{4, 3, 3\}$
6. A collection containing every positive whole number.

Solution:

1. Yes. This is a set in roster notation; it contains the elements 1, 3, 6 and 10. Since order does not matter in a set, $\{10, 6, 1, 3\}$ describes the same set.
2. Yes. We don't have a roster for this set, but we have a precise and unambiguous definition. We can look up the Green Day discography to find out exactly which elements are in this set.
3. No. This definition is ambiguous. Which albums of the 90s were the best? You will probably get a different answer from each person you ask. Since there is no definitive way to decide which elements are in this collection, it is not a set.
4. No. Elements are either included or excluded from a set. A set cannot include the same element multiple times. Since this collection has two copies of the number 3, it is not a set.
5. Yes, but only by convention: when we see the same element listed multiple times in roster notation, we'll ignore the extra copies. This convention will be useful later when we are putting sets together. So this is a set with 2 elements: it is the same as the set $\{4, 3\}$.

6. Yes. This definition is unambiguous; we know exactly which numbers are included and which are excluded. We cannot list all of the elements, though, because they go on forever. In roster notation we can indicate this with an ellipsis: the set is $\{1, 2, 3, 4, 5, 6, \ldots\}$. Using an ellipsis to continue a pattern can be confusing or even ambiguous for more complicated sets. We'll explore better notation for complicated sets and infinite sets later in this section.

8.1.2 Elements of a set

Typically we will use uppercase letters to name sets and lowercase letters to name their elements. So we could declare a set like this: Let $A = \{1, 4, 9, 16, 25\}$. Now we can treat A as an object of study. Later in this chapter we will apply operations to sets like A, for now let's see how to describe its elements.

The symbol $\in$ says that something is an element of a set. We can write $9 \in A$ since 9 is an element of our set A. We'll read this as 9 is an element of A, or 9 is in A for short. When we want to say something is not an element we cross the element symbol out. For example we write $11 \notin A$ since 11 is not an element of A.

We can work with a generic element of A like this: Let $x \in A$. Now x could be any element of A, and whatever we say about x had better be true for every element of A. So we can say that x is a square number, since every element of A is square, but not that x is even, since some of the elements of A are odd.

Example: For each set, use element notation to tell whether the given objects are elements of that set.

1. $B = \{1, 2, 3, 4\}$

   (a) 1

(b) 5

(c) 2.5

2. $C = \{1, 3, 5, 7, 9\}$

(a) 8

(b) 9

(c) 11

3. $S =$ the set of positive whole numbers.

(a) 2

(b) 3.14

(c) -9

(d) 40000

Solution:

1. $B = \{1, 2, 3, 4\}$

(a) Yes, $1 \in B$ since 1 is one of the elements listed in the roster.

(b) No, $5 \notin B$ since 5 is not listed in the roster.

(c) No, $2.5 \notin B$. Even though it's in between the elements listed, it's not listed itself so it's not part of this set.

2. $C = \{1, 3, 5, 7, 9\}$

(a) No, $8 \notin C$.

(b) Yes, $9 \in C$.

(c) No, $11 \notin C$. If this were a pattern we might guess that 11 is the next number in the sequence, but since there's no ellipsis the set stops at 9. Only the five elements listed are part of C.

3. $S =$ the set of positive whole numbers.

   (a) Yes, $2 \in S$ since 2 is positive and a whole number.

   (b) No, $3.14 \notin S$. 3.14 is positive, but it's not whole so it isn't included.

   (c) No, $-9 \notin S$. Even though it is whole, this number is not positive, so it is not included.

   (d) Yes, $40000 \in S$. This is a large number, but almost all the elements of S are even larger! No matter what element of S we pick, there are more elements of S above it than below it.

8.1.3 Commonly encountered sets

There are a few sets of numbers that show up very often in mathematics. You have probably used these sets without knowing it! They can help us decide which kinds of numbers are appropriate for a specific problem. We'll cover five sets in this subsection in increasing order of size.

First up is the **empty set** $\varnothing$. This is a set with no elements at all! In roster notation it looks like this:

$$\varnothing = \{\}$$

The empty set shows up whenever we try to satisfy an impossible condition. It is the set of all negative square numbers, for example, and it is the set of all odd numbers that are divisible by two. Whenever a problem cannot be solved, the list of solutions is the empty set.

Next up is $\mathbb{N}$, the set of *natural numbers*. These are also known as the counting numbers. When you're dealing with quantities of objects, all of the quantities are natural numbers. In roster notation $\mathbb{N}$ looks like this:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$$

We start at 0 because that is the smallest quantity of an object we can have. The ellipsis is necessary because there is no largest natural number we've counted to, we can always count one higher. In other words every natural number has a successor, a natural number that comes next. The successor of 0 is 1, the successor of 10 is 11, the successor of 1000 is 1001. We never run out of natural numbers. Sets that go on forever are called *infinite sets*.

Every natural number has a successor, but not all of them have predecessors, numbers that come before. For that property we need to expand to $\mathbb{Z}$, the *integers*. $\mathbb{Z}$ contains all positive and negative whole numbers. The symbol comes from the German word *Zahlen*, which means whole number.

$$\mathbb{Z} = \{\ldots - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

So we can use $\mathbb{Z}$ in problems where whole numbers are appropriate, and $\mathbb{N}$ when those whole numbers can't be negative.

If we add two numbers from $\mathbb{N}$, the result will always be in $\mathbb{N}$. The same goes for multiplication. If we subtract, though, we might get an answer from $\mathbb{Z}$. $\mathbb{Z}$ is self-contained when it comes to addition, multiplication, and subtraction: if we add, subtract or multiply two numbers from $\mathbb{Z}$, the answer will be in $\mathbb{Z}$.

Division is another story. Dividing 10 by 5 gives an answer still in $\mathbb{Z}$, but dividing 10 by 3 does not: the answer is a fraction. $\mathbb{Q}$ is the set of all fractions of integers. Whenever we divide one whole number by another the answer will be in $\mathbb{Q}$. (With the exception that we cannot divide by 0.)

$\mathbb{Q}$ gets its symbol from the word quotient, the answer to a division problem. It is named the *rational numbers*, from the word ratio. $\mathbb{Q}$ contains numbers like $\frac{1}{2}$, $\frac{10}{3}$ and $\frac{-19}{4}$. It also contains all the whole numbers since for example 3 is the same as $\frac{3}{1}$. Starting with numbers in $\mathbb{Q}$, we can add, subtract, multiply and divide and always remain in $\mathbb{Q}$.

There are a few useful numbers still outside $\mathbb{Q}$, so we have one more set to cover, and it contains all the others. The *real numbers*, $\mathbb{R}$, is the set containing every number on a number line. This includes all of the whole numbers and rational numbers, and in addition it includes the numbers called *irrational.* An irrational number is one that cannot be written as a fraction of whole numbers. Written as a decimal its digits go on forever without repeating. Some commonly encountered numbers that are real but not rational include:

$$\pi = 3.1415926536\ldots$$
$$\sqrt{2} = 1.4142135624\ldots$$
$$\mathrm{e} = 2.7182818285\ldots$$
$$\phi = 1.6180339888\ldots$$

We have not shown roster notation for $\mathbb{Q}$ and $\mathbb{R}$ because it is impossible to do so conveniently. Any ordered list of rational or real numbers will have omissions because of *in-betweenness*: in between any two real numbers is another real number, and in between any two rational numbers is another rational number.

For this same reason $\mathbb{Q}$ and $\mathbb{R}$ do not have successors: there's no way to say which real number comes next because there's always another number in between. 5 is not the next real number after 4, because 4.5 lies between them. 4.5 is not the successor of 4 either because

4.2 comes in between, and before that is 4.1 and 4.05. There's no next number after 4 in $\mathbb{R}$: the concept simply doesn't make sense.

Example: For each number, tell which of the five common sets it belongs to.

1. 3
2. 3.2
3. -4
4. $-1.66666\ldots$
5. $\pi + 5$

Solution:

1. (a) $3 \notin \varnothing$ since that set has no elements.
   (b) $3 \in \mathbb{N}$ since 3 is a positive whole number.
   (c) $3 \in \mathbb{Z}$ since 3 is a whole number.
   (d) $3 \in \mathbb{Q}$ since $3 = \frac{3}{1}$
   (e) $3 \in \mathbb{R}$ since 3 is on the number line.
2. (a) $3.2 \notin \varnothing$.
   (b) $3.2 \notin \mathbb{N}$ since 3.2 is not a whole number.
   (c) $3.2 \notin \mathbb{Z}$ since 3.2 is not a whole number.
   (d) $3.2 \in \mathbb{Q}$ since $3.2 = \frac{16}{5}$
   (e) $3.2 \in \mathbb{R}$ since 3.2 is on the number line.
3. (a) $-4 \notin \varnothing$.
   (b) $-4 \notin \mathbb{N}$ since -4 is not positive.

(c) $-4 \in \mathbb{Z}$ since -4 is a whole number.

(d) $-4 \in \mathbb{Q}$ since $-4 = \frac{-4}{1}$

(e) $-4 \in \mathbb{R}$ since -4 is on the number line.

4. (a) $-1.66666\ldots \notin \varnothing$.

(b) $-1.66666\ldots \notin \mathbb{N}$ since $-1.66666\ldots$ is not a whole number.

(c) $-1.66666\ldots \notin \mathbb{Z}$ since $-1.66666\ldots$ is not a whole number.

(d) $-1.66666\ldots \in \mathbb{Q}$ since $-1.66666\ldots = \frac{-5}{3}$

(e) $-1.66666\ldots \in \mathbb{R}$ since $-1.66666\ldots$ is on the number line.

5. (a) $\pi + 5 \notin \varnothing$.

(b) $\pi + 5 \notin \mathbb{N}$ since $\pi + 5$ is not positive.

(c) $\pi + 5 \notin \mathbb{Z}$ since $\pi + 5$ is a whole number.

(d) $\pi + 5 \notin \mathbb{Q}$. Subtracting two numbers in $\mathbb{Q}$ produces a result in $\mathbb{Q}$. So if $\pi + 5$ were in $\mathbb{Q}$, we could subtract 5 to get π which would then have to be in $\mathbb{Q}$. But we know $\pi \notin \mathbb{Q}$! So $\pi + 5$ cannot be in $\mathbb{Q}$ either.

(e) $\pi + 5 \in \mathbb{R}$ since $\pi + 5$ is on the number line.

8.1.4 Universal sets

One use of our common sets is to limit the scope of a math problem to the appropriate type of numbers. A **Universal set** gives the context for a problem. All numbers used in the problem, including the solutions, will come from the universal set.

If we are counting sheep, $\mathbb{N}$ is a good choice of universal set. We could have 0 sheep, or 5, or 10 or 20. We do not want to consider numbers from $\mathbb{Q}$ here, something has gone very wrong if we count $\frac{2}{3}$ of a sheep.

If we are cutting pies, $\mathbb{Q}$ is a great choice. $\mathbb{Q}$ gives us the freedom to serve $\frac{1}{4}$ or $\frac{1}{6}$ or $\frac{1}{8}$ of a pie. There's no need to divide pies up into irrational numbers, and we lack the requisite skill with a knife anyhow.

If we're doing geometry, we should work in $\mathbb{R}$. We need numbers like π for the circumference of a circle and $\sqrt{2}$ for the diagonal of a square, and we can't write those numbers as fractions.

Example: Solve the story problem once for each universal set provided.

1. Your friend picked a number between 1 and 10. Can you guess it?
   (a) $\mathbb{N}$
   (b) $\mathbb{R}$
2. A set contains two numbers. The squares of those numbers add up to 25. What is the set?
   (a) $\mathbb{N}$
   (b) $\mathbb{Z}$
   (c) $\mathbb{Q}$
   (d) $\mathbb{R}$
3. Three friends ate cake together. Alice ate twice as many cakes as Dylan. Dylan ate twice as many cakes as Carol. Together the three of them ate fewer than 10 cakes. How many cakes did each person eat?
   (a) $\mathbb{N}$
   (b) $\mathbb{Q}$

Solution:

1. Your friend picked a number between 1 and 10. Can you guess it?

   (a) Working in $\mathbb{N}$ restricts your friend's choices to positive whole numbers. If your friend picks a number between 1 and 10, it must come from the set

   $$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

   You have a 1 in 10 chance of guessing the number, which is not so unlikely. Play enough times and you will certainly win some of them.

   (b) Working in $\mathbb{R}$ is not in your favor. Since there's always another real number in between two numbers, we cannot list all of your friend's options! There are infinitely many. Your friend could be thinking of 4.5, or 6.01, or 3.99999967. They could even pick a number like $\frac{49\pi}{17}$. This time you could play forever and not guess the number once.

2. A set contains two numbers. The squares of those numbers add up to 25. What is the set?

   (a) If the numbers come from $\mathbb{N}$ there are exactly two possibilities: $\{3, 4\}$ and $\{0, 5\}$. $3^2 + 4^2 = 9 + 16 = 25$ and $0^2 + 5^2 = 0 + 25 = 25$. Any other pair of numbers will not work. Neither number can be above 5 or its square will be too big, and we can check all the pairs of numbers 5 and under.

   (b) Working in $\mathbb{Z}$ gives us access to negative whole numbers. This expands our options since for example $(-3)^2 = 3^2 = 9$. The list of possible solutions is $\{3, 4\}$, $\{-3, 4\}$, $\{3, -4\}$, $\{-3, -4\}$, $\{0, 5\}$ and $\{0, -5\}$.

   (c) In $\mathbb{Q}$ we have infinitely many options. For example you can check that $\{\frac{25}{13}, \frac{60}{13}\}$ and $\{\frac{40}{17}, \frac{75}{17}\}$ produce the desired result. Can you find the pattern to produce more such pairs?

(d) In $\mathbb{R}$ we have even more options than in $\mathbb{Q}$. We can start with any real number between -5 and 5 and solve an equation to find it a mate. Start with π for example. If $\pi^2 + x^2$ must equal 25, then

$$\begin{aligned} \pi^2 + x^2 &= 25 \\ x^2 &= 25 - \pi^2 \\ x &= \sqrt{25 - \pi^2} \end{aligned}$$

So $\{\pi, \sqrt{25 - \pi^2}\}$ is a solution. Interestingly this process is much easier than finding solutions in $\mathbb{Q}$.

3. Three friends ate cake together. Alice ate twice as many cakes as Dylan. Dylan ate twice as many cakes as Carol. Together the three of them ate fewer than 10 cakes. How many cakes did each person eat?

(a) Suppose each person ate a natural number of cakes. Let's say Carol ate c cakes with $c \in \mathbb{N}$. Dylan ate twice as many, that's $2c$ cakes. Alice ate twice as many as Dylan, so she ate $2 \cdot 2c = 4c$ cakes. Together the three of them ate $c + 2c + 4c = 7c$ cakes. If Carol ate 1 cake then the trio ate 7. But if Carol ate 2 cakes then the trio ate 14, which is more than 10 and not allowed. So the only possibility is that Carol ate 1 cake, Dylan ate 2 and Alice ate 4.

(b) If our trio can eat fractions of cakes represented by numbers in $\mathbb{Q}$ then Carol is no longer restricted to exactly 1 cake. She can eat any number greater than 0 and less than $\frac{10}{7}$. For example if Carol eats $\frac{1}{7}$ of a cake then Dylan will eat $\frac{2}{7}$ and Alice will eat $\frac{4}{7}$, for a total of exactly 1 cake consumed by the three of them. Once again there are infinitely many possible solutions in $\mathbb{Q}$.

8.1.5 Solution sets

When a problem has more than one solution, it is useful to treat the solutions as a set. The **solution set** for a problem is the set of all solutions to the problem.

Example: Write the solution set for each story problem. Assume the universal set for a problem is $\mathbb{R}$ unless otherwise specified.

1. This number's square is 16. What is the number?
2. Five times this number is four greater than three times this number. What is the number?
3. This number is five greater than its square. What is the number?
4. A rectangular patio has an area of 8 square meters. Its side lengths are whole numbers. How wide is the patio?
5. Divide this positive number by 3 and the remainder is 2. Divide it by 4 and the remainder is 3. What is the number?

Solution:

1. $4^2 = 16$ and $(-4)^2 = 16$. Either one could be the number. Our solution set is $\{4, -4\}$.
2. We can write this story as an algebra equation. If we call the number x, then five times the number is $5x$, and four more than three times the number is $4 + 3x$. So we have

$$5x = 4 + 3x$$

$$2x = 4$$

$$x = 2$$

There is only one possible solution, so the solution set only has one element. The solution set is $\{2\}$.

3. Most numbers are less than their squares. The only numbers that are greater than their squares are between 0 and 1, and none of them are five greater. This condition is impossible, there is no solution, so the solution set is $\varnothing$.

4. The area of a rectangle is length times width, so the length and width of the patio are two whole numbers whose product is 8. The possibilities are $1 \cdot 8 = 8$ and $2 \cdot 4 = 8$. Since we do not know which number is the length and which is the width, this patio's width could be 1, 2, 4 or 8 meters. The solution set is $\{1, 2, 4, 8\}$.

5. Only whole numbers have remainders when divided, so in this problem our universal set is $\mathbb{N}$. The numbers whose remainder is 2 when divided by 3 are

$$\{2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38 \ldots\}$$

The numbers whose remainder is 3 when divided by 4 are

$$\{3, 7, 11, 15, 19, 23, 27, 31, 35, 39 \ldots\}$$

Our solution set is the *intersection* of these two sets: the set of numbers that they have in common. We will study intersections and other set operations in detail in section 9.3.

Since 11 is the first set, its remainder is 2 when divided by 3, and since it's in the second set, its remainder is 3 when divided by 4. We need all of the numbers that appear in both sets. The next numbers that work are 23 and 35, but there are more. Our solution set is $\{11, 23, 35, 47, 59, 71, 83 \ldots\}$. The numbers keep going up by 12 forever, so this is an infinite set!

8.1.6 Set-builder notation

Now that we've laid the groundwork for sets it's time to develop a much more powerful way to specify them. **Set-builder notation** is a way to use propositional logic statements to tell exactly which elements are in a set - even complicated or infinite sets. Set-builder notation separates the area inside the curly braces into two halves with a vertical bar.

$$\{ \quad | \quad \}$$

In the left half we put an archetypal member of the set: a variable or expression that represents each element of the set. Usually this is just a variable. In the right half we put one or more statements about that variable that narrow down exactly what we want in our set. Suppose we write

$$A = \{ \ x \mid x < 3, x \in \mathbb{N}\}$$

The variable x is our archetype because it's on the left side. x represents any element of A. On the right side are two statements separated by a comma: the first says x is less than 3, the second says x is a natural number. A is the set of all elements x that satisfy both conditions: $A = \{0, 1, 2\}$ since those are the natural numbers that are less than 3.

Example: Convert each set from set-builder notation to roster notation.

1. $B = \{ \ x \mid x < 5, x \in \mathbb{N}\}$
2. $C = \{ \ x \mid 6 \leq x < 10, x \in \mathbb{N}\}$
3. $D = \{ \ 2x \mid x \in \mathbb{N}\}$
4. $R = \{ \ 2x + 1 \mid x \in \mathbb{N}\}$
5. $S = \{ \ 3x \mid 10 < x \leq 20, x \in \mathbb{N}\}$
6. $T = \{ \ \frac{x}{y} \mid y \neq 0, x \in \mathbb{Z}, y \in \mathbb{Z}\}$

Solution:

1. B is the set of all numbers x that are natural numbers and are less than 5. Those numbers are 0, 1, 2, 3, and 4, so $B = \{0, 1, 2, 3, 4\}$.

2. C is the set of natural numbers that are greater than or equal to 6 and are also less than 10. So, 6 is included in C but 10 is not. Counting all the natural numbers in between we get $C = \{6, 7, 8, 9\}$.

3. The only statement on the right side of the definition for D is that x must be a natural number. So x can be any number in $\{0, 1, 2, 3, 4, \ldots\}$. But this time the archetype for an element of D is $2x$. Whenever x is a natural number, $2x$ is an element of D. So we look at each natural number and multiply it by 2. The first few elements of D are $\{2 \cdot 0, 2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, \ldots\}$. So $D = \{0, 2, 4, 6, 8, \ldots\}$. D is the set of even natural numbers!

4. Just like in the last part, x can be any natural number. This time the archetype for elements of the set is $2x + 1$: for every natural number x, the number $2x + 1$ is an element of R. Here are the first few elements of R:

$$2 \cdot 0 + 1 = 1$$

$$2 \cdot 1 + 1 = 3$$

$$2 \cdot 2 + 1 = 5$$

$$2 \cdot 3 + 1 = 7$$

$$2 \cdot 4 + 1 = 9 \ldots$$

   So $R = \{1, 3, 5, 7, 9, \ldots\}$. R is the set of odd natural numbers!

5. Since our archetype here is $3x$, every element of S will be a multiple of 3. We're told that x can be any natural number greater than 10 and less than or equal to 20. So

the values for x are the whole numbers 11 through 20, and the elements of S are three times those numbers.

$$S = \{33, 36, 39, 42, 45, 48, 51, 54, 57, 60\}$$

6. In this set x and y can be any whole numbers, positive or negative, since they come from $\mathbb{Z}$. We're given one exception, that y cannot be 0. The elements of T are all of the quotients $\frac{x}{y}$. So T is the set of answers we get by dividing any two integers. We've seen this set before: T is the set of rational numbers $\mathbb{Q}$! As explained previously this set cannot be listed in roster notation in any satisfying way since there will always be more rational numbers between the ones we list. $\mathbb{Q} = \{\ \frac{x}{y} \mid y \neq 0, x \in \mathbb{Z}, y \in \mathbb{Z}\}$ is the best, most concise way to define this set.

Now that we've decoded a few of these set-builder forms, let's try to create some ourselves. Set-builder notation can do almost anything, so the solutions we give in the following example will not be unique. There are many ways to use set-builder notation to specify any particular set.

Example: For each set, find a pattern in its elements. Write one or more statements that completely describe that pattern and use them to write the set in set-builder notation.

1. $A = \{12, 13, 14, 15, 16, 17, 18\}$
2. $B = \{9, 10, 11, 12, 13, \ldots\}$
3. $C = \{7, 8, 9, 10, 11, 12, 14, 16, 17, 18\}$
4. $D = \{\ldots - 3, -2, -1, 0\}$
5. $E = \{2, 4, 6, 8, 10, \ldots 26, 28, 30\}$
6. $F = \{1, 4, 7, 10, 13, \ldots\}$

7. $G = \{2, 6, 10, 14, 18, \ldots\}$

8. $H = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$

9. $J = \{1, 4, 9, 16, 25, 36, \ldots\}$

Solution:

1. A is a set of natural numbers between 12 and 18. If we call our archetype x, we can use one statement to say x is a natural number and another to say it is between 12 and 18. This will completely specify A. So our answer is

$$A = \{\ x \mid 12 \leq x \leq 18, x \in \mathbb{N}\}$$

2. The pattern is clear: the ellipsis indicates that these numbers keep going up by one forever. So B contains every natural number starting with 9. We can write

$$B = \{\ x \mid 9 \leq x, x \in \mathbb{N}\}$$

3. Look carefully: the number 13 is missing. Roster notation is a poor choice to express C since the casual reader will not notice this omission. We can be much clearer in set-builder notation: one statement will say that elements must be between 7 and 18, a second will say that 13 is not included, and a third will say that all elements are natural numbers. Our answer looks like this:

$$C = \{\ x \mid 7 \leq x \leq 18, x \neq 13, x \in \mathbb{N}\}$$

4. The ellipsis indicates that D includes all the negative whole numbers and 0. We can get this set by starting with $\mathbb{Z}$ and disallowing all the positive numbers. So

$$D = \{\ x \mid x \leq 0, x \in \mathbb{Z}\}$$

5. Set E contains every even number from 2 to 30. We can use $2x$ as our archetype so that all elements of the set come out even, then we just need statement limiting x to the natural numbers and a statement that keeps $2x$ between 2 and 30.

$$E = \{\ 2x \mid 2 \leq 2x \leq 30, x \in \mathbb{N}\}$$

6. The pattern here is that each number is 3 greater than the one before it. The first number is 1, the second is $1 + 3$, the third is $1 + 3 + 3$, the fourth is $1 + 3 + 3 + 3$, and then $1 + 4 \cdot 3$, $1 + 5 \cdot 3$, $1 + 6 \cdot 3$ and so on. The general form of a number in this set is $1 + x \cdot 3$, where x is any whole number greater than or equal to zero. In other words, x can be any element of $\mathbb{N}$.

$$F = \{\ 1 + 3x \mid x \in \mathbb{N}\}$$

7. This set is very similar to the previous one. This time we start with 2 and count by fours, so an element of this set is two plus any number of fours, or $2 + 4x$.

$$G = \{\ 2 + 4x \mid x \in \mathbb{N}\}$$

8. This pattern might not be immediately obvious. We're not counting by twos or fours here, rather these elements all have one thing in common: these are the prime numbers. Set-builder notation can take any definition and create a set of only the things that match that definition. A prime number is defined as a number whose only factors are

1 and itself. For example 7 is equal to $1 \cdot 7$ and is not divisible by any other numbers in $\mathbb{N}$.

So if x is an element of our set, what needs to be true of x? What should we write on the right hand side? If x is the product of two natural numbers, then one of those numbers must be 1 and the other must be x.

That's an if-then statement: we can express it with a conditional. The condition is that x is the product of two natural numbers. We need to name them to refer to them. Let's write $x = m \cdot n$. The consequence is that either $m = 1$ and $n = x$ or $m = x$ and $n = 1$. We'll use a disjunction for the "or" in that sentence and conjunctions for the "ands", so we write $(m = 1 \wedge n = x) \vee (m = x \wedge n = 1)$ for the consequence. The whole if-then statement is

$$(x = m \cdot n) \rightarrow (m = 1 \wedge n = x) \vee (m = x \wedge n = 1)$$

We need this conditional to be true for any two natural numbers m and n, not just one specific pair. We accomplish this with for-all quantifiers.

$$\forall m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ (x = m \cdot n) \rightarrow (m = 1 \wedge n = x) \vee (m = x \wedge n = 1)$$

Now the conditional must be true for any m from the natural numbers and for any n from the natural numbers. If x has any factor pair other than 1 and itself, this statement will be false.

Finally, x must be a natural number, since only positive whole numbers can be prime. The numbers 0 and 1 technically fit our conditional statement but they are not prime so we must remove them by hand with $x \neq 0$ and $x \neq 1$. Now we are ready to describe our set.

$$
\begin{aligned}
H = \{\ x \mid\ & x \in \mathbb{N}, x \neq 0, x \neq 1, \\
& \forall m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ (x = m \cdot n) \rightarrow (m = 1 \wedge n = x) \vee (m = x \wedge n = 1)\}
\end{aligned}
$$

9. With the primes under our belt, the square numbers should not be so tough. This is a set of all of the squares of natural numbers, except 0. So if our archetype is x^2 and we state that $x \in \mathbb{N}$ and $x > 0$, we are done!

$$J = \{\ x^2 \mid x > 0, x \in \mathbb{N}\}$$

8.1.7 Exercises: Describing Sets

Solutions appear at the end of this textbook.

1. True or false? $7 \in \{2, 4, 6, 8, 10\}$.

2. True or false? $12 \notin \{1, 3, 5, 7, 9, 11, \ldots\}$.

3. True or false? $24 \in \{0, 3, 6, 9, 12, 15, \ldots\}$.

4. True or false? $23 \notin \{\ 5x + 3 \mid x \in \mathbb{N}\}$.

5. True or false? $27 \in \{\ x^3 \mid x \in \mathbb{Z}\}$.

6. True or false? $11 \in \mathbb{R}$.

7. True or false? $\frac{9}{10} \in \mathbb{Z}$.

8. True or false? $21 \notin \mathbb{Q}$.

9. For each scenario, state which universal set is best.

   (a) Terry is keeping track of how many apples he picked at the orchard.

   (b) Grace is measuring the dimensions of her furniture for a move.

   (c) Destiny is tracking how many seats each party wins or loses in an election.

   (d) John is cutting pizzas into equal slices to share with his friends.

10. Write this set in roster form: $K = \{\ x \mid 7 < x \leq 15, x \in \mathbb{N}\}$.

11. Write this set in roster form: $L = \{\ 10x \mid x \leq 5, x \in \mathbb{N}\}$.

12. Write this set in set-builder notation: $M = \{21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$.

13. Write this set in set-builder notation: $N = \{5, 10, 15, 20, 25, 30, \ldots\}$.

8.2 Subsets and Cardinality

8.2.1 Subsets

We've seen how to talk about individual members of sets. For example, we can say $3 \in A$ to mean that 3 is an element of the set A. But what if we want to talk about whole portions of sets? That's where the concept of *subsets* comes in.

Suppose we have two sets S and T. S is a *subset* of T if T contains at least every element of S. We write $S \subseteq T$ to say that S is a subset of T. Whenever S is a subset of T we can also say T is a *superset* of S since it contains every element of S.

For a technical definition of **subset** we can say $S \subseteq T$ means $\forall x \ \ (x \in S) \rightarrow (x \in T)$. For every object x, if x is an element of the subset S then x has to be an element of the superset T.

Example: Tell whether one set is a subset of the other.

1. $S = \{1, 3, 5\}$ and $T = \{1, 2, 3, 4, 5\}$
2. $U = \{2, 4, 6, 8\}$ and $V = \{4, 5, 6, 7, 8, 9, 10\}$
3. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{10\}$
4. $C = \{\ x \mid x \in \mathbb{N}, 20 \leq x \leq 30\}$ and $D = \{\ x \mid x \in \mathbb{N}, 10 \leq x \leq 40\}$
5. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$
6. $\varnothing$ and $P = \{3, 4, 5\}$
7. $M = \{\ 2x \mid x \in \mathbb{N}\}$ and $N = \{\ 4x \mid x \in \mathbb{N}\}$
8. $X = \{3, 6, 9\}$ and $Y = \{3, 6, 9\}$

Solution:

1. The elements of S are 1, 3, and 5. Checking the roster for T we see that $1 \in T$, $3 \in T$ and $5 \in T$. So every element of S is also included in T: We write $S \subseteq T$, that S is a subset of T.

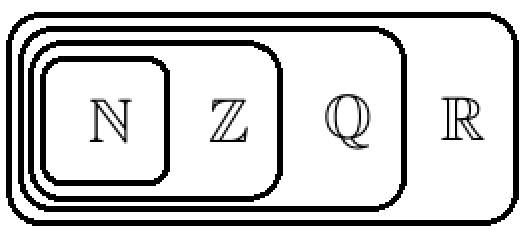
2. No, $U \nsubseteq V$. Since $2 \in U$ but $2 \notin V$, U cannot be a subset of V. The superset must contain every element of the subset, not even one can be missing.

3. Yes, $B \subseteq A$. The only element of B is 10, and 10 is an element of A. So B is a subset of A.

4. C is the set of natural numbers between 20 and 30. D is the set of natural numbers between 10 and 40. Every number that's between 20 and 30 is also in between 10 and 40. So since every element found in C can also be found in D, we say $C \subseteq D$.

5. Since $\mathbb{Z}$ contains every whole number and every element of $\mathbb{N}$ is a whole number, $\mathbb{N} \subseteq \mathbb{Z}$. Every element n of $\mathbb{Z}$ is also a rational number since n can be written $\frac{n}{1}$, so $\mathbb{Z} \subseteq \mathbb{Q}$. Finally, every element of $\mathbb{Q}$ is on the number line, so $\mathbb{Q} \subseteq \mathbb{R}$.

   Putting it all together we can say that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

   In addition this implies that $\mathbb{N}$ is a subset of $\mathbb{Q}$ and $\mathbb{R}$, and that $\mathbb{Z}$ is a subset of $\mathbb{R}$. The sets fit inside each other like nesting dolls.

6. Remember that $\varnothing$ is the empty set, a set with no elements. Here it helps to look at the technical definition of a subset: $\forall x \ (x \in \varnothing) \rightarrow (x \in P)$. There are no counterexamples to this statement: there are no elements that are present in $\varnothing$ but absent in P, because $\varnothing$ has no elements! Therefore $\varnothing \subseteq P$. In fact, every set is a superset of $\varnothing$.

7. M is the set of natural multiples of 2: $M = \{0, 2, 4, 6, 8, 10, \ldots\}$. N is the set of natural multiples of 4: $N = \{0, 4, 8, 12, 16, \ldots\}$. Every multiple of 4 is also a multiple of 2, that is, every element of N is also found in M. So $N \subseteq M$.

8. Every element of X is also found in Y, so $X \subseteq Y$. And every element of Y is also found in X, so $Y \subseteq X$! Whenever $X \subseteq Y$ and $Y \subseteq X$ are both true we can conclude that $Y = X$.

8.2.2 Proper subsets

In the previous example we saw that when two sets are identical, each is a subset of the other. That means that for any set A, we can say that $A \subseteq A$! That's because every element of A is also found in A. Sometimes we wish to avoid this behavior, at these times we will speak of *proper subsets.*

A **proper subset** of Y is a subset of Y that is also not equal to Y. Its symbol is $\subset$. In other words, we can write $X \subset Y$ whenever $X \subseteq Y$ and $X \neq Y$. The situation is analogous to the less than symbol: $\subset$ is to $\subseteq$ as $<$ is to $\leq$, as illustrated here:

$$
\begin{aligned}
4 &< 5 & \{1,2\} &\subset \{1,2,3\} \\
4 &\leq 5 & \{1,2\} &\subseteq \{1,2,3\} \\
5 &\not< 5 & \{1,2,3\} &\not\subset \{1,2,3\} \\
5 &\leq 5 & \{1,2,3\} &\subseteq \{1,2,3\}
\end{aligned}
$$

We can use $\subset$ or $\subseteq$ interchangeably when the sets are not equal, but we can only use $\subseteq$ when the sets are equal. That's because $\subset$ indicates that one is a subset of the other and that they are not equal.

Example: Tell whether one set is a subset or a proper subset of the other.

1. $H = \{2, 3\}$ and $J = \{1, 2, 3, 4\}$
2. $K = \{1, 5, 6\}$ and $L = \{6, 5, 1\}$
3. $M = \{2, 8, 4\}$ and $N = \{4, 8, 4, 2, 4\}$
4. $P = \{\ x \mid x > 0, x \in \mathbb{N}\}$ and $\mathbb{N}$
5. $Q = \{\ 2x \mid x \in \mathbb{N}\}$ and $R = \{\ x \mid x = 2y, y \in \mathbb{N}\}$

Solution:

1. Every element of H can be found in J so $H \subseteq J$. Since $H \neq J$ we can also say that $H \subset J$.
2. Order doesn't matter in a set. Since these sets contain the same elements, they are the same set. So we can say that $K \subseteq L$ and $L \subseteq K$, but neither is a proper subset of the other because they are equal.
3. Recall that we ignore repetition in roster notation. $\{4, 2, 4, 8, 4\}$ describes the same set as $\{4, 2, 8\}$. M and N contain the same three elements, so they are equal to each other. Therefore $M \subseteq N$ and $N \subseteq M$, but neither is a proper subset of the other.
4. P includes the condition $x \in \mathbb{N}$, so every element of P is found in $\mathbb{N}$. Hence $P \subseteq \mathbb{N}$. Furthermore $0 \in \mathbb{N}$ but $0 \notin P$, so the sets are not equal and we can also say $P \subset \mathbb{N}$.
5. Q is the set containing two times every natural number. R contains every number that is equal to two times a natural number. These are two ways of describing the same set in set-builder notation. So, $Q = R$ which means that $Q \subseteq R$ and $R \subseteq Q$. Since the sets are equal, $Q \not\subset R$ and $R \not\subset Q$.

8.2.3 Cardinality

Cardinality refers to the size of a set. For finite sets cardinality is just the number of elements the set has. The cardinality of infinite sets is more complicated, we'll discuss it in section 5 of this chapter. The symbol $n()$ refers to the cardinality of a set, so $n(A)$ is the cardinality of the set A.

Example: For each set, state its cardinality.

1. $A = \{3, 6, 12\}$
2. $B = \{2, 4, 8, 16, 32\}$
3. $C = \{3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5\}$
4. $D = \{\ x \mid x < 10, x \in \mathbb{N}\}$
5. $E = \{\ x \mid 15 \leq x < 28, x \in \mathbb{N}\}$
6. $F = \{\ x \mid 31 \leq x < 185, x \in \mathbb{N}\}$
7. $G = \{\ x \mid 41 < x < 97, x \in \mathbb{N}\}$
8. $H = \{\ x \mid 21 \leq x \leq 103, x \in \mathbb{N}\}$
9. $J = \{\ x \mid -51 \leq x < 22, x \in \mathbb{Z}\}$
10. $K = \{\ x \mid x \leq 100, x \in \mathbb{N}\}$
11. $L = \{\ x \mid x \geq 100, x \in \mathbb{N}\}$

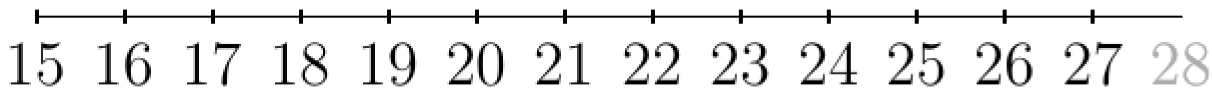
Solution:

1. Since A has 3 elements, we say $n(A) = 3$. A has a cardinality of 3.
2. Since B has 5 elements, $n(B) = 5$.

3. We ignore repetition in roster form. C only has 3 elements: 3, 4 and 5. So $n(C) = 3$.

4. D contains all the natural numbers less than 10. We can count these without too much trouble: $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and so $n(D) = 10$.

5. Listing this set or the next in roster form and counting is not an attractive proposition, so we will develop a shortcut.



   We've labeled a number line with the elements of E and drawn a tick mark for each element. 28 is grayed out because it is not an element: E contains only $15 \le x < 28$. We say this inequality includes its left endpoint but not its right.

   Subtracting two numbers gives their distance on the number line. $28 - 15 = 13$, so there are 13 units between 15 and 28 on this line. Each unit is represented by the horizontal line between one tick mark and the next. From 15 to 28 there are 13 dashes.

   If we pair each tick mark with the dash to its right, there are no dashes or ticks left over. So there are the same number of dashes as tick marks. There are 13 dashes, so there must be 13 tick marks and therefore 13 elements in the set. $n(E) = 13$.

6. We will use the shortcut developed in the previous problem: since $31 \le x < 185$ the number of elements is $185 - 31 = 154$. $n(F) = 154$.

7. If this set had the rule $41 \le x < 97$ there would be $97 - 41 = 56$ elements. But its rule is $41 < x$ instead of $41 \le x$, so the left endpoint 41 is not included. With the element 41 missing there is one fewer element than we counted, and the actual number is $n(G) = 55$.

8. If this set had the rule $21 \le x < 103$ there would be $103 - 21 = 82$ elements. But we are given $x \le 103$ instead of $x < 103$, so an extra element is included: the right

endpoint, 103. We counted 82 elements without the right endpoint, so when we include it too there are $n(H) = 83$ elements.

9. Since $\mathbb{Z}$ is a set of whole numbers just like $\mathbb{N}$ our same number line shortcut will work here. The number of elements is $22 - -51 = 73$. We write $n(J) = 73$.

10. We're missing some information to apply our subtraction shortcut, but we can recover. The natural numbers start at 0, so for any subset of $\mathbb{N}$ we can say the elements follow $x \geq 0$. That means that in the statements for K we can rewrite $x \leq 100$ as $0 \leq x \leq 100$. This is the same kind of inequality we were given for set H, so we follow the same method: $100 - 0 = 100$, and since both endpoints of the inequality are included we add one more element. $n(K) = 101$.

11. There is no upper limit given for elements of L. The list of elements keeps going on forever: $L = \{100, 101, 102, 103, 104, \ldots\}$. So this is an infinite set. We don't have the tools yet to tell the cardinality of infinite sets, so we will have to come back to L in a later section.

8.2.4 Power sets and counting subsets

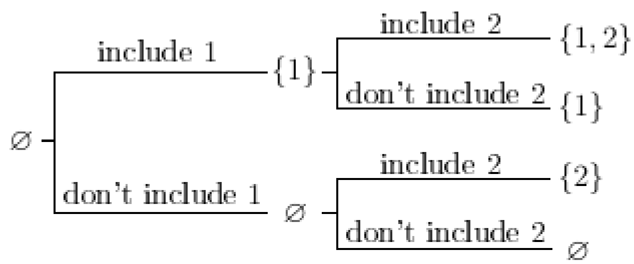
Let S be a set. The **power set** of S is the set containing all the subsets of S. Its symbol is 2^S. For a technical definition we can write $2^S = \{\ A \mid A \subseteq S\}$, that is, the power set of S is the set whose elements are any set A that is a subset of S. For example, let $T = \{1, 2\}$. Then

$$2^T = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$$

Every element of 2^T is itself a set. Each of these sets is a subset of T, and there are no subsets of T missing.

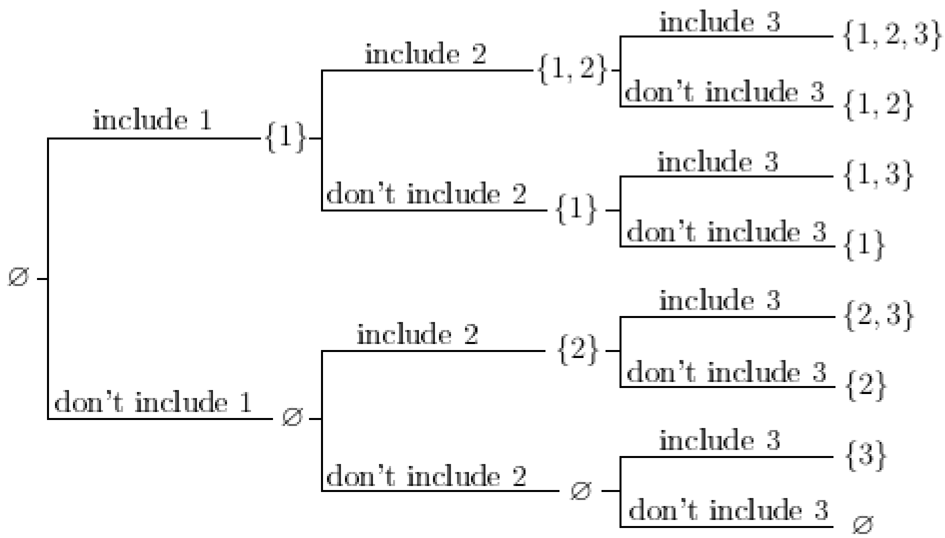
We will demonstrate that we have listed every possible subset of T by showing a process for listing them exhaustively. When creating a subset of T, we have two choices to make.

Should 1 be included? And should 2 be included? Since we have two yes-or-no choices, there are exactly four options.



This decision tree illustrates our options. Once we've decided for each element of T whether it will be included or not, there are no more choices to make: our subset is uniquely determined.

If we add a third element to our set to make $\{1, 2, 3\}$, we add another yes-or-no choice to our decision tree:



The new choice to make doubles our options from 4 to 8. Each additional element will double our options again: a set with 4 elements has 16 subsets, a set with 5 elements has 32 subsets. In general the rule for counting subsets is

$$n\left(2^S\right) = 2^{n(S)}$$

If a set has k elements, it will have 2^k subsets.

Example:

1. How many subsets does $\{5, 8, 9\}$ have?
2. How many subsets does $\{1, 2, 3, 4, 5, 6\}$ have?
3. How many subsets does $T = \{\ x \mid 12 \leq x < 22, x \in \mathbb{N}\}$ have?
4. How many proper subsets does $K = \{1, 2, 3\}$ have?
5. How many proper subsets does $N = \{2, 3, 5, 7, 11, 13\}$ have?

6. How many proper subsets does $V = \{\ x \mid 1 \leq x < 10, x \in \mathbb{N}\}$ have?

7. Suppose that $A = 2^{\varnothing}$, $B = 2^{A}$, $C = 2^{B}$ and $D = 2^{C}$. What is $n(D)$?

Solution:

1. The situation is exactly as in the decision tree for $\{1, 2, 3\}$ above. We have three yes-or-no choices to make. Only the names of the elements have changed.

$$2^{\{1,2,3\}} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$
$$2^{\{5,8,9\}} = \{\varnothing, \{5\}, \{8\}, \{9\}, \{5, 8\}, \{5, 9\}, \{8, 9\}, \{5, 8, 9\}\}$$

   Since $\{5, 8, 9\}$ has 3 elements, it has $2^3 = 8$ subsets.

2. Because this set has 6 elements, it has $2^6 = 64$ subsets.

3. We can count the elements of T using our shortcut from the previous example: since $22 - 12 = 10$, T has 10 elements. If $n(T) = 10$ then $n\left(2^T\right)$ must equal $2^{10} = 1024$. T has 1024 subsets.

4. The subsets of K are $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ and $\{1, 2, 3\}$. Of these only $\{1, 2, 3\}$ is not a proper subset, since it is equal to K. The other seven subsets are proper subsets. So K has 7 proper subsets, one less than the number of subsets altogether.

5. The number of proper subsets for N will again be one fewer than the total number of subsets. Since N has 6 elements, it has $2^6 = 64$ subsets and $2^6 - 1 = 63$ proper subsets.

6. Using the same shortcut we used for T we see that V has $10 - 1 = 9$ elements. It will therefore have $2^9 - 1 = 511$ proper subsets.

7. It is a good thing we have a formula: listing out the elements of D would be tedious and error-prone. $\varnothing$ has 0 elements. Since $A = 2^{\varnothing}$, A has $2^0 = 1$ element. Then $B = 2^A$

so B has $2^1 = 2$ elements, $C = 2^B$ so C has $2^2 = 4$ elements and finally $D = 2^C$ so D has $2^4 = 16$ elements. $n(D) = 16$.

8.2.5 Exercises: Subsets and Cardinality

Solutions appear at the end of this textbook.

1. True or false? $\{2, 4, 6, 8, 10\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

2. True or false? $\{1, 3, 11\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

3. True or false? $\{3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$?

4. True or false? $\{3, 6, 12, 15\} \subseteq \{\ 3x \mid x \in \mathbb{N}\}$?

5. True or false? $\{1, 7, 10\} \subseteq \{1, 7, 10\}$?

6. True or false? $\{1, 7, 10\} \subset \{1, 7, 10\}$?

7. True or false? $\{2, 12\} \subset \{2, 4, 6, 8, 10, 12, 14, \ldots\}$?

8. What is $n\left(\{1, 2, 3, 5, 8, 13, 21, 34\}\right)$?

9. What is $n\left(\{\ x \mid 3 \leq x < 25, x \in \mathbb{N}\}\right)$?

10. What is $n\left(\{\ x \mid 5 < x < 21, x \in \mathbb{N}\}\right)$?

11. What is $n\left(\{\ x \mid 100 \leq x \leq 1000, x \in \mathbb{N}\}\right)$?

12. Write $2^{\{5,10\}}$ in roster notation.

13. How many subsets does $\{2, 4, 6, 8, 10\}$ have?

14. How many subsets does $\{1, 3, 9, 27\}$ have?

15. How many proper subsets does $\{1, 2, 4, 8, 16, 32\}$ have?

16. How many proper subsets does $\{18, 21\}$ have?

8.3 Set Operations

Recall that an *operator* takes one or two numbers or objects and combines them into something new. In arithmetic the operators were $+, -, \times$ and $\div$. For example addition can take two numbers, say 2 and 3, and produce a new number: $2+3=5$. In propositional logic the operators were $\sim, \wedge, \vee, \rightarrow$ and $\leftrightarrow$, these work on statements instead of numbers. Now we will look at operators for sets. These will combine or manipulate old sets into new ones.

We will study four set operators in this section:

1. The union $\cup$ makes a new set that contains all the elements of both inputs.

2. The intersection $\cap$ makes a new set that contains only the shared elements of its inputs.

3. The complement $'$ makes the opposite of a set.

4. Set subtraction $\setminus$ removes one set from another.

These operators are closely linked to the conjunction, disjunction and negation from propositional logic.

8.3.1 Unions

Suppose A and B are sets. The **union** of A and B is a new set that contains every element of A and every element of B. Its notation is $A \cup B$.

For a formal definition we can say that $A \cup B = \{\ x \mid (x \in A) \vee (x \in B)\}$. When x is an element of A or an element of B or both, it will be an element of $A \cup B$.

Unions in roster form

Example: Write the union of A and B.

1. $A = \{1, 2, 4\}$ and $B = \{3, 6, 9\}$.

2. $A = \{5, 10, 20\}$ and $B = \{10, 20, 50\}$.

3. $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3\}$.

Solution:

1. $A \cup B$ is the set containing all elements of A and all elements of B. Adding all of these to one set we get $A \cup B = \{1, 2, 4, 3, 6, 9\}$. It's fine to leave the set like this since order inside roster notation doesn't matter, but for easy readability we might rewrite it $A \cup B = \{1, 2, 3, 4, 6, 9\}$.

   A and B are called *disjoint* sets because they have no elements in common. Since A contributes 3 elements and B contributes 3 different elements, $A \cup B$ has $3 + 3 = 6$ elements.

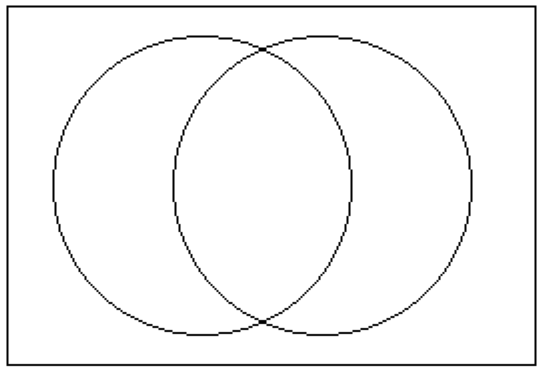
2. As above we can start by writing all the elements of A and then all the elements of B inside a single roster. $A \cup B = \{5, 10, 20, 10, 20, 50\}$. This is a correct answer since duplicate elements in roster form are ignored. (This is why that rule exists!) It's strongly preferred however to list each element only once, again for readability. So we will write $A \cup B = \{5, 10, 20, 50\}$.

   A and B are not disjoint: they share the elements 10 and 20. So although A has 3 elements and B has 3 elements, $A \cup B$ has only 4 elements instead of 6. That's because the 2 shared elements are only counted once each.

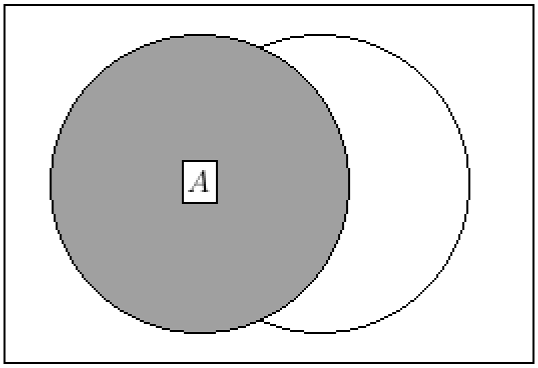
3. $A \cup B$ will contain every element of A, so it includes 1, 2, 3, 4 and 5. It will also include every element of B, so it includes 2 and 3. But 2 and 3 were already included because of A! If disjoint sets are one extreme, this is the other: B is a subset of A. The overlap with A covers all of B. So $A \cup B$ only has five elements: $A \cup B = \{1, 2, 3, 4, 5\}$.

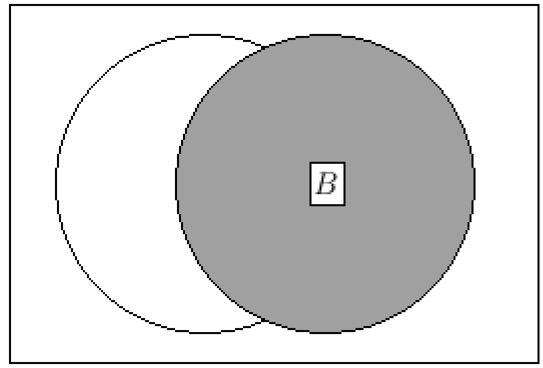
Unions with Venn diagrams

We can use a *Venn diagram* to represent a union. A Venn diagram is a tool for visualizing two sets that partially overlap. Each circle represents one of the sets. The leaf shape in the center is where the two sets overlap.

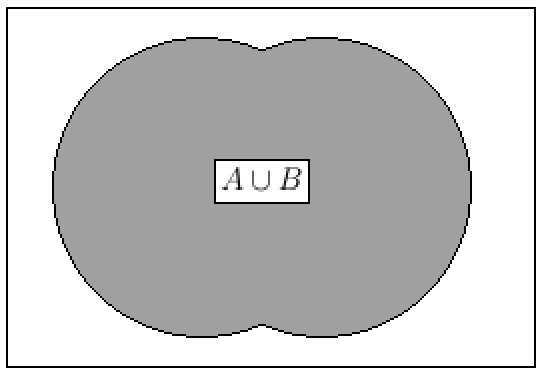


Since we're visualizing $A \cup B$ we will call one set A and the other set B.



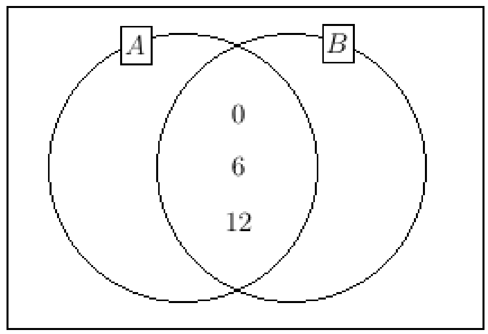


Then the union of the two sets looks like this:

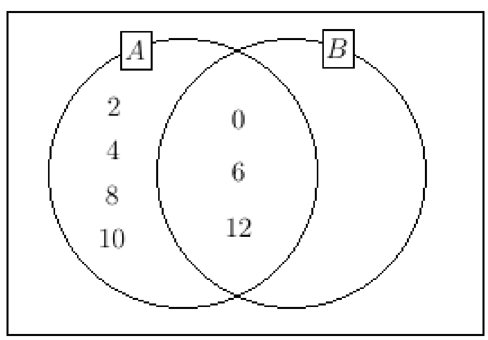


Example: Fill in a Venn diagram for the sets $A = \{0, 2, 4, 6, 8, 10, 12\}$ and $B = \{0, 3, 6, 9, 12\}$.

Solution: It's always a good idea to start in the center when filling out a Venn diagram: once the central elements are situated, placing the rest becomes much easier. The center of the diagram is where sets A and B overlap. It holds the elements that belong to both sets because it is inside both circles. The elements 0, 6 and 12 belong to both A and B.

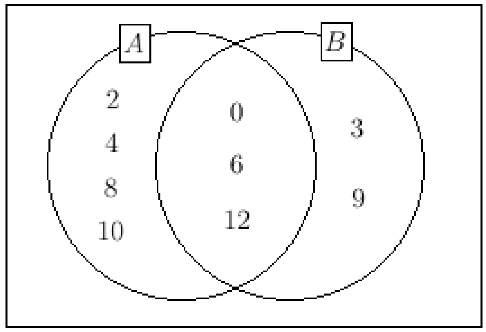


Once those elements common to both sets are placed in the center we can place the remainders of the sets. The other elements of A are $2, 4, 8$ and 10. They go in the part of A that does not overlap B.



The remaining elements of B are 3 and 9. They go in the part of B that does not overlap A.

From the Venn diagram we can read the elements $A \cup B$. They are all of the elements inside either circle. $A \cup B = \{2, 4, 8, 10, 0, 6, 12, 3, 9\}$.



Unions in set-builder form

The union operator has a special relationship to the disjunction (the "or″ operator) from propositional logic. The union of two sets consists of all the elements that are inside one set, or the other, or both. The disjunction is inclusive, meaning it says one statement, or the other, or both are true. The logic of "one or the other or both″ is common to both operators! The disjunction shows up explicitly when we take the union of sets in set-builder notation.

Example: Suppose that $A = \{\ x \mid x \leq 5, x \in \mathbb{N}\}$ and $B = \{\ x \mid x \geq 15, x \in \mathbb{N}\}$. What is $A \cup B$?

Solution: We can find the solution one way by converting to roster notation and using the same technique as before. $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{15, 16, 17, 18, 19, 20, \ldots\}$. When we put all the elements from A and B into one roster we get

$$A \cup B = \{0, 1, 2, 3, 4, 5, 15, 16, 17, 18, 19, 20, \ldots\}.$$

A more interesting solution is to use the formal definition of union:

$$A \cup B = \{\ x \mid\ (x \in A) \vee (x \in B)\}$$

According to the set-builder notation for A, saying that $x \in A$ is the same as saying $x \leq 5$. Saying that $x \in B$ is equivalent to saying $x \geq 15$. When we substitute those statements in for $x \in A$ and $x \in B$ in the formal definition, we get

$$A \cup B = \{\ x \mid\ (x \leq 5) \vee (x \geq 15)\}$$

$A \cup B$ is the set of elements that satisfy the condition for A, satisfy the condition for B, or do both! We can check this conclusion against our roster solution: $A \cup B$ is the set of natural numbers that are less than or equal to 5 or are greater than or equal to 15.

Example: For each pair of sets, find the union.

1. $J = \{\ x \mid 20 \leq x < 40, x \in \mathbb{N}\}$ and $K = \{\ x \mid 30 \leq x < 50, x \in \mathbb{N}\}$
2. $B = \{\ x \mid x < 10, x \in \mathbb{N}\}$ and $C = \{\ x \mid x = 2y, y \in \mathbb{N}\}$
3. $X = \{\ 2x \mid x \in \mathbb{N}\}$ and $Y = \{\ 2x + 1 \mid x \in \mathbb{N}\}$.

Solution:

1. J is the set of natural numbers between 20 and 39. K is the set of natural numbers between 30 and 49. So $J \cup K$ is the set of natural numbers that are between 20 and 39, or between 30 and 49, or both. All of the numbers in the twenties are in J, all of the forties are in K, and the thirties are in both sets: so all of the twenties, thirties and forties are in $J \cup K$.

$$J \cup K = \{\ x \mid 20 \leq x < 50 \in \mathbb{N}\}$$

2. B contains numbers less than 10. C contains multiples of 2. So $B \cup C$ contains numbers that are less than 10 or even or both. Inside the universe $\mathbb{N}$:

$$\begin{aligned} B &= \{\ x \mid (x < 10)\} \\ C &= \{\ x \mid (x = 2y)\} \\ B \cup C &= \{\ x \mid (x < 10) \vee (x = 2y)\} \\ B \cup C &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20, 22, \ldots\} \end{aligned}$$

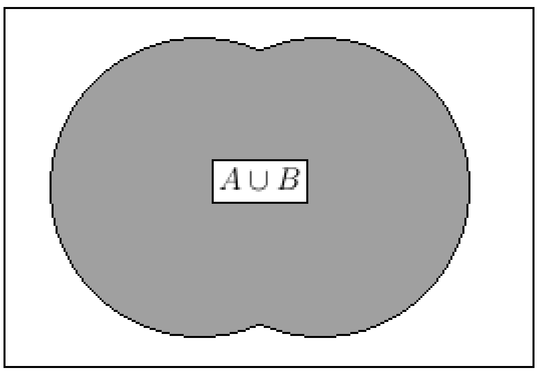
3. X is the set of even natural numbers. Y is the set of odd natural numbers. So $X \cup Y$ is the set of natural numbers that are either even or odd: that's every natural number! So $X \cup Y = \mathbb{N}$.

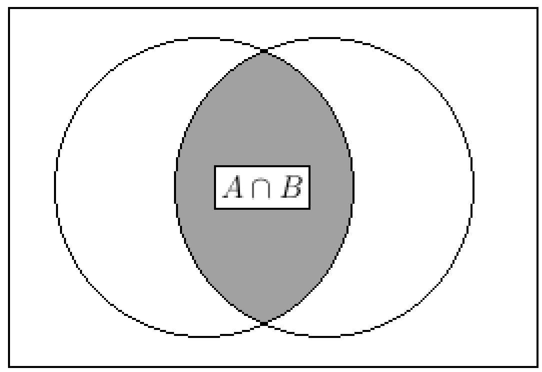
8.3.2 Intersections

The **intersection** of two sets contains only the elements they have in common. Its symbol is $\cap$: to take the intersection of sets A and B we write $A \cap B$. An element is in the intersection $A \cap B$ only when it is an element of both A and B. We can write this formally:

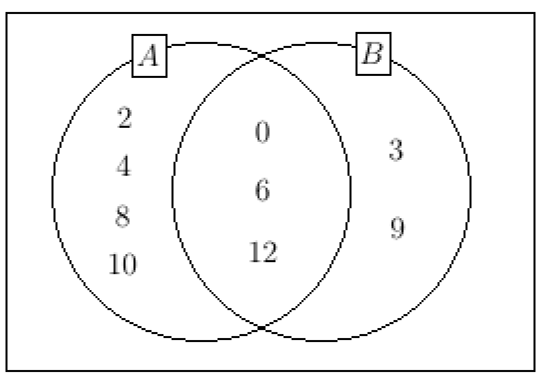
$$A \cap B = \{ \ x \mid \ (x \in A) \wedge (x \in B)\}$$

Whereas elements in the union may come from either set, elements in the intersection must be found in both sets.





So in the example where $A = \{0, 2, 4, 6, 8, 10, 12\}$ and $B = \{0, 3, 6, 9, 12\}$, the intersection $A \cap B$ is $\{0, 6, 12\}$ since those are the three elements that show up in both rosters.



Example: For each pair of sets, find the intersection.

1. $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

2. $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$.

3. $E = \{0, 1, 2, 3, 4, 5\}$ and $F = \{2, 3, 5\}$.

4. K is the set of blue animals and L is the set of animals with wings.

5. Inside the universal set $\mathbb{N}$, $H = \{\ x \mid x < 15\}$ and $J = \{\ x \mid x > 5\}$.

6. Inside the universal set $\mathbb{N}$, $P = \{\ x \mid x < 20\}$ and $Q = \{\ x \mid x < 10\}$.

7. Inside the universal set $\mathbb{N}$, $R = \{\ x \mid x > 30\}$ and $S = \{\ x \mid x < 20\}$.

8. $T = \{\ 3x \mid x \in \mathbb{N}\}$, $V = \{\ 5x \mid x \in \mathbb{N}\}$

Solution:

1. The intersection of A and B contains only the elements that are found in both sets. 1 and 2 are elements of A but not B, so they do not appear in $A \cap B$. 5 and 6 are in B, but not in A, so they do not appear in $A \cap B$ either. Because 3 and 4 are the elements common to both sets, $A \cap B = \{3, 4\}$

2. C and D are disjoint: they do not have any elements in common. Since there are not any elements that belong to both sets, the intersection has no elements: it is the empty set. We write $C \cap D = \varnothing$.

3. Notice that every element of F can also be found in E. We can write $F \subseteq E$, meaning F is a subset of E. When we take the intersection we are looking for elements common to both sets. Since every element of F is also in E, every element of F is in the intersection: $E \cap F = F$.

   To summarize: when $F \subseteq E$, $F \cap E = F$ and $F \cup E = E$.

4. K is the set of blue animals and L is the set of animals with wings. $K \cap L$ is the set of animals who belong to both sets, so it is the set of blue animals with wings.

5. Since H is the set of natural numbers less than 15 and J is the set of natural numbers greater than 5, their intersection $H \cap K$ is the set of natural numbers that are both less than 15 and greater than 5. $H \cap K = \{\, x \mid 5 < x < 15\}$. The same line of reasoning in set-builder notation is illuminating: by definition

$$H \cap J = \{\, x \mid \ (x \in H) \wedge (x \in J)\}.$$

Now $x \in H$ is equivalent to $x < 15$ and $x \in J$ is equivalent to $x > 5$. If we substitute these statements into the definition we get

$$H = \{\, x \mid x < 15\}$$
$$J = \{\, x \mid x > 5\}$$
$$H \cap J = \{\, x \mid \ (x < 15) \wedge (x > 5)\}$$

So the intersection of the sets is the same as the conjunction of their definitions!

6. P is the set of natural numbers less than 20. Q is the set of natural numbers less than 10. Elements of $P \cap Q$ must be both less than 20 and less than 10: only numbers less than 10 fit that description. $P \cap Q = \{\, x \mid x < 10\}$. Notice that this is identical to the set Q because $Q \subseteq P$ we have that $P \cap Q = Q$.

7. R is the set of numbers greater than 30. S is the set of numbers less than 20. Elements of $R \cap S$ must be both greater than 30 and less than 20. Since no numbers fit that description, the intersection is empty. In other words, since R and S are disjoint, $R \cap S = \varnothing$.

8. T is the set of multiples of 3. V is the set of multiples of 5. Therefore each element of $T \cap V$ must be both a multiple of 3 and a multiple of 5. If a number is divisible by both 3 and 5 then it must be divisible by $3 \cdot 5 = 15$ So $T \cap V$ is the set of multiples of 15. $T \cap V = \{\, 15x \mid x \in \mathbb{N}\}$!

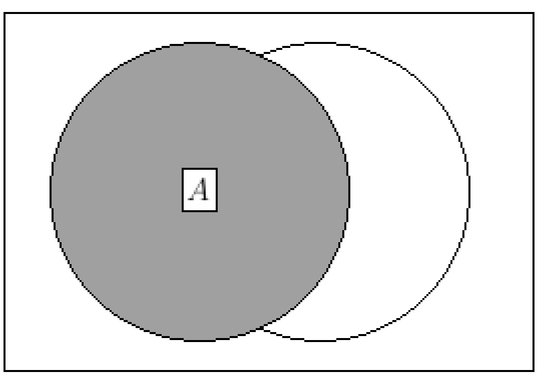
We can verify this conclusion by listing the sets in roster form:

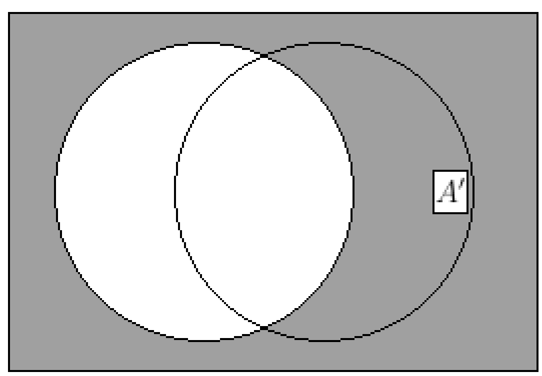
$$
\begin{aligned}
T &= \{\mathbf{0}, 3, 6, 9, 12, \mathbf{15}, 18, 21, 24, 27, \mathbf{30}, 33, 36, 39, 42, \mathbf{45}, 48, 51, 54, 57, \mathbf{60}, \ldots\} \\
V &= \{\mathbf{0}, 5, 10, \mathbf{15}, 20, 25, \mathbf{30}, 35, 40, \mathbf{45}, 50, 55, \mathbf{60}, \ldots\} \\
T \cap V &= \{\mathbf{0}, \mathbf{15}, \mathbf{30}, \mathbf{45}, \mathbf{60}, \ldots\}
\end{aligned}
$$

8.3.3 Complements

The **complement** of a set A is the set of all elements from the universal set that are not in A. Its symbol is A'. We read this notation "A complement". Formally we can say that $A' = \{\ x \mid \sim(x \in A)\}$.

Note that these definitions only make sense in the context of a universal set. For example if the setting of the problem is the universal set $\mathbb{N}$, then A' is the set of all the natural numbers that are not in A. Without a universal set it doesn't make sense to say "all elements not in A". Where would those elements come from? We can put almost anything into a set so there is no way to list all the possibilities.





In this diagram the rectangle represents the universal set.

Example: Working inside the universal set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, write the complement of each of the following.

1. $A = \{0, 1, 2, 3\}$
2. $B = \{0, 1, 2, 3, 5, 7, 8, 9\}$
3. $C = \{0, 2, 4, 6, 8\}$
4. $D = \{1, 3, 5, 7, 9\}$
5. S
6. $\varnothing$

Solution:

1. A' is the set of all elements of the universe S that are not in A. It will not include 0, 1, 2 or 3, since those are in A, but it will include every other element of S. So $A' = \{4, 5, 6, 7, 8, 9\}$.

2. Since B contains almost all of S, B' will only have a couple of elements. The only elements of S that are not in B are 4 and 6. So $B' = \{4, 6\}$.

3. C contains all of the even elements of S, so C' will contain all the elements of S that are not even. The elements of the universe S that are not found in C are 1, 3, 5, 7 and 9, so $C' = \{1, 3, 5, 7, 9\}$.

4. D is the complement of C from the previous part. Since D contains all the elements that are not in C, that means C contains all the elements that are not in D! $D' = C = \{0, 2, 4, 6, 8\}$.

   We can say that D and C complement each other. They are disjoint sets whose union is the entire universal set. Complementary sets always come in these pairs: a set is always its complement's complement.

For any set A, $(A')' = A$: the complement of the complement is the original set. The complement operator inherits this behavior from the negation operator: we define $A' = \{\ x \mid \sim(x \in A)\}$, so

$$(A')' = \{\ x \mid \sim(x \in A)\}'$$
$$(A')' = \{\ x \mid \sim\sim(x \in A)\}$$
$$(A')' = \{\ x \mid\ (x \in A)\} = A$$

To get from the second to the third line we used the fact that $\sim\sim p \equiv p$. Operators that cancel themselves out like the negation and the complement are called *idempotent*.

5. S' is the set of all elements that are inside the universe but not in S. However, S contains every element in the universe, so there are no other elements left! Hence its complement is the empty set: $S' = \varnothing$

6. We know by idempotency that since $\varnothing = S'$, $\varnothing' = S$. We can check this conclusion with the definition of the complement. $\varnothing'$ is the set of elements that are in the universe S but not in $\varnothing$. Since there are no elements in $\varnothing$, its complement contains every element in the universe.

Example: Working inside the universal set $\mathbb{N}$, write the complement of each of the following.

1. $T = \{1, 2, 3, 4, 5\}$

2. $X = \{\ x \mid x \geq 10\}$

3. $V = \{\ x \mid x < 20\}$

4. $W = \{\ x \mid x \leq 50\}$

5. $Y = \{\ 3x \mid x \in \mathbb{N}\}$

Solution:

1. T' is the set of natural numbers that do not belong to T. This includes 0 and every number above 5. So $T' = \{0, 6, 7, 8, 9, 10, 11, \ldots\}$.

2. X' is the set of elements that are in $\mathbb{N}$ but not in X. Since X contains every number greater than or equal to 10, X' contains every number less than 10. We write $X' = \{\ x \mid x < 10\}$.

3. Using the formal definition we can say that $V' = \{\ x \mid \sim(x < 20)\}$. The complement of V is the set of natural numbers that are not less than 20. Therefore they must be greater than or equal to 20. $V' = \{\ x \mid x \geq 20\}$.

4. Since W is the set of numbers less than or equal to 50, W' is the set of numbers greater than 50. We found the complement by negating the definition of the set. $W' = \{\ x \mid x > 50\}$.

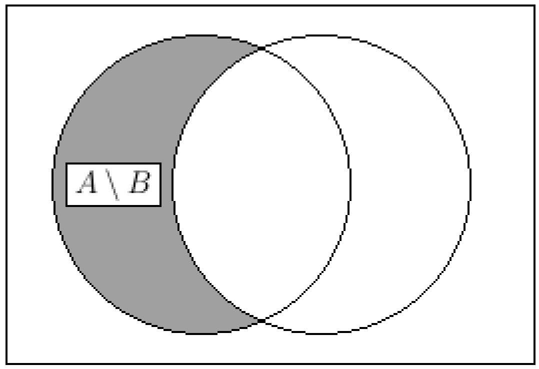
5. Since Y is the set of natural numbers that are divisible by 3, Y' is the set of natural numbers that are not divisible by 3.

$$Y' = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, \ldots\}$$

8.3.4 Set subtraction

The complement operator allows us to exclude a set, to consider all the elements in the universe that do not belong to the set. In essence we are subtracting the set from the universe.

In contexts where no universal set is specified, we cannot use the complement because it is undefined what we are subtracting from. In those situations we will instead use **set subtraction**. When we subtract B from A we are left with all the elements of A that cannot be found in B. In other words we remove all the elements of B from A, and the result is whatever elements are left. We write $A \setminus B$ and read this as A minus B. For a formal definition we write $A \setminus B = \{\ x \mid x \in A, x \notin B\}$.



When a universal set is defined, subtracting B from the universal set gives B'.

Example: Perform set subtraction on each pair of sets.

1. $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find $A \setminus B$.

2. $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find $B \setminus A$.

3. $A = \{1, 2, 3, 4, 5\}$ and $C = \{10, 20, 30, 40, 50\}$. Find $A \setminus C$.

4. $A = \{1, 2, 3, 4, 5\}$ and $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find $A \setminus D$.

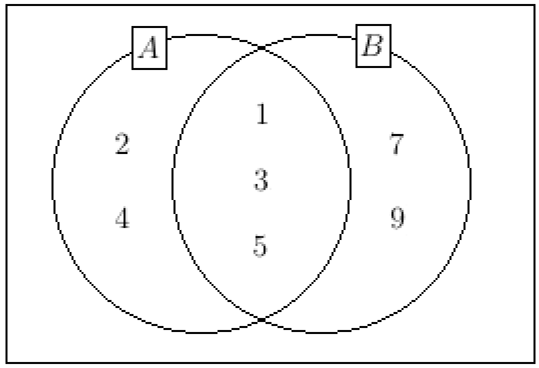
5. Find $\mathbb{Z} \setminus \mathbb{N}$.

6. $T = \{\ x \mid x < 50, x \in \mathbb{N}\}$ and $V = \{\ x \mid 25 \leq x < 75, x \in \mathbb{N}\}$. Find $T \setminus V$.

7. $T = \{\ x \mid x < 50, x \in \mathbb{N}\}$ and $V = \{\ x \mid 25 \leq x < 75, x \in \mathbb{N}\}$. Find $V \setminus T$.

Solution:

1. $A \setminus B$ is the subset of A that doesn't contain any elements from B. Of the elements of A, 1, 3 and 5 are also found in B but 2 and 4 are not. So $A \setminus B = \{2, 4\}$.



2. $B \setminus A$ contains all the elements of B that are not also elements of A. Of the elements of B, 1, 3 and 5 are also found in A but 7 and 9 are not. So $B \setminus A = \{7, 9\}$.

   Be careful! $A \setminus B$ and $B \setminus A$ are two different sets. Just like subtraction in arithmetic, order matters for set subtraction.

3. $A \setminus C$ is what remains of A after every element from C has been removed. In this case A and C are disjoint sets: A does not contain any elements from C. There is nothing to remove, so $A \setminus C = A$.

4. A is a subset of D. That is, every element of A can also be found in D. So when we remove all of these elements from A, we are left with nothing. $A \setminus D = \varnothing$.

5. $\mathbb{Z}$ is the set of all whole numbers. $\mathbb{N}$ contains the positive whole numbers and 0. When we remove those from $\mathbb{Z}$ we are left with only negative numbers. $\mathbb{Z} \setminus \mathbb{N} = \{-1, -2, -3, -4, -5, -6, \ldots\}$.

6. T contains all the natural numbers less that 50. V contains numbers from 25 to 75. When we take V away, what's left in T is the numbers less than 25. $T \setminus V = \{\ x \mid x < 25, x \in \mathbb{N}\}$.

7. V contains numbers from 25 to 75. When we take away all the numbers less than 50, we're left with only the numbers from 50 to 75. $V \setminus T = \{\ x \mid 50 \leq x < 75, x \in \mathbb{N}\}$.

8.3.5 Combining set operations

When we need to perform multiple set operations in one problem, we'll use the following order of operations:

1. Expressions in parentheses.
2. Complements.
3. All other operators.

The order of operations does not distinguish between unions, intersections and set subtraction so we have to use parentheses to indicate which goes first in an expression. Expressions such as $A \cup B \cap C$ that do not indicate an order are invalid because they produce two different results depending on the order of evaluation.

Example: Working inside the universal set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and using

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{4, 5, 6, 7, 8\}$$
$$C = \{2, 4, 6, 8, 10\}$$

Evaluate each of the following expressions.

1. $A \cap B'$
2. $(A \cup B) \cap C'$
3. $A' \cap B'$
4. $(A \cup B)'$
5. $A' \cap B' \cap C'$

6. $A \cap B' \cap C$

Solution:

1. According to the order of operations we should perform the complement before the intersection. B' is the set of elements in S but not in B: so $B' = \{1, 2, 3, 9, 10\}$. Now to find $A \cap B'$ we look for elements that are in both A and B'. Those elements are 1, 2 and 3 so $A \cap B' = \{1, 2, 3\}$.

   Another way to solve this problem is to use the connection of set theory to propositional logic. We can combine the definitions of the complement and intersection:

$$A \cap B = \{\ x \mid\ (x \in A) \wedge (x \in B)\}$$
$$B' = \{\ x \mid \sim (x \in B)\}$$
$$A \cap B' = \{\ x \mid\ (x \in A) \wedge \sim (x \in B)\}$$

   We see that $\cap$ functions like $\wedge$ and the complement functions like a negation. So $A \cap B'$ is the set of elements that are in A and not in B. Of the elements of A, 4 and 5 are in B but 1, 2 and 3 are not. $A \cap B' = \{1, 2, 3\}$.

2. As in the previous problem we can translate this expression to a logical rule by treating $\cap$ as $\wedge$, $\cup$ as $\vee$, and the complement as a negation.

$$(A \cup B) \cap C' = \{\ x \mid\ [(x \in A) \vee (x \in B)] \wedge \sim (x \in C)\}$$

   We are looking for elements that are in either A or B but not in C. Since C contains all the even elements in the universe we are looking for odd elements in A or B: we find 1, 3 and 5 in A and 5 and 7 in B. So $(A \cup B) \cap C' = \{1, 3, 5, 7\}$.

We can check our work by working through the order of set operations in roster form. First we perform operations in parentheses:

$$A \cup B = \{1,2,3,4,5\} \cup \{4,5,6,7,8\} = \{1,2,3,4,5,4,5,6,7,8\} = \{1,2,3,4,5,6,7,8\}.$$

Then we take the negation of C: since $C = \{2,4,6,8,10\}$, $C' = \{1,3,5,7,9\}$.

Finally we perform the intersection: $\{1,2,3,4,5,6,7,8\} \cap \{1,3,5,7,9\}$ contains only the elements common to both sets: 1, 3, 5 and 7. So $(A \cup B) \cap C' = \{1,3,5,7\}$.

3. We can interpret $A' \cap B'$ as $\{\ x \mid \sim(x \in A) \wedge \sim(x \in B)\}$: it is the set of elements not in A and not in B. Only 9 and 10 are not found in either set, so $A' \cap B' = \{9,10\}$.

   Alternatively we can work through the set order of operations:

$$\begin{aligned} A' &= \{6,7,8,\mathbf{9},\mathbf{10}\} \\ B' &= \{1,2,3,\mathbf{9},\mathbf{10}\} \\ A' \cap B' &= \{9,10\} \end{aligned}$$

4. By set operations: $A \cup B = \{1,2,3,4,5,6,7,8\}$. $(A \cup B)' = \{1,2,3,4,5,6,7,8\}' = \{9,10\}$. Notice that this answer is the same as in the previous problem. It's not a coincidence! Translating to propositional logic we have

$$\begin{aligned} A' \cap B' &= \{\ x \mid \sim(x \in A) \wedge \sim(x \in B)\} \\ (A \cup B)' &= \{\ x \mid \sim((x \in A) \vee (x \in B))\} \end{aligned}$$

   De Morgan's law states that

$$\sim p \wedge \sim q \equiv \sim(p \vee q)$$

So if $p = (x \in A)$ and $q = (x \in B)$ we get

$$\sim (x \in A) \wedge \sim (x \in B) \equiv \sim ((x \in A) \vee (x \in B))$$

which tells us that $A' \cap B' = (A \cup B)'$. So De Morgan's laws apply in set theory as well as propositional logic! De Morgan's laws are:

$$\sim p \wedge \sim q \equiv \sim (p \vee q) \qquad A' \cap B' = (A \cup B)'$$
$$\sim p \vee \sim q \equiv \sim (p \wedge q) \qquad A' \cup B' = (A \cap B)'$$

5. $A' \cap B' \cap C'$ is the set of elements that are not in A and not in B and not in C. The only element that is absent from all three sets is 9, so $A' \cap B' \cap C' = \{9\}$.

6. $A \cap B' \cap C$ is the set of elements that are in A, not in B and in C. The only element that is in both A and C but not in B is 2, so $A \cap B' \cap C = \{2\}$.

8.3.6 Exercises: Set Operations

Solutions appear at the end of this textbook.

1. Write the union: $\{1, 2, 4, 8, 16\} \cup \{2, 4, 6, 8, 10, 12, 14, 16\}$.
2. Write the union: $\{1, 3, 7, 8, 9\} \cup \{2, 4, 7, 9, 10\}$.
3. Write the union: $\{\ x \mid 5 \leq x \leq 15\} \cup \{\ x \mid 10 \leq x \leq 20\}$.
4. Write the intersection: $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \cap \{3, 6, 9, 12, 15, 18, 21\}$.
5. Write the intersection: $\{1, 3, 5, 7, 9, 11, 13, 15\} \cap \{2, 6, 10, 14, 18, 22\}$.
6. Write the intersection: $\{\ x \mid 5 \leq x \leq 15\} \cap \{\ x \mid 10 \leq x \leq 20\}$.
7. Write the complement of $\{1, 3, 4, 6, 7\}$ in the universe $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
8. Write the complement of $\{0, 2, 4, 6, 8\}$ in the universe $\mathbb{N}$.
9. Write the complement of $\{\ x \mid x > 400\}$ in the universe $\mathbb{N}$.
10. Write the difference: $\{2, 3, 5, 7, 11, 13\} \setminus \{0, 2, 4, 6, 8, 10, 12, 14\}$.
11. Write the difference: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \setminus \{2, 3, 5, 7, 11, 13, 17, 23\}$.
12. Write the difference: $\{\ 2x + 1 \mid x \in \mathbb{N}\} \setminus \{\ 4x + 1 \mid x \in \mathbb{N}\}$.
13. Inside the universe $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ suppose that

    L = {2, 3, 5, 7} M = {1, 2, 6, 7, 8} N = {4, 6, 7, 8, 9, 10}

    (a) Find $L' \cap (M \cup N)$

    (b) Find $(L \cup M) \cap (L \cup N)$

    (c) Find $(L \cap M) \cup N$

8.4 Survey Problems

8.4.1 The union cardinality formula

Recall that the *cardinality* of a set is a measure of its size. For finite sets, the cardinality is the number of elements in the set. So for example if $A = \{1, 2, 3, 4, 5, 6\}$ then we write $n(A) = 6$ because A has 6 elements. Sets come in all sizes, from the empty set with $n(\varnothing) = 0$ up to sets like $\mathbb{N}$ and $\mathbb{R}$ with infinitely many elements. In this section we will work with finite sets to see how unions and intersections change their cardinality.

If $A \subseteq B$, then every element of A must also be found in B, so A cannot have more elements than B.

$$A \subseteq B \to n(A) \leq n(B).$$

If A is a subset of B then the cardinality of A is less than or equal to the cardinality of B.

An intersection contains only the elements that were included in both sets. So if $x \in A \cap B$ then $x \in A$. That means that $A \cap B$ is a subset of A. We can conclude that

$$n(A \cap B) \leq n(A).$$

Taking an intersection reduces the number of elements or leaves it the same.

Unions work the other way: they increase the number of elements. Every element of A is included in $A \cup B$, so $A \subseteq A \cup B$ and therefore

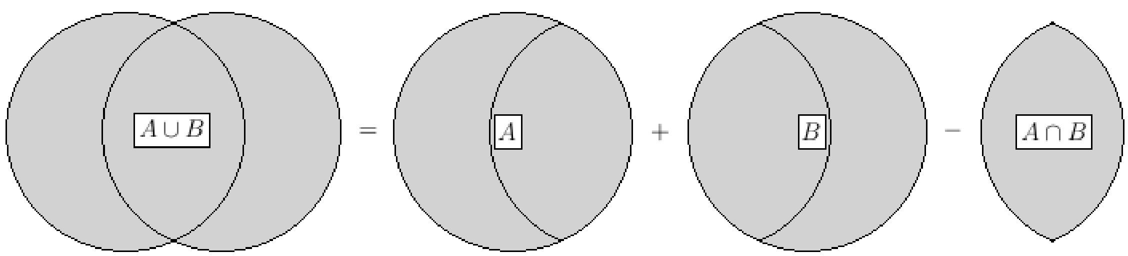
$$n(A) \leq n(A \cup B).$$

By how much does taking a union increase cardinality? The simplest case is when A and B are disjoint. For example if $A = \{1, 2, 3\}$ and $B = \{7, 8, 9, 10\}$ then $A \cup B =$

$\{1, 2, 3, 7, 8, 9, 10\}$. Because A and B do not share any elements the union has as many as both of them combined: $n(A) + n(B) = 3 + 4 = 7$.

When there is an overlap between A and B the union will be smaller. Say that $A = \{4, 5, 6, 7, 8\}$ and $B = \{7, 8, 9, 10\}$. Then $A \cup B = \{4, 5, 6, 7, 8, 9, 10\}$. It only has 7 elements even though $n(A) + n(B) = 9$ because there's an overlap of 2 elements. The elements 7 and 8 are included twice in A and B, but we only count them once in $n(A \cup B)$. If we want the cardinality of $A \cup B$, we can add the sizes of A and B, but we have to subtract the size of the overlap.

The overlap is the set of elements that are in both A and B. We have a name for this set: it is $A \cap B$! So if we want the size of the union, we can add the sizes of the two sets and then subtract the size of their intersection.



$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Example: Use the union cardinality formula to solve each problem.

1. Suppose that $n(A) = 15$, $n(B) = 12$, and $n(A \cap B) = 4$. How many elements are in $A \cup B$?

2. Twenty-four of the residents of Smalltown own cats. Thirty-seven residents own dogs. Eleven residents own both kinds of pet. How many residents own at least one cat or dog?

3. Suppose that $n(A \cup B) = 44$, $n(B) = 29$, and $n(A \cap B) = 12$. How many elements are in A?

4. The Smalltown Driver's Club has 60 members, and all of them can drive. 54 of the members can drive with an automatic transmission. 38 members can drive both automatic and stick shift. How many members can drive stick?

5. Eight of your classmates speak French and thirteen speak Spanish. In total seventeen people in the class speak at last one of these languages. How many speak both?

6. This January it snowed 11 days. 9 days were rainy and 3 had both rain and snow.

   (a) How many days had no rain or snow?

   (b) How many days had snow but no rain?

Solution:

1. We are given values for all three terms on the right side of the union cardinality formula. All we need to do is plug them in and simplify.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A \cup B) = 15 + 12 - 4$$
$$n(A \cup B) = 23.$$

   There are 23 elements in the union of A and B.

2. We can interpret this story as a union cardinality problem just like the previous problem. Let's say C is the set of cat-owners. We know $n(C) = 24$. The set of dog owners can be D, with $n(D) = 37$. The eleven residents who own both kinds of pet are the

intersection of the cat-owners C and the dog-owners D, so $n(C \cap D) = 11$. We're ready to plug in our numbers:

$$\begin{aligned} n(C \cup D) &= n(C) + n(D) - n(C \cap D) \\ n(C \cup D) &= 24 + 37 - 11 \\ n(C \cup D) &= 50. \end{aligned}$$

$C \cup D$ is the set of Smalltown residents who own a cat or a dog or both. So there are 50 such people.

3. The union cardinality formula has four unknown quantities. Since we have been given values for three of them, we can plug those in and solve for the fourth.

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 44 &= n(A) + 29 - 12 \\ 44 &= n(A) + 17 \\ 44 - 17 &= n(A) \\ 27 &= n(A). \end{aligned}$$

So A has 27 elements. Whatever equation or formula you've got, as long as you can plug in all but one of the quantities, you can always solve for the last one.

4. We'll solve this story problem using the method from the previous part. Let's say A is the set of automatic drivers and S is the set of stick drivers, so we know $n(A) = 54$ and we want to solve for $n(S)$. Since all 60 members of the club drive we can say $n(A \cup S) = 60$ since they must drive at least one type of vehicle. We're told 38 drive

both, so $n(A \cap S) = 38$. We've got values for 3 of the four quantities in the equation, so it's time to plug them in.

$$\begin{aligned} n(A \cup S) &= n(A) + n(S) - n(A \cap S) \\ 60 &= 54 + n(S) - 38 \\ 60 &= 16 + n(S) \\ 60 - 16 &= n(S) \\ 44 &= n(S). \end{aligned}$$

Forty-four of the club members know how to drive stick.

5. Lets say S is the set of Spanish speakers and F is the set of french speakers. We are given that $n(F) = 8$ and $n(S) = 13$. The seventeen people who speak at least one of the languages are the elements of $F \cup S$ since they speak French or Spanish or both. So $n(F \cup S) = 17$. With these three quantities known we can solve for the fourth.

$$\begin{aligned} n(F \cup S) &= n(F) + n(S) - n(F \cap S) \\ 17 &= 8 + 13 - n(F \cap S) \\ 17 - 8 - 13 &= -n(F \cap S) \\ -4 &= -n(F \cap S) \\ 4 &= n(F \cap S). \end{aligned}$$

You have four classmates who speak both Spanish and French.

6. (a) We can call the set of snowy days S and the set of rainy days R. The days that are both rainy and snowy are the elements of $S \cap R$. So we know that $n(S) = 11$, $n(R) = 9$ and $n(S \cap R) = 3$. Plugging these in we find:

$$n(R \cup S) = n(R) + n(S) - n(R \cap S)$$
$$n(R \cup S) = 9 + 11 - 3$$
$$n(R \cup S) = 17.$$

So there were 17 days in January with snow or rain or both. But this isn't our final answer! The question asks how many days had neither, and for that we need one more piece of information. January has 31 days. The set of days in January is our universal set for this problem. If 17 of those days had precipitation then the other 14 did not. Fourteen days is our final answer. These days are the set $(R \cup S)'$.

(b) We want to know the size of $S \setminus R$, the set of snowy days with no rain. $S \cap R$ and $S \setminus R$ are disjoint sets: either a snowy day has rain or it doesn't. Together $S \setminus R$ and $S \cap R$ make up all of S: every snowy day is either rainy or not rainy.

$$n(S \setminus R) + n(S \cap R) = n(S)$$
$$n(S \setminus R) + 3 = 11$$
$$n(S \setminus R) = 11 - 3 = 8.$$

Since three of the snowy days had rain, the other eight did not.

8.4.2 The Venn Diagram method

In the previous scenario about snow days, we needed to use more algebra every time we had a new question to answer. A different way to approach these problems is to fill out a Venn diagram with all possible information about the scenario first, and then use it to answer any and all questions.

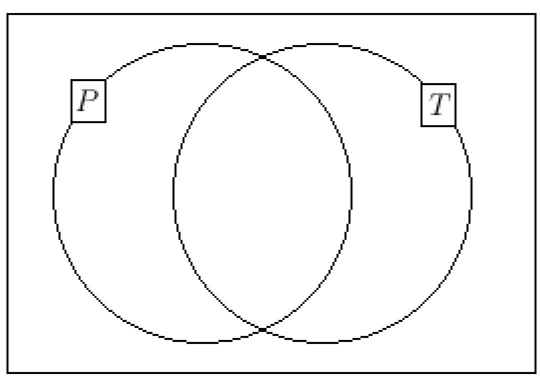
Example: One hundred students were asked about their food preferences for an upcoming party. Eighty-one said they would be happy with pizza, sixty-eight wanted tacos, and sixty students said they would be happy with either food.

1. How many students want neither food?

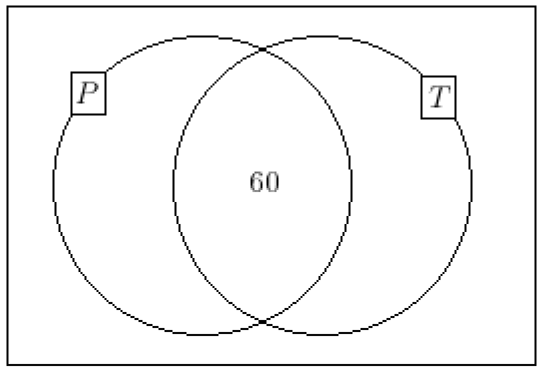
2. How many students want pizza but not tacos?

3. How many students want tacos but not pizza?

Solution: We could answer each of these questions separately with the union cardinality formula, but a Venn diagram will allow us to answer all of them at once. Let T be the set of students who want tacos and P be the set of students who want pizza. Then $n(P) = 81$, $n(T) = 68$ and $n(P \cap T) = 60$.

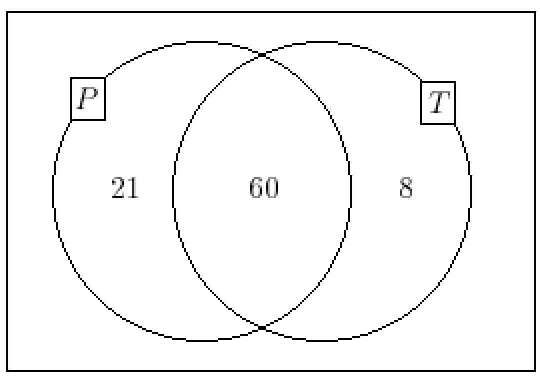


We're not going to place the numbers 81 and 68 directly on this diagram. Instead we want to put a number in each *region*. The boundaries of this diagram close off four regions: the first is the overlap between the two circles, the second and third are the remaining crescent portions of the circles, and the fourth is the space outside either circle. We will fill them in that order, from the inside to the outside. The only region whose population we already know is the center. Since $n(P \cap T) = 60$ we will put a 60 in the middle of the diagram.

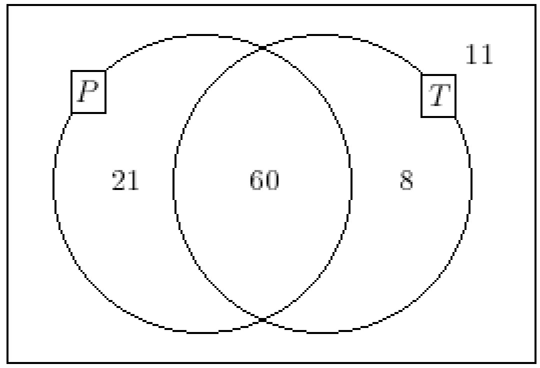


Now we can fill the crescent shapes. 81 students want pizza, but we do not put an 81 in the crescent. 81 is the number of students who fit in the whole circle P. That circle is made up of two regions: the center and a crescent. Since 60 students are already placed in the center, if we put another 21 in the crescent, there will be 81 total in the circle P.

Following the same logic, since $n(T) = 68$, there are $68 - 60 = 8$ taco-loving students left to be placed: they go in the right crescent. Now the T circle adds up to 68 students total.



Finally we need to fill the outside region. This space is for members of the universal set, the 100 students surveyed, who have not been placed in the first three regions. We have already placed $21 + 60 + 8 = 89$ students, so there are $100 - 89 = 11$ remaining. These eleven students go in the outside region. Now the four regions add to 100 students.



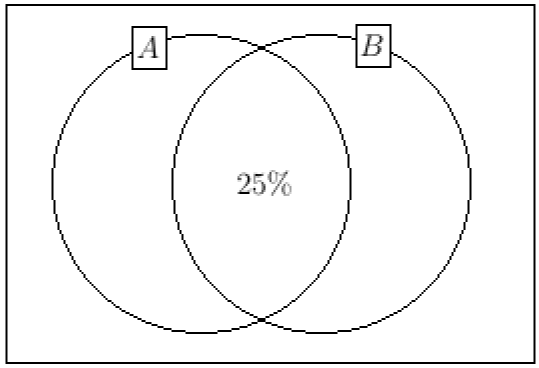
With the diagram filled we are ready to take questions.

1. How many students want neither food? The 11 students who are outside both circles.
2. How many students want pizza but not tacos? These are the 21 students in the left crescent.
3. How many students want tacos but not pizza? These are the 8 students in the right crescent.

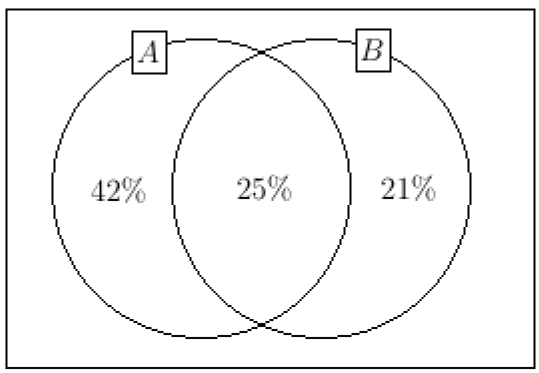
Example: Researchers conducted a survey of public opinion on upcoming ballot measures. 67% of those surveyed said they would vote yes on proposition A. 46% would vote yes on proposition B, and 25% would vote yes on both measures.

1. What percent of the public will vote no on A but yes on B?
2. What percent will vote yes on at least one measure?
3. What percent will vote yes on one of the measures but not both?

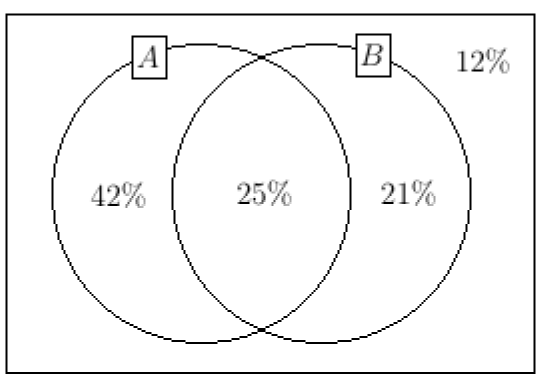
Solution: We set up a Venn diagram for support of A and B. When solving survey problems we always start in the center of the diagram and work our way out. We're told that 25% would vote yes on both measures, so we put that number in the center region.



67% of the respondents would vote yes on prop A. We need the A circle to add up to 67, and 25 is already placed in the center. So there's $67 - 25 = 42$ left to place: we put 42% in the left crescent. 46% of respondents would vote yes on prop B. Since $46 - 25 = 21$, we put 21% in the right crescent. That way the two regions in circle B will add up to 46%.



Finally we know that the whole diagram should add to 100%. We've already placed $42 + 25 + 21 = 88$ in the first three regions, so there's $100 - 88 = 12$ left. We put 12% in the outside region.



Now we can answer the questions given.

1. 21% of the public will vote no on A but yes on B. This is the right crescent.
2. What percent will vote yes on at least one measure? This is $n(A \cup B)$. $A \cup B$ covers all three regions in the circles, so we add $42 + 25 + 21 = 88$. 88% of the respondents will vote yes on at least one measure.
3. What percent will vote yes on one of the measures but not both? These are the individuals in the two crescents. $42\% + 21\% = 63\%$.

8.4.3 Surveys with three sets

With three sets instead of two these problems take more time, but the underlying strategy does not change. We will draw a Venn diagram with three overlapping circles and fill it out starting in the center and working our way to the outside.

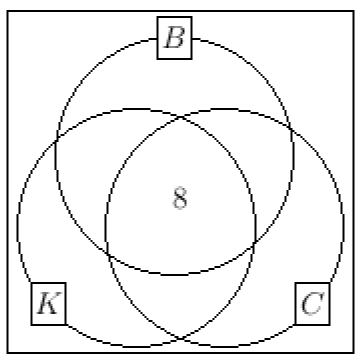
Example: A survey was taken of 100 fifth graders to determine their methods of transportation.

- 58 kids said they owned a bicycle.
- 44 owned a skateboard.
- 52 owned a scooter.
- 19 kids owned both a bike and a skateboard.
- 29 owned a bike and a scooter.
- 24 owned a scooter and a skateboard.
- 8 owned all three.

Use this information to answer the following questions.

1. How many students have a bicycle only?
2. How many students have a bike or a scooter but not a skateboard?
3. How many students have a bike and a scooter but not a skateboard?
4. How many students have none of the three vehicles?
5. How many students have at least two of the vehicles?

Solution: We'll use set B for students with bicycles, K for skateboards, and C for scooters. Like before, we'll start in the center of a Venn diagram and work our way out. The very center of this diagram is a region corresponding to $B \cap K \cap C$. We'll put an 8 in this region since there are 8 students who own all three vehicles and are therefore members of all three sets.

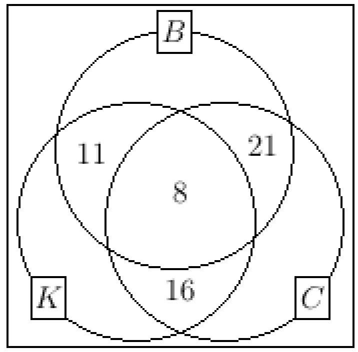


Next we can handle the three regions immediately adjacent to the center. Each of these regions is part of the intersection of two sets.

The small region up and to the left of the center is in $B \cap K$. It shares $B \cap K$ with the center region: together these two regions make up all of the overlap between B and K. We're told that 19 kids own both a bike and a skateboard, so $n(B \cap K) = 19$. Since 8 of these students are already situated in the center, we put the other 11 in this small region.

By the same reasoning, 8 of the 29 students with a bike and a scooter are already in the center, so the other 21 go in the small region above and to its right. Similarly the remaining

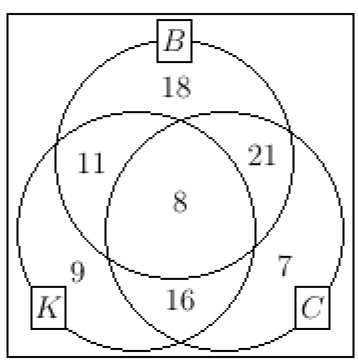
16 of the 24 students with both a skateboard and a scooter go in the small region below the center, so that the two regions that make up $K \cap C$ add to 24 as desired.



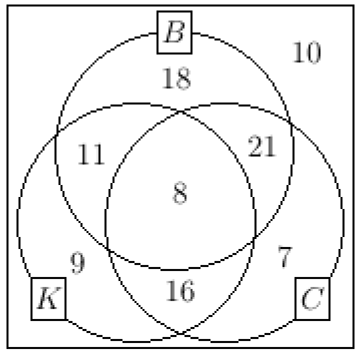
There are three more regions inside the circles in this diagram. These correspond to the kids with only one vehicle.

The unfilled region inside circle B represents kids with a bicycle but no scooter or skateboard. The whole circle B should contain the 58 kids who own bicycles. $11 + 8 + 21 = 40$ of them were already placed in earlier steps, so we should put the other $58 - 40 = 18$ in this region.

Similarly, circle C should hold all 52 of the kids with scooters. There are already $21 + 8 + 16 = 45$ kids in the circle, so we'll put the other $52 - 45 = 7$ in the unfilled region. Circle K should hold all 44 students with skateboards. It already contains $11 + 8 + 16 = 35$ kids, so the other $44 - 35 = 9$ go in its unfilled region.



Finally the entire diagram should hold all 100 students who were surveyed. In the seven regions we have already filled there are $18 + 11 + 8 + 21 + 9 + 16 + 7 = 90$ students, so the other $100 - 90 = 10$ will go in the outside region.



Now we are ready to answer all of the questions attached to this scenario.

1. How many students have a bicycle only? The region at the top of circle B is outside of circles K and C. It holds the 18 students with only a bicycle.

2. How many students have a bike or a scooter but not a skateboard? We're looking for regions that are in B or in C or both, but are not in K. There are three such regions, and they contain $18 + 21 + 7 = 46$ students.

3. How many students have a bike and a scooter but not a skateboard? Now we need a region that is in both B and C, but is not in K. The only such region holds 21 students.

4. How many students have none of the three vehicles? These are the 10 students in the outside region.

5. How many students have at least two of the vehicles? These students are in at least two overlapping circles. They can be found in the four regions in the middle of the diagram: $11 + 8 + 21 + 16 = 56$ students.

8.4.4 Exercises: Survey Problems

Solutions appear at the end of this textbook.

1. 21 of your friends have seen *The Princess Bride* and 16 have seen *A Knight's Tale*. If 10 of them have seen both movies, how many have seen at least one of them?

2. You keep track of which of your friends has an X-Box and who has a Playstation. 5 friends have both, 13 have one or the other. If 8 have Playstations, how many have X-Boxes?

3. You have 28 papers to write. 18 still need outlines written, 14 need their bibliographies written, and 7 need both. How many of your papers already have both their outline and bibliography written?

4. Thirty-eight students live on your floor. 23 are majoring in arts, 25 are majoring in humanities, and 17 are double majors in arts and humanities.

   (a) Fill out a Venn diagram for this scenario.

   (b) How many students on the floor are not arts or humanities majors?

   (c) How many students are majoring in humanities but not arts?

   (d) How many students are majoring in either humanities, arts or both?

5. 127 student athletes responded to a survey.

   - 68 play soccer.
   - 57 play volleyball.
   - 76 play basketball.
   - 36 play soccer and volleyball.
   - 43 play soccer and basketball.

- 40 play volleyball and basketball.
- 24 play all three sports.

(a) Fill out a Venn diagram for this scenario.

(b) How many students only play basketball?

(c) How many students play volleyball and basketball?

(d) How many students play soccer or volleyball but not basketball?

(e) How many students don't play soccer, volleyball or basketball?

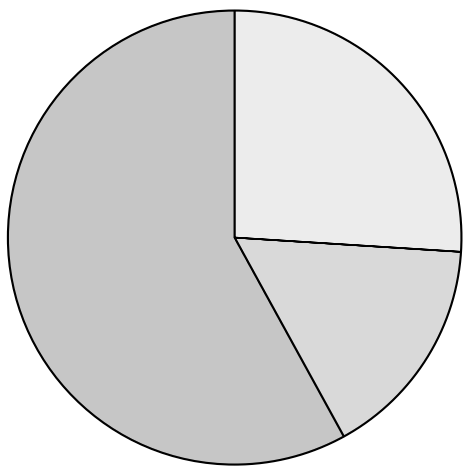
(f) How many students play basketball and also play soccer or volleyball?

Answers to Try This On Your Own Problems

Section 1.1: Describe the pattern found in the following sequence of numbers and then find in the next two values: 80, -40, 20, -10. **ANSWER: each value is divided by -2. The next two values are 5 and -2.5.**

Section 1.2: Round 136.295 to the nearest hundred, one, and tenth. **ANSWERS: 100, 136 and 136.3**

Section 1.3: Estimate the percent size of each pie slice. **ANSWERS: about 27%, 15% and 58%**



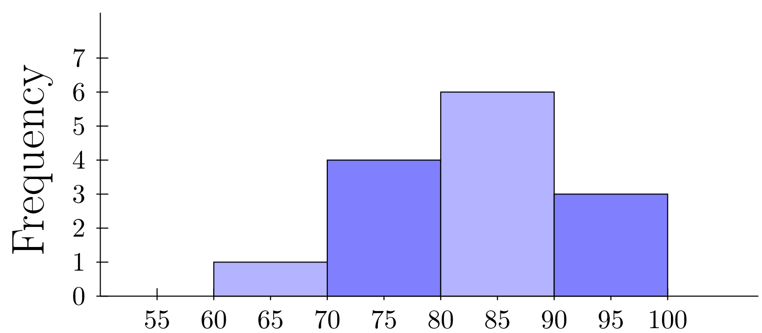
Section 1.4: Edera works as a salesperson and gets paid \$500 per month plus \$100 for each car she sells. If she makes \$1,600 in a month, how many cars did she sell? Setup a formula and solve. **ANSWER: $p = 500 + 100c$ and she sells $c = 11$ cars**

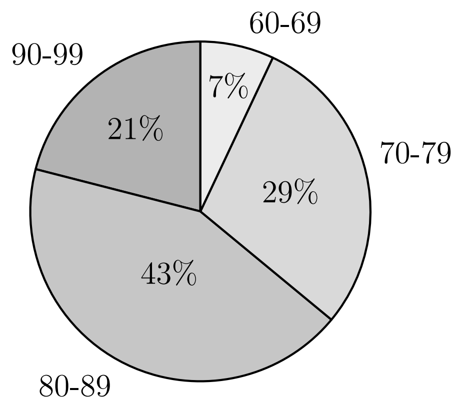
Section 2.1: For the following scenario, describe the population of interest, describe the sample, state the parameter of interest, and the statistic that was calculated. A farm wants to track the weight gain of their chickens after they switched to a new feed. The farm has over 10,000 chickens. They isolated 200 chickens and weighed them before the switch, then every week for the next 10 weeks. At the end of 10 weeks, the 200 isolated chickens gained an average of 1.2 pounds. **ANSWERS: Population is all 10,000 chickens, sample is the 200 isolated chickens, parameter is weight gain, the statistic calculated is 1.2 average weigh gain**

Would the following sample be representative of the population? A teacher would like to know how students feel about the new math curriculum. They selected a sample of students from the ones who are failing the class and come for extra help. **ANSWER: No, the sample would not be representative. The students who are failing are more likely to not like the class or find it difficult. Students who are doing well in the class should be included in the sample.**

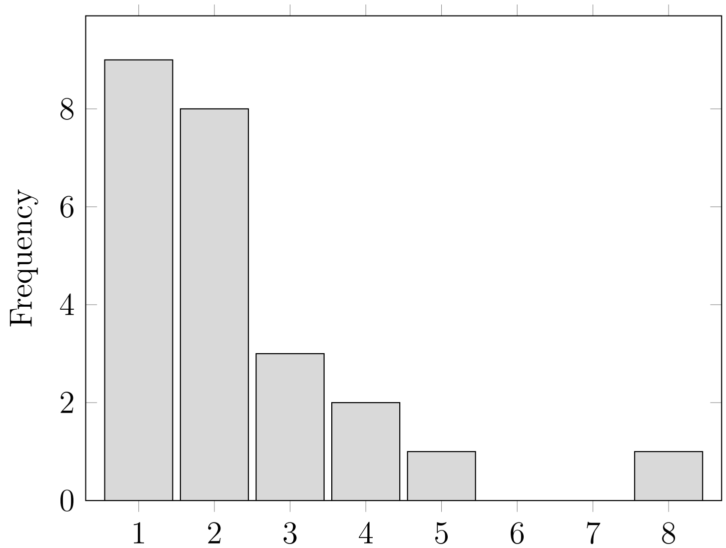
Section 2.2: The grades on a science final exam were 75, 83, 96, 82, 90, 78, 60, 76, 82, 71, 92, 86, 83, 88. Create a table with frequencies and relative frequencies using the intervals 60-69, 70-79, 80-89, 90-99. Then sketch a frequency histogram and a relative frequency pie chart. **ANSWERS:**

Grades	Frequency	Relative Frequency
60-69	1	7%
70-79	4	29%
80-89	6	43%
90-99	3	21%
Total	14	100%





What are the characteristics of the following graph? Examine the spread, symmetry, and outliers.



ANSWERS: The peak of the graph is at data values 1 and 2. The outlier is the one value at 8. It is not spread out very much, since most of the data is concentrated near left peak. The graph is right skewed.

Section 2.3: The grades from a sample of a science final exam were 75, 83, 96, 82, 90, 78, 60, 76, 82, 71, 92, 86, 83, 88. Calculate the mean, median, and mode. **ANSWERS: mean = 81.6, med = 82.5, mode = 82 and 83**

Section 2.4: A runner kept a record of their miles ran for the past 6 training sessions. The miles were 9, 3, 6, 8, 6, 10. Calculate the mean, range, and standard deviation of this sample. **ANSWERS: mean = 7, range = 7, std dev = 2.5**

Section 2.5: Calculate the z-score of a woman who is 5 feet tall if the mean height is 65 inches and standard deviation is 3 inches. Is she unusually short or not? Round Z to two decimal places. **ANSWER:** $z = \frac{60-65}{3} = -1.67$**, she is a bit short but within the usual values of -2 to 2.**

The birth weights of babies in South America are normally distributed, with a mean of $\mu = 3,100$ grams and a standard deviation of $\sigma = 400$ grams. Find the percentage of babies born with a weight more than 2,300 grams. **ANSWER:** $47.5 + 50 = 97.5\%$

Section 3.1: In each situation below, calculate the probability, deciding whether to use empirical or theoretical probability.

1. 200 people are at a banquet and 8 people are at your table including you. What is the probability that someone at your table is chosen at random to win a prize out of the entire banquet? **ANSWER: Theoretical** $P(win) = \frac{8}{200} = 0.04 = 4\%$

2. Danny has played 20 tennis matches this season and has won 17 of them. What is the probability that he wins his next match? **ANSWER: Empirical** $P(win) = \frac{17}{20} = 0.85 = 85\%$

If a team has a $\frac{1}{3}$ chance of making the playoffs, find the probability they do not make the playoffs. **ANSWER: Using complement rule** $P(notmakingplayoffs) = 1 - P(playoffs) = 1 - \frac{1}{3} = \frac{2}{3} = 0.67 = 67\%$

Section 3.2: If a room is being decorated by choosing one of five colors for the walls, one of three choices of carpet, and one of four furniture sets, how many ways are there to create a look for the room? **ANSWER:** $5 * 3 * 4 = 60$ **different looks**

For a lottery in which you pick five numbers from 1 to 50, how many different sets can you pick if they can be in any order, and if they must be in a specific order? **ANSWER: There are** ${}_{50}C_5 = 2,118,760$ **different sets in any order and** ${}_{50}P_5 = 254,251,200$ **different ordered sets.**

Section 3.3: A football team has 42 players. There are 18 players who play offense, 20 players who play defense, and 10 players who play on special teams. Six of the offensive players play both offense and special teams. Find the probability that a player is on the offense or special teams. **ANSWER: P(O or S) = P(O) + P(S) - P(O and S)** $= \frac{22}{42} = 0.52 = 52\%$

A particular game has the prize distribution shown below. Find the expected value of a prize. **ANSWER: $19**

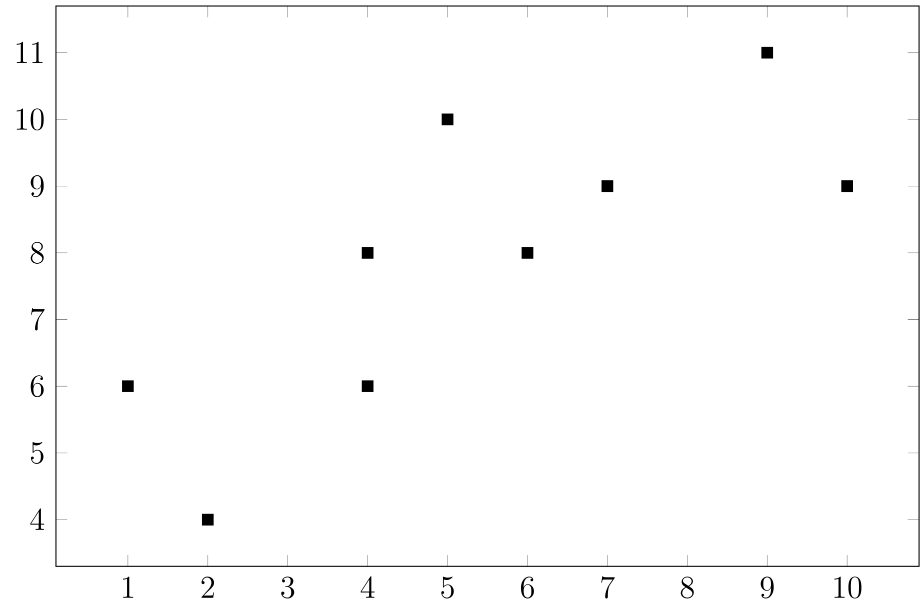
Prize Amount	$0	$25	$100	$500
Probability	0.7	0.2	0.09	0.01

Section 5.1: For the function $h(t) = 4\sqrt{5-t} - 1$, find $h(1), h(4)$, and $h(-11)$. **ANSWERS:** 15, 3**, the last one is undefined since there is a negative under the square root.**

Section 5.2: Create a scatterplot for the data below. Is the linear correlation positive or negative, weak or strong?

X	1	2	4	4	5	6	7	9	10
Y	6	4	6	8	10	8	9	11	9

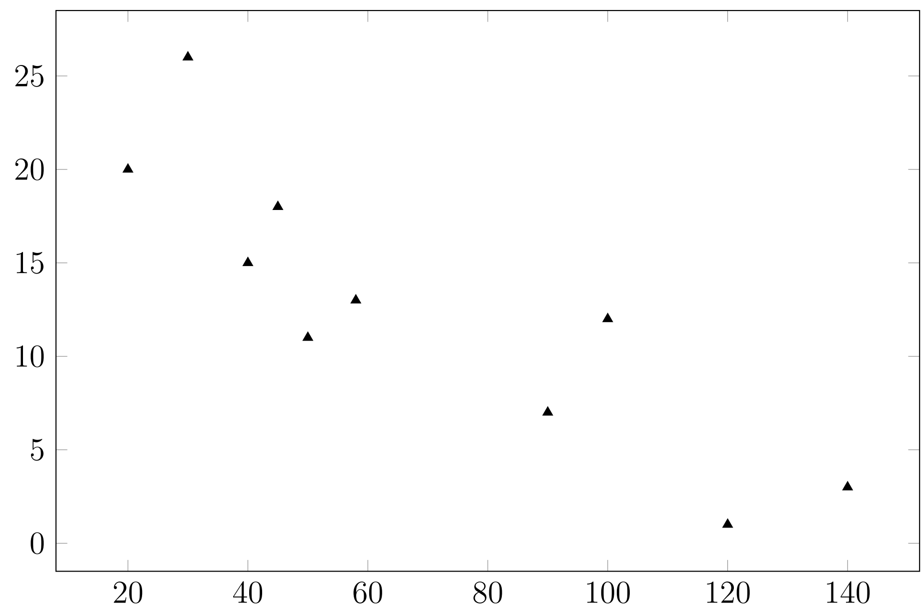
ANSWER: The scatterplot is below. Overall there seems to be a somewhat weak linear pattern in a positive direction.



Section 5.3: The table below shows data from ten people of their average monthly spending on fast food, as well as their average number of days of exercise each month. Create a scatterplot and find the correlation coefficient r. Then find the regression equation and use it to forecast the number of days of exercise output for the fast food value \$70. Is that prediction interpolation or extrapolation? Do you notice a pattern between fast food spending and exercise? What could explain the pattern?

fast food \$	20	40	58	50	140	30	90	45	100	120
exercise days	20	15	13	11	3	26	7	18	12	1

ANSWER: The scatterplot is below. Overall there seems to be a mildly strong linear pattern in a negative direction. This is confirmed by $r = -0.879$. **The regression equation is** $y = -0.166x + 24.1$. **The prediction is** $y = -0.166(70) + 24.1 = 12.5$ **days of exercise in a month with** \$70 **fast food spending. This is interpolation, since** 70 **is in the range of the data (20 to 140). It seems that as fast food spending increases, the number of days of exercise goes down. This could be because people who eat a lot of fast food are not as health conscious as those who eat better, and they would then not exercise as much on average.**



Section 6.1: Convert 7 yards to inches and 310,000 cm to hectometers.

ANSWERS: 252 **in. and** 31 **hm**

Convert 8 yards to centimeters using the conversion $1ft = 30.5\ cm$ **ANSWER:** 732 cm

Section 6.2: Convert 5 gallons into tablespoons. **ANSWER:** $1,280$ **Tbsp.**

Section 6.3: Convert $20°F$ to Celsius. Round to whole degree. **ANSWER:** $-7°C$

Section 1.1 solutions

1. This is deductive reasoning. Given the first general statement being true, then the second statement is just one of the specific examples that fall under the first one.

2. This is inductive reasoning since it is only for two specific examples. There is no evidence that it applies to all graduates from Harvard.

3. When 8 is divided by 6, it results in $\frac{8}{6} = 1.33$ which is not whole.

4. When you add zero to itself, the results is also zero, which is not greater.

5. The numbers are increasing, but the amount of increase goes up by one each time: $1 + 1 = 2$, $2 + 2 = 4$, $4 + 3 = 7$, $7 + 4 = 11$, $11 + 5 = 16$, etc. The next values would be $16 + 6 = 22$ and $22 + 7 = 29$.

6. The days are increasing by three each time, so the next values would be Tuesday and Friday.

Section 1.2 solutions

1. It is closest to $2,000,000$, $1,840,000$, $1,843,000$, $1,842,510$

2. Values are 6 , 6.21 , 6.2082

3. Rina spent approximately $86 + 50 + 6 = \$142$ on car expenses.

4. Estimate is $400 + 600 + 900 = 1,900$ miles.

5. The answer could be correct, if the original number was close to $1,200$. For example, if the original number was $1,197$ then it would be closer to $1,200$ than $1,190$.

Section 1.3 solutions

1. The graphs cross between years 24 and 26, so we can estimate 25 years. The cost is between 30,000 and 35,000. Since it is closer to 35,000, we can estimate the cost as \$34,000.

2. At age 3, the height appears to be 90 cm. At age 6, 110 cm. The growth is $110 - 90 = 20$ cm.

3. The bars appear to go up to the number of trees of approximately 80, 30, 80 and 50. Adding these gives 240 total trees as an estimate. You might have slightly higher or lower total, which is perfectly fine.

4. Staring at twelve o'clock and going clockwise, the first slice is more than a quarter of the pie. It appears about a third of it, so a good estimate would be 35% (or anything within a few from that). The second slice appears a bit less than a quarter of the pie, so maybe 20%. These two estimates add to 55% which fits the fact that they combine to just over half of the pie. The third slice appears to be well over 25% but not quite half a pie, so a good estimate could be 40%. That would leave about 5% for the last slice to total 100%.

Section 1.4 solutions

1. For two weeks, the initial cost is $210*2 = 420$ and the mileage adds another $400*0.28 = 112$ making the total cost $420 + 112 = \$532$. The other data here is irrelevant to the cost.

2. Store A cost is $24 * \$0.50 + 2 * \$3 = \$12 + \$6 = \$18$. Store B cost is $5 * \$5.50 + \$8 = \$11 + \$8 = \$19$. therefore, store A is the better deal.

3. Average is calculated by adding all values and dividing by how many there are. So we can setup the average formula using x for the missing grade as $\frac{95 + 91 + 88 + 82 + x}{5} = 90$. Combining grades gives $\frac{356 + x}{5} = 90$. To solve for x we multiply both sides by 5, then subtract 356, to get $x = 94$.

Section 2.1 solutions

1. The population is all homeschool science textbooks in the United States. The sample is the 15 science books obtained. The (unknown) parameter is the population average price of all the books, which they hope to determine. The statistic measured is the average price of the 15 books = $52.

2. The variables are: name, height, weight, eye color, hair color, and page-hits. The corresponding values are: Sean Higgins, 5ft.10in., 185 lbs., Green, Red, and 142. Name, eye color, and hair color, are qualitative (categories). Height, weight, and number of page-hits are quantitative (measures or counts).

3. A census is a gathering of information from the entire population of people or things that is being studied. The US government only does a census every ten years, because it takes so much time, money, and staff to complete. Technically, the US census is not a true complete census. It is impossible to keep track of everyone in the country at one exact moment in time. There are people being born and dying every day, criminals or illegal aliens hiding who don't want to be found, and some people who ignore requests or lie about their information.

4. Which samples are representative of their populations, which are not? Explain why.

   (a) Not representative since the people watching the top ten youtube videos will likely watch more than most teenagers. They need to include teenagers who do a variety of activities, have jobs, etc.

   (b) Yes representative since they pick many different types of employees from several different locations.

Section 2.2 solutions

1. Since we are trying to summarize, a small number of groups makes it easy to see the big picture. If a data set has 1000 values, using 100 groups would be so large and cumbersome, it would not be a summary and difficult to see anything.

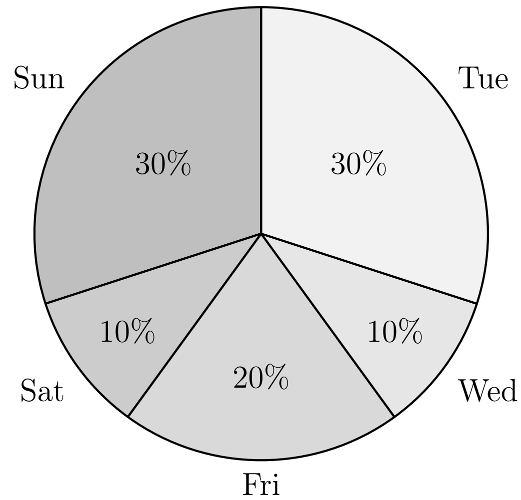
2. When computing relative frequencies, the total should equal 100% or very close. 100% means all the data has been accounted for.

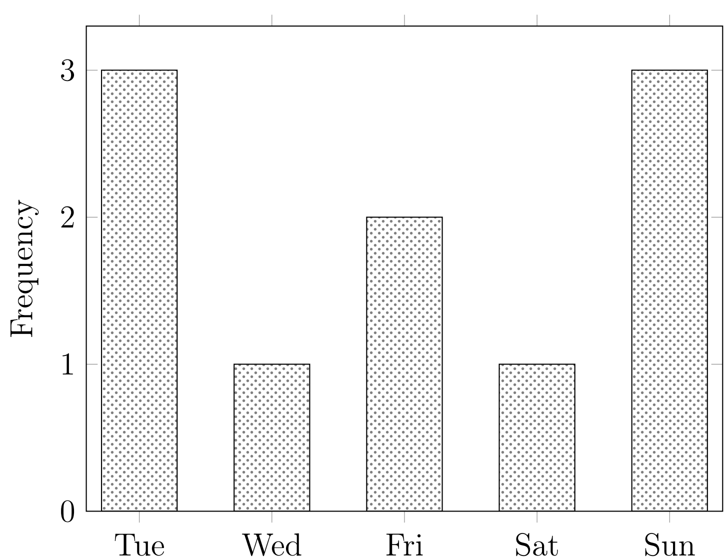
3. A bar graph can be in any order, but a Pareto chart has the bars shown in size order. A Pareto chart cannot be done from quantitative data, since numbers must go in numerical order of the classes (intervals) and the graph is a histogram.

4. It rained ten days in this month. The distribution of which days it rained would be as follows:

Day	Frequency	Relative Frequency
Tuesday	3	$\frac{3}{10} = 0.3 = 30\%$
Wednesday	1	$\frac{1}{10} = 0.1 = 10\%$
Friday	2	$\frac{2}{10} = 0.2 = 20\%$
Saturday	1	$\frac{1}{10} = 0.1 = 10\%$
Sunday	3	$\frac{3}{10} = 0.3 = 30\%$
Total	10	100%

5. The pie chart and bar graph for which days it rained are:

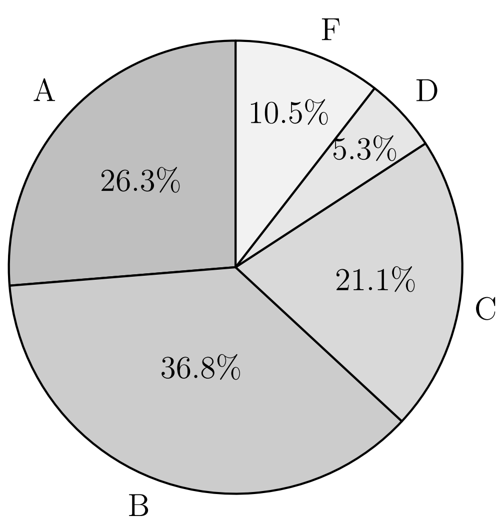


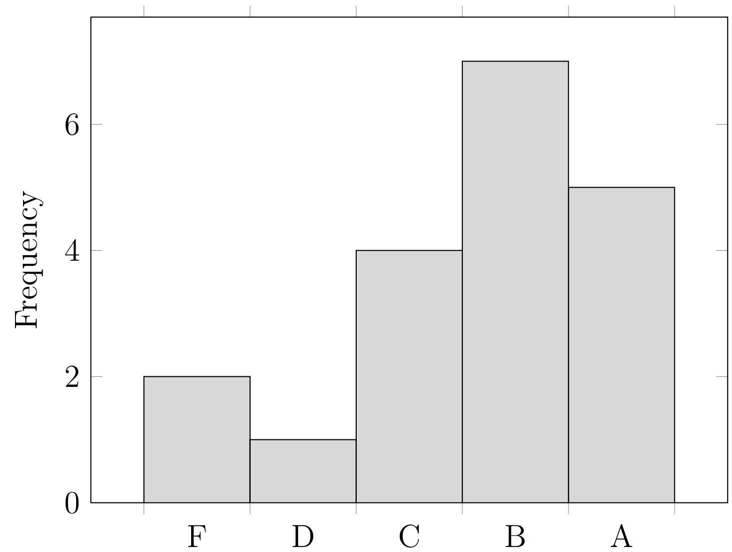


6. For the classes, we can use the common grading scale F (<60), D (60-69), C (70-79), B (80-89), and A (90+). If your school has a different scale, that would be fine also. The distribution is shown below.

Grade	Frequency	Relative Frequency
F (<60)	2	$\frac{2}{19} = 0.10526 = 10.5\%$
D (60-69)	1	$\frac{1}{19} = 0.05263 = 5.3\%$
C (70-79)	4	$\frac{4}{19} = 0.21053 = 21.1\%$
B (80-89)	7	$\frac{7}{19} = 0.36842 = 36.8\%$
A (90+)	5	$\frac{5}{19} = 0.26316 = 26.3\%$
Total	19	100.0%

7. The pie chart and histogram for the history grades are:





8. The center is at 6. The graph is not spread out that much, it is concentrated from 5-8, with a single outlier at 1. The graph is left-skewed semi-bell shape.

9. The graph has a truncated vertical axis, making the first bar appear much smaller and the third bar much larger than the others. The first bar height is around 50, the second bar around 60, not really that much of a difference. The second bar appears to be twice the size of the first, this is misleading. Also there is no title and the categories are vague. This is a very bad graph.

Section 2.3 solutions

1. Mean $= \frac{\sum x}{N} = \frac{210}{11} = 19.0909$, rounded is 19.1. After data is put in order, the median is the 6th value, 19. There are two values which are repeated four times each, so there are two modes 18 and 19.

2. The median and mode are usually unaffected by extreme values, since the median is in the middle (not at extremes) and the extreme values usually don't occur often. The mean is found from the sum of all values, one extreme value can affect the mean drastically.

3. No, if a data set has no data values repeated, they all occur once only, then there is no mode. The mode is the data value that is REPEATED the most. There can be a tie if two or more data values are repeated the same amount of times as the most often.

4. The midpoints of the classes ($\widehat{x}$) are: $12, 17, 22, 27, 32, 37$. Then the formula would be $\frac{\sum \widehat{x} f}{\sum f} = \frac{12(12)+17(5)+22(7)+27(2)+32(6)+37(3)}{12+5+7+2+6+3} = \frac{740}{35} = 21.14286$, rounded to 21.1.

5. Here we are looking for the average grade points, so the data values are the grade points. The credit amounts are the weights.

$$GPA = \frac{3.0(3) + 2.0(3) + 4.0(3) + 4.0(1) + 4.0(3)}{3 + 3 + 3 + 1 + 3} = \frac{40}{13} = 3.077.$$

6. The values in order are $5, 7, 9, 14, 14, 19, 20, 22, 23, 27$. There are ten values, so there are two of them tied for the middle at the fifth and sixth positions, 14 and 19. The median is $\frac{14+19}{2} = 16.5$.

Section 2.4 solutions

1. Minimum is 17, maximum is 25, and range is $25 - 17 = 8$.

2. Setup as range = max - min, so $83 = 124 - x$, then $x = 124 - 83 = 41$ is the minimum.

3. The mean is 40, the deviations form the mean are $-30, -15, 0, 10, 35$, the squared deviations are $900, 225, 0, 100, 1225$. The variance is $\frac{2450}{4} = 612.5$ and standard deviation is $\sqrt{612.5} = 24.75$ years.

4. Compute coefficient of variation for each. For shot put, $cv = \frac{5.5}{38} * 100\% = 14.47\%$. For gymnastics, $cv = \frac{1.4}{8.45} * 100\% = 16.56\%$. The gymnastics scores are more spread out relative to the size of the values.

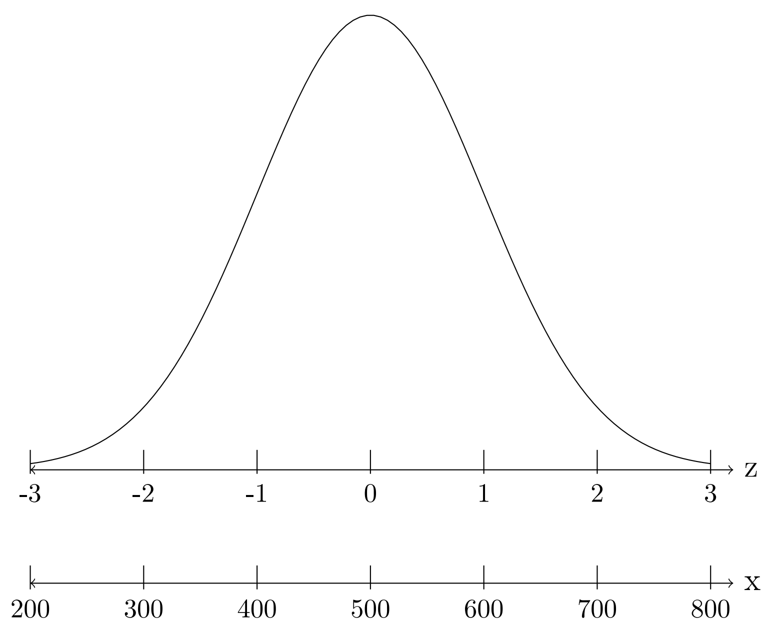
5. The mean is 485, the deviations form the mean are $-135, -65, -85, 35, 85, 165$, the squared deviations are $18225, 4225, 7225, 1225, 7225, 27225$. The variance is $\frac{65,350}{6} = 10,891.7$ and standard deviation is $\sqrt{10891.7} = 104.4$ pounds.

Section 2.5 solutions

1. From $z = -1$ and $z = 1$ is 68%, from $z = 1$ and $z = 2$ is another 13.5%, and from $z = 2$ and $z = 3$ is 2.35%. Added together these result in $68 + 13.5 + 2.35 = 83.85\%$

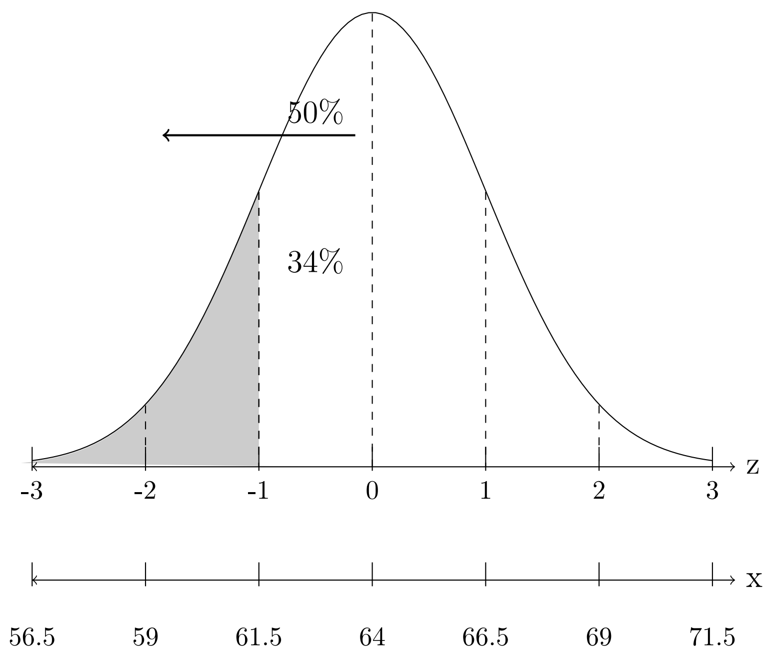
2. From $z = -3$ and $z = -2$ is 2.35% and from $z = -1$ and $z = -2$ is another 13.5%. Added together these result in $13.5 + 2.35 = 15.85\%$

3. SAT test scores for English are shown on the x-axis below the standard normal distribution bell curve.



4. Draw a bell curve with standard z-axis from -3 to 3 and below that, an x-axis with heights that correspond to the z marks. The mean height of 64 will go below $z = 0$, one standard deviation higher ($64 + 2.5 = 66.5$ inches) will go below $z = 1$, etc. Do similar process on left side, subtracting standard deviation to go under the negative z-values.

Change 5 feet 1-1/2 inches into 61.5 inches. So we are looking for the slice of the graph below 61.5, which is to the left $z = -1$. The area is the lower half (50%) minus the section from $z = -1$ to 0 of 34%. The answer is $50 - 34 = 16\%$. The graph is shown below.



5. The weight of 3.9 kg is 2 standard deviations above average, so it corresponds with $z = +2$. The percentage to the right of $z = 2$ will be the upper half area minus the 34% and 13.5% areas, which is $50 - 34 - 13.5 = 2.5\%$. This is the section from $z = 2$ to 3 plus the tiny bit extra beyond 3.

6. The score of 280 corresponds with $z = -2$ and 480 with $z = 0$, so the percentage between them is $13.5 + 34 = 47.5\%$

Section 3.1 solutions

1. List the sample spaces for the following experiments:

   (a) SS = { H, T }, $N = 2$.

   (b) We need to look at what happens for each flip and combine them.

   SS = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }, $N = 8$.

   (c) $SS = \{1, 2, 3, 4, 5, 6\}$, $N = 6$.

   (d) The sums range from 2 (rolling a 1+1) up to 12 (rolling 6+6)

   $SS = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $N = 11$.

   (e) Using letter abbreviations for the 7 rainbow colors and seasons,

   SS = { Rw, Ow, Yw, Bw, Gw, Iw, Vw, Rsp, ..., Rsm, ..., Rf, Of, Yf, Bf, Gf, If, Vf }, $N = 28$.

2. Probabilities must be between 0 and 1 (or 0% and 100%), so the valid values are 0.35, 0.004, and $\frac{3}{8}$. All of the others are either negative or greater than 1.

3. The event 'even' consists of the outcomes 2, 4, and 6. This is three out of the 6 possible outcomes, so $P(even) = \frac{3}{6} = 0.5 = 50\%$. $P(3) = \frac{1}{6} = 0.167 = 17\%$. P(>2) = P(3 or 4 or 5 or 6) = $\frac{4}{6} = 0.667 = 67\%$.

4. Probability of making a shot, based on his data, is $\frac{5}{12} = 0.417 = 42\%$. Subjectively, you might believe this to be low percentage and he might not make the team.

5. The probability of precipitation equals 0.45 (the sum of all three given). By the complement rule, the probability of no precipitation = 1 - probability of precipitation $= 1 - 0.45 = 0.55 = 55\%$.

6. The theoretical probabilities of rolling each number are all equal to $\frac{1}{6}$ or about 17%. Your empirical probabilities are computed by dividing the count of how often a number was rolled, by the total of 15. Your values most likely vary, some smaller than 17% and some larger. If they are close to the theoretical, you got lucky. If they are not, that is because your rolls are random and with only 15 rolls, the law of large numbers does not work very well.

Section 3.2 solutions

1. Using the Fundamental Counting Principle, the number of outfits is $5 \times 4 \times 3 = 60$.

2. They have 10 movies for first choice, nine remaining for second, and eight for third. The number of movie combinations is $10 \times 9 \times 8 = 720$.

3. We can setup the equation $10 * 5 * d = 200$ where d is the unknown number of desserts. Then solve $50d = 200$ by dividing both sides by 50 to get $d = 4$ desserts to choose from.

4. Assuming digits can be repeated, then each digit has ten choices, 0 through 9. The number of possible sets of 5 digits is $10 * 10 * 10 * 10 * 10 = 100,000$, so the probability of matching the one exact set of winning digits is $\frac{1}{100,000}$

5. There are four people who could finish first, then three remaining to finish second, etc., therefore the number of ways for the four runners to finish is $4 * 3 * 2 * 1 = 24$. If Kenji finishes first then there are $3 * 2 * 1 = 6$ ways for the other three to finish behind him. The probability of Kenji winning is then $\frac{6}{24} = \frac{1}{4} = 25\%$

6. Compute the following: ${}_8C_3 = \frac{8!}{5!3!} = 56$ ${}_{11}C_9 = \frac{11!}{2!9!} = 55$ ${}_7P_4 = \frac{7!}{3!} = 840$ ${}_8P_8 = \frac{8!}{0!} = 40,320$ ${}_5C_1 = \frac{5!}{4!1!} = 5$

7. How many sets of 5 numbers can be played? This is a combination so ${}_{39}C_5 = \frac{39!}{5!34!} = 575,757$ different sets of numbers can be played.

Section 3.3 solutions

1. Mutually exclusive means that the events cannot happen at the same time. One example is rolling a 3 on a die and a 5 on a die. Another example would be the experiment picking a name for a raffle winner, with events picking female and picking male.

2. We add up all of the physics or engineering majors, but make sure not to double count the 14 double majors. There are $32+(112-14) = 130$ in that group. For total students we add all three majors without double counting, $32 + (49 - 8) + (112 - 14) = 171$. Now P(Phys or Eng)= $\frac{130}{171} = 0.760 = 76\%$.

3. Using addition rule, $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B) = 0.5 + 0.7 - 0.3 = 0.9$.

4. Setting up the addition rule, $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$, we get $0.85 = 0.65 + P(B) - 0.25$. By simplifying and solving, we get $P(B) = 0.45$

5. These are dependent events, the first card picked affects what cards are left and so affects the second pick. We use the multiplication rule $P(\text{red6 and red6}) = P(\text{1st red6}) \cdot P(\text{2nd red6} \mid \text{1st red6}) = \frac{2}{52} \cdot \frac{1}{51} = \frac{1}{1326} = 0.00075 = 0.075\%$, which is a very small chance.

6. By reasoning, we know the ten of hearts is one out of 26 red cards. By formula, $P(\text{10hearts}|\text{red}) = \dfrac{P(\text{10hearts and red})}{P(red)} = \frac{\frac{1}{52}}{\frac{26}{52}} = \dfrac{1}{26} = 0.038 = 3.8\%$

7. One example is rolling two dice and getting a 3 on one die and a 5 on the other die. Another example is randomly picking a person and determining that they work at Home Depot and like vanilla ice cream. Where you work has no effect on what ice cream you like.

8. Two events can never be both mutually exclusive and independent. If they are mutually exclusive, the occurrence of one automatically affects the other (it prohibits the other). Once you know a coin lands heads up, then tails has no chance (until the next flip).

9. This is not a valid probability distribution. The probabilities themselves are valid, but the total only sums to 0.97 or 97%.

10. $E(x) = \frac{752,944}{800,000}(0) + \frac{45,000}{800,000}(5) + \frac{2,000}{800,000}(100) + \frac{50}{800,000}(2500) + \frac{5}{800,000}(20000) + \frac{1}{800,000}(100000) = \$0.94$, so the lottery expects to pay out \$0.94 on average for each play of the game. In order to make a profit, they need to charge more than that, maybe \$1.00 or \$2.00.

Section 4.1 solutions

1. Convert the following numbers from decimal form to percent form:

   - $19.78 \times 100\% = 1,978\%$
   - $1.12 \times 100\% = 112\%$
   - $0.57 \times 100\% = 57\%$
   - $0.0031 \times 100\% = 0.31\%$

2. Convert the following numbers from percent form to decimal form:

   - $25\% \div 100 = 0.25$
   - $148\% \div 100 = 1.48$
   - $0.156\% \div 100 = 0.00156$
   - $17.34\% \div 100 = 0.1734$

3. Convert the following numbers from fraction form to percent form, rounded to two decimal places as necessary:

   - $\frac{2}{5} = 2 \div 5 = 0.4,\ 0.4 \times 100\% = 40\%$
   - $1\ \frac{7}{8} = \frac{15}{8} = 15 \div 8 = 1.875,\ 1.875 \times 100\% = 187.5\%$
   - $\frac{137}{50} = 137 \div 50 = 2.74,\ 2.74 \times 100\% = 274\%$
   - $\frac{15}{16} = 15 \div 16 = 0.9375,\ 0.9375 \times 100\% = 93.75\%$

4. 15% of $200 = 0.15 \times 200 = 30$

5. 34% of $50 = 0.34 \times 50 = 17$

6. $75 = 0.30 * B, \qquad \frac{75}{0.30} = B, \qquad B = 250$

7. $48 = 0.12 * B, \qquad \frac{48}{0.12} = B, \qquad B = 400$

8. $28 = P * 200, \qquad \frac{28}{200} = P, \qquad P = 0.14 \times 100\% = 14\%$

9. Tax is $0.06 * 153.25 = \$9.20$, the total after tax is $153.25 + 9.20 = \$162.45$

10. Discount is $0.25 * 23,050 = \$5,762.50$, the sale price is $23.050 - 5,762.5 = \$17,287.50$

11. Discount is $0.33 * 899.99 = \$297.00$, the sale price is $899.99 - 297 = \$602.99$.

    Tax is $0.09 * 602.99 = \$54.27$, the total after tax is $602.99 + 54.27 = \$657.26$

12. $\frac{10,000}{45,000} * 100\% = 0.222 * 100\% = 22.2\%$ increase.

13. The home's value dropped $157,000 - 135,000 = \$22,000$.

    Percent decrease is $\frac{22,000}{157,000} * 100\% = 0.140 * 100\% = 14\%$ decrease.

Section 4.2 solutions

1. $I = Prt = 500 * 0.04 * 15 = \$300$, the total value is $500 + 300 = \$800$

2. $I = Prt = 1,000 * 0.03 * 5 = \$150$, the total payback is $1,000 + 150 = \$1,150$

3. Using $A = P(1 + rt)$ we get $3,000 = P(1 + 0.06 * 10)$,
   so $3,000 = P(1.6)$ and $P = \dfrac{3000}{1.6} = \$1,875$

4. Using $A = P(1 + rt)$ we get $12,000 = P(1 + 0.02 * 15)$,
   so $12,000 = P(1.3)$ and $P = \dfrac{12000}{1.3} = \$9,230.77$

5. Using $A = P(1 + rt)$ we get $4,000 = 3,200(1 + r * 4)$, so $4,000 = 3,200(1 + 4r)$.
   Divide by 3,200 to get $1.25 = 1 + 4r$, then $0.25 = 4r$ and $r = 0.0626 = 6.25\%$

6. Option 1: $A = 5000(1 + .06 * 7) = \$7,100$,
   Option 2: $A = 5000(1 + .07 * 6) = \$7,100$, they are the same.

7. If you invest $10,000 into an account offering 5% compound interest for 8 years, what is the future value of the account subject to the following compounding periods/options:

   - Annually: $n = 1$, $A = P\left(1 + \frac{r}{n}\right)^{nt} = 10,000\left(1 + \frac{0.05}{1}\right)^{1*8} = \$14,774.55$
   - Semi-annually: $n = 2$, $A = 10,000\left(1 + \frac{0.05}{2}\right)^{12*8} = \$14,845.06$
   - Monthly: $n = 12$, $A = 10,000\left(1 + \frac{0.05}{12}\right)^{12*8} = \$14,905.85$
   - Weekly: $n = 52$, $A = 10,000\left(1 + \frac{0.05}{52}\right)^{52*8} = \$14,915.38$
   - Daily: $n = 365$, $A = P10,000\left(1 + \frac{0.05}{365}\right)^{365*8} = \$14,917.84$
   - Compounded Continuously: $A = Pe^{rt} = 10,000e^{0.05*8} = 14.918.25$

8. Option 1: $A = 7,500\left(1 + \frac{0.0675}{12}\right)^{12*8} = \$12,850.59$

   Option 2: $A = 7,700e^{0.065*8} = 12,951.61$, which is better by \$101.02

9. Setup $14,000 = P\left(1 + \frac{0.055}{52}\right)^{52*9}$ to get $14,000 = P(1.64006915)$, then $P = \$8,536.23$

10. Setup $150,000 = Pe^{0.10*5}$ to get $150,000 = P(1.648721271)$, then $P = \$90,979.60$

Section 4.3 solutions

1. The total future value is $A = \dfrac{5000\left[\left(1+\dfrac{0.08}{1}\right)^{1*30}-1\right]}{\left(\dfrac{0.08}{1}\right)} = \$566,416.06$. The amount deposited into the account is $5000 * 30 = \$150,000$ The growth/return is $566,416.06 - 150,000 = 416,416.06$

2. The total future value is $A = \dfrac{250\left[\left(1+\dfrac{0.11}{12}\right)^{12*40}-1\right]}{\left(\dfrac{0.11}{12}\right)} = \$2,150,031.80$. The amount deposited into the account is $250 * 12 * 40 = \$120,000$ The growth/return is $2,150,031.80 - 120,000 = 2,030,031.80$

3. Suppose you invest \$50 into an IRA (individual retirement account) each month. If you average 12% return on your investments, how much money will be in the IRA after 35 years? How much of that total is pure return/growth? The total future value is $A = \dfrac{50\left[\left(1+\dfrac{0.12}{12}\right)^{12*35}-1\right]}{\left(\dfrac{0.12}{12}\right)} = \$321,547.97$. The amount deposited into the account is $50 * 12 * 35 = \$21,000$ The growth/return is $321,547.97 - 21,000 = 300,547.97$

4. Investments in order form least risk to highest risk: CD's, Bonds, index funds, single stocks.

Section 5.1 solutions

1. All functions are relations, is a true statement. The definition of a function states that it is a special relation, so it is always a relation. All relations are functions, is false. Some relations, such as circles, do not meet the special conditions of a function, and don't pass the vertical line test.

2. List out the domain and range of the following relations. Do they represent functions?

   (a) $D = \{2, 3, 4, 5, 6\}$ $R = \{3, 6, 7\}$ This is a function, each value in the domain has only ONE matching value in the range.

   (b) $D = \{-1, 0, 2, 4, 278\}$ $R = \{-1, 0, 2, 4, 278\}$ This is a function, each value in the domain has only ONE matching value in the range.

   (c) $D = \{2, 3, 5\}$ $R = \{taco, burrito, enchilda, fajitas\}$ This is NOT a function, since the value 2 in the domain has TWO matching values in the range.

3. $f(0) = 0^2 - 3(0) + 1 = 1$

   $f(-2) = (-2)^2 - 3(-2) + 1 = 11$

   $f(\frac{1}{4}) = (\frac{1}{4})^2 - 3(\frac{1}{4}) + 1 = \frac{1}{16} - \frac{3}{4} + 1 = \frac{5}{16}$

4. Graphs (a) and (c) are functions, since every vertical line will intersect the graph at most once. Graph (b) is NOT a function, since there are many vertical lines which will intersect the graph twice.

Section 5.2 solutions

1. Correlation is when two variables tend to be related. Certain values of one variable tend to be paired with certain values of the other. For positive correlation, as one goes up (increases), the other tends to go up. For negative correlation, as one goes up, the other tends to go down (decreases).

2. Match the most likely linear correlation values to the graphs below.

   $r = +0.7 \Rightarrow b \quad r = +0.99 \Rightarrow a \quad r = -0.4 \Rightarrow c \quad r = +0.15 \Rightarrow e \quad r = -0.86 \Rightarrow d$

   $r = 0 \Rightarrow f$

3. We can add extra rows to organize the calculations.

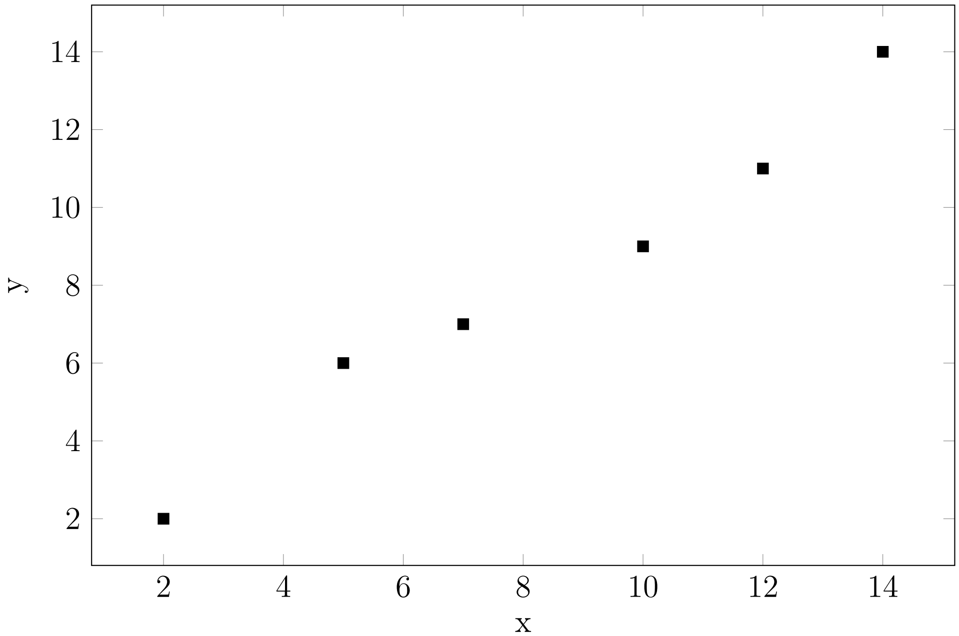
x	2	5	7	10	12	14	$\sum x = 50$
y	2	6	7	9	11	14	$\sum y = 49$
x^2	4	25	49	100	144	196	$\sum x^2 = 518$
y^2	4	36	49	81	121	196	$\sum y^2 = 487$
xy	4	30	49	90	132	196	$\sum xy = 501$

Now we input the value $n = 6$ and all of the sums into the formula to get:

$$r = \frac{6 * 501 - 50(49)}{\sqrt{[6 * 518 - 50^2]\,[6 * 487 - 49^2]}}$$

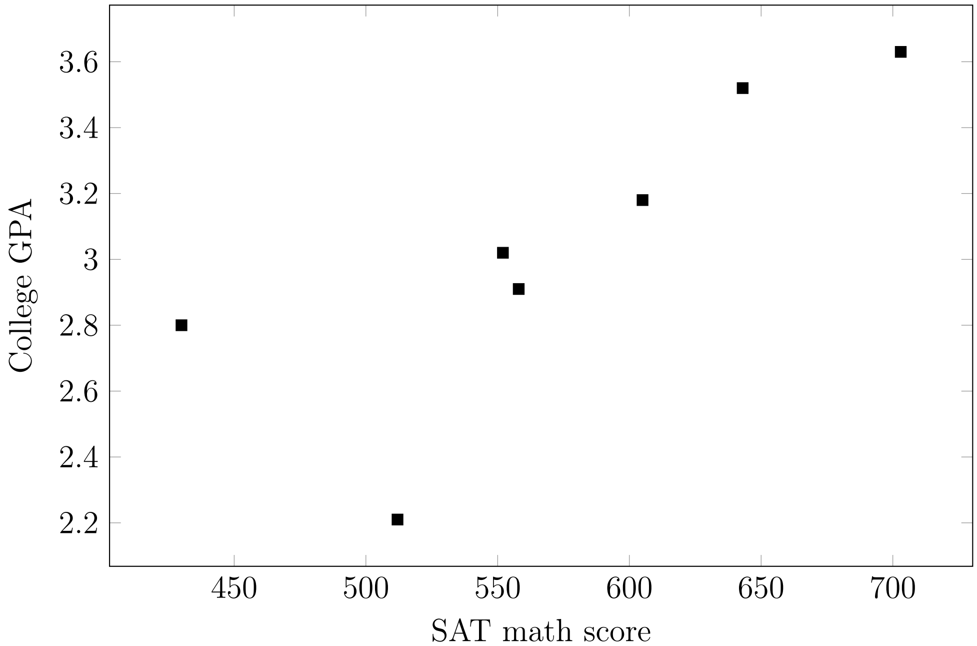
$$= \frac{3006 - 2450}{\sqrt{[3108 - 2500][2922 - 2401]}} = \frac{556}{562.82} = +0.988$$

The scatterplot will look like this:



4. On TI-83/84 series calculators, go to STAT EDIT menu, enter the SAT data into L_1. Then move over to L_2 and input the GPA data exactly in order. Press STAT, CALC menu, scroll down to item LinReg(ax+b) and press ENTER. Type L_1, L_2 then hit enter again. The results you should see are $r = +0.793$ (rounded). The value $r = +0.793$ is a fairly strong positive correlation between SAT math score and college GPA. It is probably that the students who get higher scores, work harder, study well, and are more interested in learning, so they do well in college.

The scatterplot will look like this:



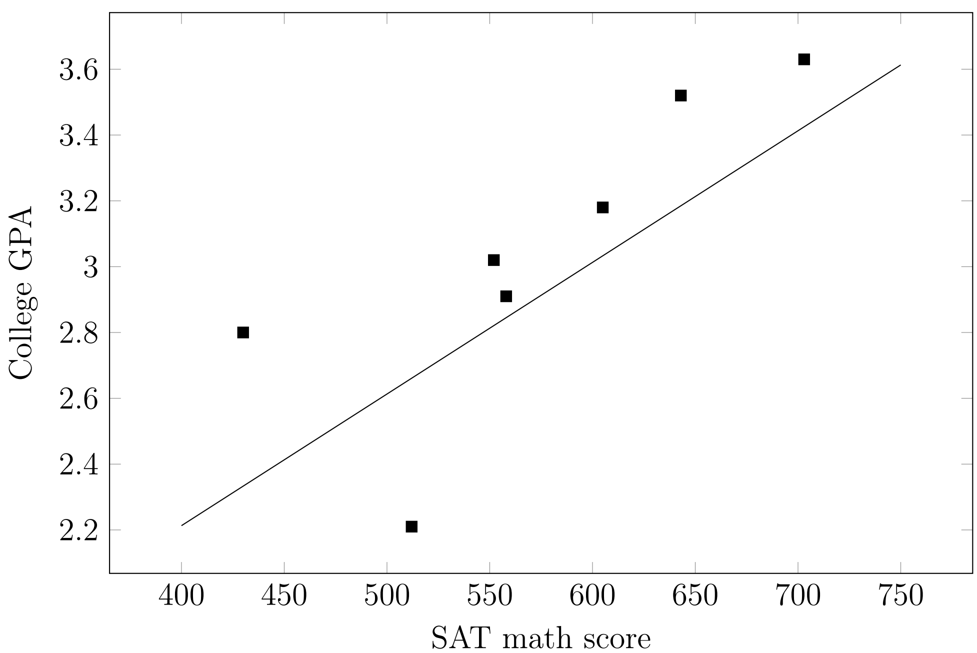
Section 5.3 solutions

1. This is a positive linear pattern.

2. This is an upward exponential "J" pattern.

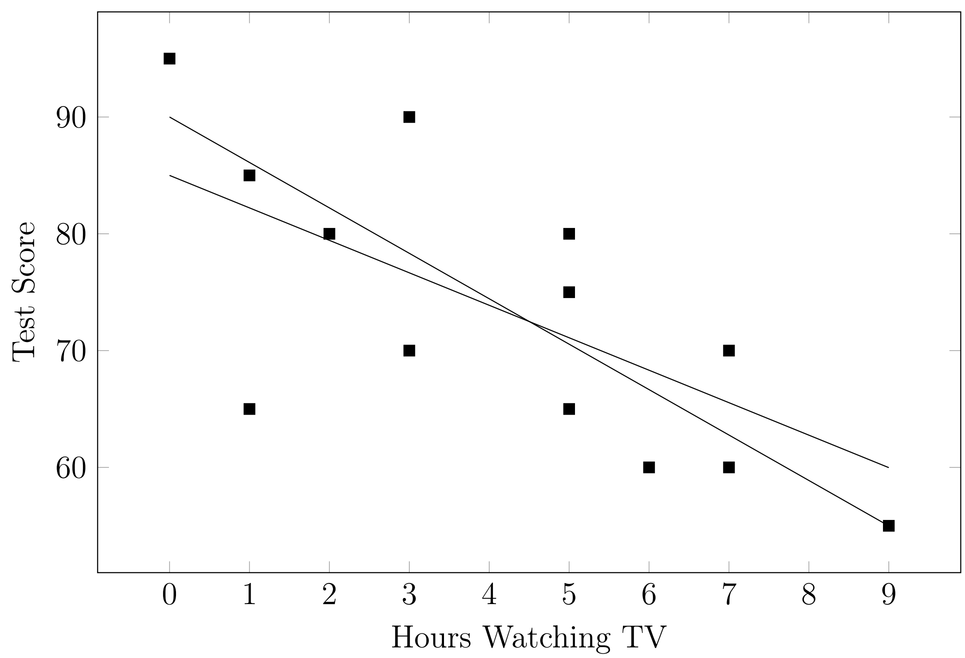
3. This is an upside down quadratic "U" pattern.

4. The regression line is $y = 0.004x + 0.613$ (rounded). These values imply that for every point your SAT math score increases, your college GPA will increase by 0.004, and that with a zero on SAT math, you should still be able to get a 0.613 GPA. In reality, the lowest SAT score is 200, but the equation follows that pattern if we could project down to zero. The regression line is shown along the scatterplot points below.



5. The predictions are $y = 0.004(760) + 0.613 = 3.653$ GPA with an SAT math score of 760, and $y = 0.004(500) + 0.613 = 2.613$ GPA with an SAT math score of 500. The first one is extrapolation, since 760 is outside the range of the data (430 to 703). The second is interpolation, since 500 is in the range.

6. Two possible regression lines are shown below. Other similar lines would be good fits as well. There appears to be a somewhat strong negative correlation between the variables. More TV watching tends to match with lower scores. It is reasonable to assume that watching a lot of TV, takes time away from studying, and therefore causes lower scores (on average). There are exceptions, but this is a general fact backed up by much research.



7. A positive correlation coefficient r corresponds with a positive slope. A negative correlation coefficient r corresponds with a negative slope. The values have no relation, only the sign, $\pm$.

Section 6.1 solutions

1. $237,600\ ft\left(\dfrac{1\ mile}{5,280\ ft}\right) = \dfrac{237,600}{5,280} = 45\ miles$

2. $22\ yd\left(\dfrac{3\ ft}{1\ yd}\right)\left(\dfrac{12\ in}{1\ ft}\right) = \dfrac{22*3*12}{1} = 792\ \text{ inches}$

3. Deci means one-tenth and Hecto means one Hundred.

4. $7\ km\left(\dfrac{1,000\ m}{1\ km}\right)\left(\dfrac{100\ cm}{1\ m}\right) = 7(1,000)(100) = 700,000\ cm$

5. $45,000\ mm\left(\dfrac{1\ m}{1,000\ mm}\right) = \dfrac{45,000}{1,000} = 45\ \text{ meters}$

6. Multiply by unit fractions to get to centimeters, then over to inches, then yards. Yards are much smaller than kilometers, so the answer should be a much larger number.

$$4\ km\left(\frac{1,000\ m}{1\ km}\right)\left(\frac{100\ cm}{1\ m}\right)\left(\frac{1\ in}{2.54\ cm}\right)\left(\frac{1\ ft}{12\ in}\right)\left(\frac{1\ yd}{3\ ft}\right)$$

$$= 4\ \cancel{km}\left(\frac{1,000\ \cancel{m}}{1\ \cancel{km}}\right)\left(\frac{100\ \cancel{cm}}{1\ \cancel{m}}\right)\left(\frac{1\ \cancel{in}}{2.54\ \cancel{cm}}\right)\left(\frac{1\ \cancel{ft}}{12\ \cancel{in}}\right)\left(\frac{1\ yd}{3\ \cancel{ft}}\right)$$

$$= \frac{4(1,000)(100)}{2.54(12)(3)}yds$$

$$= 4,374.5\ yds$$

7.

$$18\ ft\left(\frac{12\ in}{1\ ft}\right)\left(\frac{2.54\ cm}{1\ in}\right)\left(\frac{1\ dm}{10\ cm}\right)$$

$$= 18\ \cancel{ft}\left(\frac{12\ \cancel{in}}{1\ \cancel{ft}}\right)\left(\frac{2.54\ \cancel{cm}}{1\ \cancel{in}}\right)\left(\frac{1\ dm}{10\ \cancel{cm}}\right)$$

$$= \frac{18(12)2.54}{10}yds$$

$$= 54.864\ dm$$

Section 6.2 solutions

1. One square yard is $3\ ft \times 3\ ft = 36\ in \times 36\ in = 1,296\ in^2$.

2. Multiply by unit fractions to get to feet and inches, then convert over to centimeters, then to kilometers.

$$3\ miles^2 \left(\frac{5,280\ ft}{1\ mile}\right)^2 \left(\frac{12\ in}{1\ ft}\right)^2 \left(\frac{2.54\ cm}{1\ in}\right)^2 \left(\frac{1\ m}{100\ cm}\right)^2 \left(\frac{1\ km}{1,000\ m}\right)^2$$

$$= 3\ \cancel{miles^2} \left(\frac{27,878,400\ \cancel{ft^2}}{1\ \cancel{mile^2}}\right) \left(\frac{144\ \cancel{in^2}}{1\ \cancel{ft^2}}\right) \left(\frac{6.4516\ \cancel{cm^2}}{1\ \cancel{in^2}}\right) \left(\frac{1\ m^2}{10,000\ \cancel{cm^2}}\right) \left(\frac{1\ km^2}{1,000,000\ \cancel{m^2}}\right)$$

$$= \frac{3(27,878,400)(144)(6.4516)}{10,000(1,000,000)} km^2$$

$$= 7.8\ km^2$$

3. The first is $1.5\ acres \left(\frac{43,560\ ft^2}{1\ acre}\right) = 1.5(43,560)\ ft^2 = 65,340\ ft^2$

The second part is $65,340\ ft^2 \left(\frac{12\ in}{1\ ft}\right)^2 \left(\frac{2.54\ cm}{1\ in}\right)^2 \left(\frac{1\ m}{100\ cm}\right)^2 \left(\frac{1\ are}{100\ m^2}\right)$

$$= 65,340\ ft^2 \left(\frac{144\ in^2}{1\ ft^2}\right) \left(\frac{6.4516\ cm^2}{1\ in^2}\right) \left(\frac{1\ m^2}{10,000\ cm^2}\right) \left(\frac{1\ are}{100\ m^2}\right)$$

$$= 65,340\ ft^2 \left(\frac{144\ \cancel{in^2}}{1\ \cancel{ft^2}}\right) \left(\frac{6.4516\ \cancel{cm^2}}{1\ \cancel{in^2}}\right) \left(\frac{1\ \cancel{m^2}}{10,000\ \cancel{cm^2}}\right) \left(\frac{1\ are}{100\ \cancel{m^2}}\right)$$

$$= \frac{65,340(144)(6.4516)}{10,000(100)}\ ares$$

$$= 60.7\ ares$$

4. We go from cubic inches to cubic centimeters and millimeters.

$$3\ in^3 \left(\frac{2.54\ cm}{1\ in}\right)^3 \left(\frac{10\ mm}{1\ cm}\right)^3$$

$$= 3\ in^3 \left(\frac{16.387\ cm^3}{1\ in^3}\right) \left(\frac{1,000\ mm^3}{1\ cm^3}\right)$$

$$= 3(16.387)(1,000)\ mm^3$$

$$= 49,161\ mm^3$$

5. $1\ pint \left(\frac{16\ fl.oz}{1\ pint}\right) \left(\frac{2\ Tbsp}{1\ fl.oz}\right) = 16(2) = 32\ Tbsp$

6. $1\ gal \left(\frac{1\ ft^3}{7.5\ gal}\right) \left(\frac{12\ in}{1\ ft}\right)^3 = 1\ gal \left(\frac{1\ ft^3}{7.5\ gal}\right) \left(\frac{1,728\ in^3}{1\ ft^3}\right) = \frac{1 * 1,728}{7.5} = 230.4\ in^3$

7. We go from cubic millimeters to cubic centimeters, then Liters.

$$12,500\ mm^3 \left(\frac{1\ cm}{10\ mm}\right)^3 \left(\frac{1\ mL}{1\ cm^3}\right) \left(\frac{1\ L}{1,000\ mL}\right)$$

$$= 12,500\ mm^3 \left(\frac{1\ cm^3}{1,000\ mm^3}\right) \left(\frac{1\ mL}{1\ cm^3}\right) \left(\frac{1\ L}{1,000\ mL}\right)$$

$$= \frac{12,500}{1,000(1,000)}\ L$$

$$= 0.0125\ L$$

Section 6.3 solutions

1. $12\ lbs\left(\dfrac{16\ oz}{1\ lbs}\right) = 12 * 16 = 192$ ounces

2. $13,600\ cg\left(\dfrac{1\ g}{100\ cg}\right)\left(\dfrac{1\ kg}{1,000\ g}\right) = \dfrac{13,600}{100,000} = 0.136\ kg$

3. $9,000\ lbs\left(\dfrac{1\ ton}{2,000\ lbs}\right) = \dfrac{9,000}{2,000} = 4.5$ tons

4. $6.75\ g\left(\dfrac{1,000\ mg}{1\ g}\right) = 6,750\ mg$

5. We go from tonnes to kilograms to pounds, then ounces.

$$1\ T\left(\frac{1,000\ kg}{1\ T}\right)\left(\frac{2.2\ lbs}{1\ kg}\right)\left(\frac{16\ oz}{1\ lb}\right)$$

$$= 1\ \cancel{T}\left(\frac{1,000\ \cancel{kg}}{1\ \cancel{T}}\right)\left(\frac{2.2\ \cancel{lbs}}{1\ \cancel{kg}}\right)\left(\frac{16\ oz}{1\ \cancel{lb}}\right)$$

$$= 1,000(2.2)(16)\ oz$$

$$= 35,200\ oz$$

6. $86\ lbs\left(\dfrac{1\ kg}{2.2\ lbs}\right) = \dfrac{86}{2.2} = 39.09\ kg$

7. Plug in $C = 28$ to get $F = \frac{9}{5}(28) + 32 = 82.4°$

8. Setup and solve for C:

$$-10 = \frac{9}{5}C + 32$$

$$-10 - 32 = \frac{9}{5}C$$

$$-42\left(\frac{5}{9}\right) = \left(\frac{\cancel{5}}{\cancel{9}}\right)\frac{\cancel{9}}{\cancel{5}}C$$

$$-\frac{210}{9} = C$$

$$C = -23.3°$$

Section 7.1 solutions

1. Yes.
2. No.
3. No.
4. Yes.
5. 3 is greater than 5. False.
6. A rectangle with height 2 inches and width 4 inches has an area of 8 square inches. True.
7. This bolt is not over-tightened.
8. I did not see him finish the cake.
9. (a) $\forall x\ p(x)$: All of the parts are missing from the kit.
   (b) $\exists x\ p(x)$: There is a part missing from the kit.
   (c) $\forall x\ {\sim}p(x)$: Of all of the parts, none are missing from the kit.
   (d) $\exists x\ {\sim}p(x)$: There is at least one part that is not missing from the kit.
10. Some of us have not taken the training.
11. None of the beams have rotted.
12. Some of this is relevant.
13. All of your packages could be delivered.

Section 7.2 solutions

1. (a) $p \wedge q$: This item is damaged and it is being sold at a discount.

   (b) $\sim p \wedge q$: This item is not damaged but it is being sold at a discount.

   (c) $p \wedge \sim q$: This item is damaged but it is not being sold at a discount.

2. True.

3. False.

4. True.

5. (a) $p \vee q$: The toner is low or the paper tray is empty.

   (b) $\sim p \wedge (q \vee r)$: The toner is not low, but the paper tray is empty or the paper is jammed.

   (c) $\sim r \wedge q \wedge p$: The paper is not jammed, but the paper tray is empty and the toner is low.

Section 7.3 solutions

1. Column order may vary but truth values must match for each column.

p	q	$p \vee q$	$\sim(p \vee q)$	$p \vee \sim(p \vee q)$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	F
F	F	F	T	T

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

p	q	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$(p \wedge q) \vee (p \wedge \sim q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F

p	q	r	$p \wedge q$	$\sim(p \wedge q)$	$\sim r$	$\sim(p \wedge q) \vee \sim r$	$\sim(\sim(p \wedge q) \vee \sim r)$
T	T	T	T	F	F	F	T
T	T	F	T	F	T	T	F
T	F	T	F	T	F	T	F
T	F	F	F	T	T	T	F
F	T	T	F	T	F	T	F
F	T	F	F	T	T	T	F
F	F	T	F	T	F	T	F
F	F	F	F	T	T	T	F

2. We didn't go to the theater or we went to the museum.

3. The ball didn't land in quadrant three and it didn't land in quadrant four either.

4. It's not true that I have money or fame.

5. The album is not too short or it's not too long.

Section 7.4 solutions

1. (a) If it is after 9 PM then the library is closed.

   (b) If it is before 9 PM the library is open.

   (c) The library is closed if and only if it is after 9 PM.

2. (a) $\sim r \wedge q \rightarrow p$

   (b) $r \rightarrow \sim p$

   (c) $p \rightarrow q$

3. (a) For $(p \leftrightarrow q) \wedge (q \rightarrow \sim p)$:

p	q	$\sim p$	$p \leftrightarrow q$	$q \rightarrow \sim p$	$(p \leftrightarrow q) \wedge (q \rightarrow \sim p)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	T	T	T

   (b) For $(q \rightarrow p) \wedge q$:

p	q	$q \rightarrow p$	$(q \rightarrow p) \wedge q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	T	F

4. If I can buy bread then I have the money.

5. If I do not send you an update I did not hear bad news.

6. Somebody saw who did it and didn't tell me.

7. Some numbers, you square them and they don't get bigger.

Section 8.1 solutions

1. False.
2. True.
3. True.
4. False.
5. True.
6. True.
7. False.
8. False.
9. (a) $\mathbb{N}$
   (b) $\mathbb{R}$
   (c) $\mathbb{Z}$
   (d) $\mathbb{Q}$
10. $K = \{8, 9, 10, 11, 12, 13, 14, 15\}$
11. $L = \{0, 10, 20, 30, 40, 50\}$
12. Answers vary, one possibility is $M = \{\ x \mid 21 \leq x \leq 30, x \in \mathbb{N}\}$.
13. Answers vary, one possibility is $N = \{\ 5x \mid x > 0, x \in \mathbb{N}\}$.

Section 8.2 solutions

1. True.
2. False.
3. True.
4. True.
5. True.
6. False.
7. True.
8. 8
9. 22
10. 15
11. 901
12. $2^{\{5,10\}} = \{\varnothing, \{5\}, \{10\}, \{5, 10\}\}$
13. 32
14. 16
15. 63
16. 3

Section 8.3 solutions

1. $\{1, 2, 4, 6, 8, 10, 12, 14, 16\}$

2. $\{1, 2, 3, 4, 7, 8, 9, 10\}$

3. $\{\ x \mid 5 \leq x \leq 20\}$

4. $\{6, 12, 18\}$

5. $\varnothing$

6. $\{\ x \mid 10 \leq x \leq 15\}$

7. $\{0, 2, 5, 8, 9\}$

8. $\{1, 3, 5, 7, 9, 10, 11, 12, 13, 14, 15, \ldots\}$

9. $\{\ x \mid x \leq 400\}$

10. $\{3, 5, 7, 11, 13\}$

11. $\{1, 4, 6, 8, 9, 10\}$

12. $\{3, 7, 11, 15, 19, 23, 27, \ldots\}$

13. (a) $\{1, 4, 6, 8, 9, 10\}$

    (b) $\{2, 3, 5, 6, 7, 8\}$

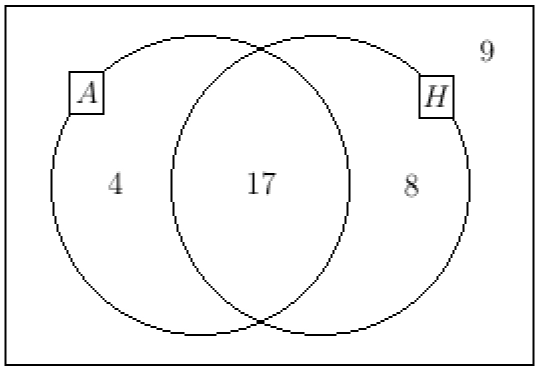
    (c) $\{2, 4, 6, 7, 8, 9, 10\}$

Section 8.4 solutions

1. 27 friends have seen at least one.

2. 10 friends have X-Boxes.

3. 3 papers have both written.

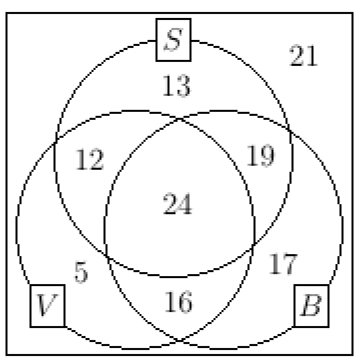


4. (a)

   (b) 9.

   (c) 8.

   (d) 29.



5. (a)

   (b) 17.

   (c) 40.

(d) 30.

(e) 21.

(f) 59.

Index